Chapter 1

1.1. A finite element is a small body or unit interconnected to other units to model a larger structure or system.

1.2. Discretization means dividing the body (system) into an equivalent system of finite elements with associated nodes and elements.

1.3. The modern development of the finite element method began in 1941 with the work of Hrennikoff in the field of structural engineering.

1.4. The direct stiffness method was introduced in 1941 by Hrennikoff. However, it was not commonly known as the direct stiffness method until 1956.

1.5. A matrix is a rectangular array of quantities arranged in rows and columns that is often used to aid in expressing and solving a system of algebraic equations.

1.6. As computer developed it made possible to solve thousands of equations in a matter of minutes.

1.7. The following are the general steps of the finite element method.

Step 1

Divide the body into an equivalent system of finite elements with associated nodes and choose the most appropriate element type.

Step 2

Choose a displacement function within each element.

Step 3

Relate the stresses to the strains through the stress/strain law—generally called the constitutive law.

Step 4

Derive the element stiffness matrix and equations. Use the direct equilibrium method, a work or energy method, or a method of weighted residuals to relate the nodal forces to nodal displacements.

Step 5

Assemble the element equations to obtain the global or total equations and introduce boundary conditions.

Step 6

Solve for the unknown degrees of freedom (or generalized displacements).

Step 7

Solve for the element strains and stresses.

Step 8

Interpret and analyze the results for use in the design/analysis process.

1.8. The displacement method assumes displacements of the nodes as the unknowns of the problem. The problem is formulated such that a set of simultaneous equations is solved for nodal displacements.

1.9. Four common types of elements are: simple line elements, simple two-dimensional elements, simple three-dimensional elements, and simple axisymmetric elements.

1.10 Three common methods used to derive the element stiffness matrix and equations are

(1) direct equilibrium method

(2) work or energy methods

(3) methods of weighted residuals

1.11. The term ‘degrees of freedom’ refers to rotations and displacements that are associated with each node.

1.12. Five typical areas where the finite element is applied are as follows.

(1) Structural/stress analysis

(2) Heat transfer analysis

(3) Fluid flow analysis

(4) Electric or magnetic potential distribution analysis

(5) Biomechanical engineering

1.13. Five advantages of the finite element method are the ability to

(1) Model irregularly shaped bodies quite easily

(2) Handle general load conditions without difficulty

(3) Model bodies composed of several different materials because element equations are evaluated individually

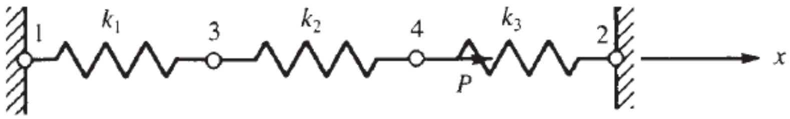
(4) Handle unlimited numbers and kinds of boundary conditions

(5) Vary the size of the elements to make it possible to use small elements where necessary

Chapter 2

2.1

(a)



$$[k^{(1)}] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix}$$

$$[k_3{}^{(3)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}$$

$$[K] = [k^{(1)}] + [k^{(2)}] + [k^{(3)}]$$

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix}$$

(b) Nodes 1 and 2 are fixed so $u_1 = 0$ and $u_2 = 0$ and $[K]$ becomes

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ P \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\{F\} = [K]\,\{d\} \Rightarrow [K]^{-1}\,\{F\} = [K]^{-1}\,[K]\,\{d\}$$

$$\Rightarrow\ [K]^{-1}\,\{F\} = \{d\}$$

Using the adjoint method to find $[K^{-1}]$

$$C_{11} = k_2 + k_3 \qquad C_{21} = (-1)^3\,(-k_2)$$

$$C_{12} = (-1)^{1+2}\,(-k_2) = k_2 \qquad C_{22} = k_1 + k_2$$

$$[C] = \begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix} \text{ and } C^T = \begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}$$

$$\det [K] = |\,[K]\,| = (k_1 + k_2)\,(k_2 + k_3) - (-k_2)\,(-k_2)$$

$$\Rightarrow\quad |\,[K]\,| = (k_1 + k_2)\,(k_2 + k_3) - k_2^2$$

$$[K^{-1}] = \frac{[C^T]}{\det K}$$

$$[K^{-1}] = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}}{(k_1 + k_2)(k_2 + k_3) - k_2^2} = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}\begin{Bmatrix} 0 \\ P \end{Bmatrix}}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$\Rightarrow\quad u_3 = \frac{k_2\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$\Rightarrow\quad u_4 = \frac{(k_1 + k_2)\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

(c) In order to find the reaction forces we go back to the global matrix $F = [K]\,\{d\}$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

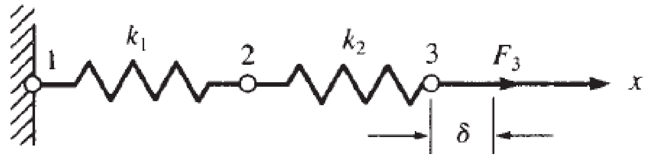
$$F_{1x} = -k_1\,u_3 = -k_1\,\frac{k_2\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$\Rightarrow F_{1x} = \frac{-k_1\,k_2\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$F_{2x} = -k_3\,u_4 = -k_3\,\frac{(k_1 + k_2)\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$\Rightarrow F_{2x} = \frac{-k_3\,(k_1 + k_2)\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

2.2



$$k_1 = k_2 = k_3 = 1000\ \frac{\text{lb}}{\text{in.}}$$

$$[k^{(1)}] = \begin{array}{c} \begin{array}{cc} (1) & (2) \end{array} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array} \begin{array}{c} (1) \\ (2) \end{array} ; \quad [k^{(2)}] = \begin{array}{c} \begin{array}{cc} (2) & (3) \end{array} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array} \begin{array}{c} (2) \\ (3) \end{array}$$

By the method of superposition the global stiffness matrix is constructed.

$$[K] = \begin{array}{c} \begin{array}{ccc} (1) & (2) & (3) \end{array} \\ \begin{bmatrix} k & -k & 0 \\ -k & k+k & -k \\ 0 & -k & k \end{bmatrix} \end{array} \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \Rightarrow [K] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

Node 1 is fixed $\Rightarrow u_1 = 0$ and $u_3 = \delta$

$$\{F\} = [K]\ \{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = \delta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 \\ \delta \end{Bmatrix} \Rightarrow \begin{Bmatrix} 0 = 2k\,u_2 - k\delta \\ F_{3x} = -k\,u_2 + k\delta \end{Bmatrix}$$

$$\Rightarrow \quad u_2 = \frac{k\delta}{2k} = \frac{\delta}{2} = \frac{1\text{ in.}}{2} \Rightarrow u_2 = 0.5''$$

$$F_{3x} = -\,k\,(0.5'') + k\,(1'')$$

$$F_{3x} = (-\,1000\ \frac{\text{lb}}{\text{in.}})\,(0.5'') + (1000\ \frac{\text{lb}}{\text{in.}})\,(1'')$$

$$F_{3x} = 500\text{ lbs}$$

Internal forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 0.5'' \end{Bmatrix}$$

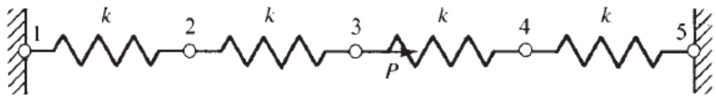
$$\Rightarrow \quad f_{1x}^{(1)} = (-\,1000\ \frac{\text{lb}}{\text{in.}})\,(0.5'') \Rightarrow f_{1x}^{(1)} = -\,500\text{ lb}$$

$$f_{2x}^{(1)} = (1000\ \frac{\text{lb}}{\text{in.}})\,(0.5'') \Rightarrow f_{2x}^{(1)} = 500\text{ lb}$$

Element (2)

$$\begin{Bmatrix} f_{2x}{}^{(2)} \\ f_{3x}{}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 = 0.5'' \\ u_3 = 1'' \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}{}^{(2)} = -500 \text{ lb} \\ f_{3x}{}^{(2)} = 500 \text{ lb} \end{matrix}$$

2.3



(a) $[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

By the method of superposition we construct the global $[K]$ and knowing $\{F\} = [K]\,\{d\}$ we have

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = P \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 = 0 \end{Bmatrix}$$

(b) $$\begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \Rightarrow \begin{aligned} 0 &= 2ku_2 - ku_3 && (1) \\ P &= -ku_2 + 2ku_3 - ku_4 && (2) \\ 0 &= -ku_3 + 2ku_4 && (3) \end{aligned}$$

$$\Rightarrow \quad u_2 = \frac{u_3}{2} \; ; \; u_4 = \frac{u_3}{2}$$

Substituting in the second equation above

$$P = -k\,u_2 + 2k\,u_3 - k\,u_4$$

$$\Rightarrow \quad P = -k\left(\frac{u_3}{2}\right) + 2k\,u_3 - k\left(\frac{u_3}{2}\right)$$

$$\Rightarrow \quad P = k\,u_3$$

$$\Rightarrow \quad u_3 = \frac{P}{k}$$

$$u_2 = \frac{P}{2k} \; ; \; u_4 = \frac{P}{2k}$$

(c) In order to find the reactions at the fixed nodes 1 and 5 we go back to the global equation $\{F\} = [K]\,\{d\}$

$$F_{1x} = -k\,u_2 = -k\,\frac{P}{2k} \Rightarrow F_{1x} = -\frac{P}{2}$$

$$F_{5x} = -k\,u_4 = -k\,\frac{P}{2k} \Rightarrow F_{5x} = -\frac{P}{2}$$

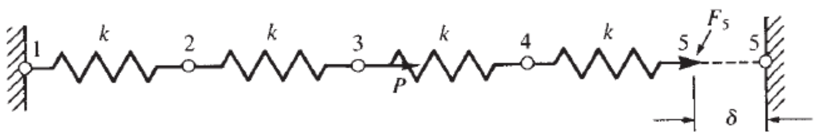
Check

$$\Sigma F_x = 0 \Rightarrow F_{1x} + F_{5x} + P = 0$$

$$\Rightarrow -\frac{P}{2} + \left(-\frac{P}{2}\right) + P = 0$$

$$\Rightarrow 0 = 0$$

2.4



(a) $[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

By the method of superposition the global $[K]$ is constructed.

Also $\{F\} = [K]\,\{d\}$ and $u_1 = 0$ and $u_5 = \delta$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 0 \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = \delta \end{Bmatrix}$$

(b) $0 = 2k\,u_2 - k\,u_3$ (1)

$0 = -k u_2 + 2k\,u_3 - k\,u_4$ (2)

$0 = -k\,u_3 + 2k\,u_4 - k\,\delta$ (3)

From (2)

$u_3 = 2\,u_2$

From (3)

$$u_4 = \frac{\delta + 2\,u_2}{2}$$

Substituting in Equation (2)

$$\Rightarrow -k\,(u_2) + 2k\,(2\,u_2) - k\left(\frac{\delta + 2u_2}{2}\right)$$

$$\Rightarrow -u_2 + 4\,u_2 - u_2 - \frac{\delta}{2} = 0 \Rightarrow u_2 = \frac{\delta}{4}$$

$$\Rightarrow u_3 = 2\,\frac{\delta}{4} \Rightarrow u_3 = \frac{\delta}{2}$$

$$\Rightarrow u_4 = \frac{\delta + 2\,\frac{\delta}{4}}{2} \Rightarrow u_4 = \frac{3\,\delta}{4}$$

(c) Going back to the global equation

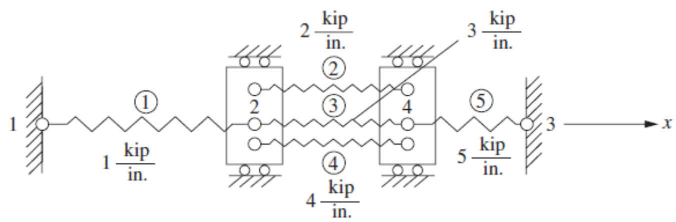
$$\{F\} = [K]\{d\}$$

$$F_{1x} = -k\ u_2 = -k\frac{\delta}{4} \Rightarrow F_{1x} = -\frac{k\,\delta}{4}$$

$$F_{5x} = -k\,u_4 + k\,\delta = -k\left(\frac{3\,\delta}{4}\right) + k\,\delta$$

$$\Rightarrow\ F_{5x} = \frac{k\,\delta}{4}$$

2.5



$$[k^{(1)}] = \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}; \quad [k^{(2)}] = \begin{array}{c} \begin{array}{cc} u_2 & u_4 \end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \begin{array}{c} \begin{array}{cc} u_2 & u_4 \end{array} \\ \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \end{array}; \quad [k^{(4)}] = \begin{array}{c} \begin{array}{cc} u_2 & u_4 \end{array} \\ \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array}$$

$$[k^{(5)}] = \begin{array}{c} \begin{array}{cc} u_4 & u_3 \end{array} \\ \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \end{array}$$

Assembling global [K] using direct stiffness method

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+2+3+4 & 0 & -2-3-4 \\ 0 & 0 & 5 & -5 \\ 0 & -2-3-4 & -5 & 2+3+4+5 \end{bmatrix}$$

Simplifying

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 10 & 0 & -9 \\ 0 & 0 & 5 & -5 \\ 0 & -9 & -5 & 14 \end{bmatrix} \frac{\text{kip}}{\text{in.}}$$

2.6 Now apply + 3 kip at node 2 in spring assemblage of P 2.5.

$$\therefore\ F_{2x} = 3\text{ kip}$$

$$[K]\{d\} = \{F\}$$

$[K]$ from P 2.5

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 10 & 0 & -9 \\ 0 & 0 & 5 & -5 \\ 0 & -9 & -5 & 14 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 3 \\ F_3 \\ 0 \end{Bmatrix} \qquad \text{(A)}$$

where $u_1 = 0$, $u_3 = 0$ as nodes 1 and 3 are fixed.

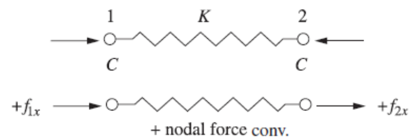
Using Equations (1) and (3) of (A)

$$\begin{bmatrix} 10 & -9 \\ -9 & 14 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 0 \end{Bmatrix}$$

Solving

$$u_2 = 0.712 \text{ in.}, \quad u_4 = 0.458 \text{ in.}$$

2.7



$$f_{1x} = C, \quad f_{2x} = -C$$

$$f = -k\delta = -k(u_2 - u_1)$$

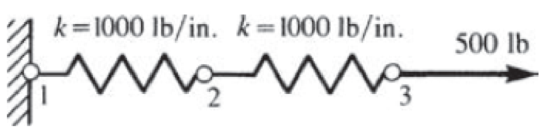
$$\therefore \quad f_{1x} = -k(u_2 - u_1)$$

$$f_{2x} = -(-k)(u_2 - u_1)$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore \quad [K] = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{matrix} \text{same as for} \\ \text{tensile element} \end{matrix}$$

2.8



$$k_1 = 1000\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \; k_2 = 1000\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

So

$$[K] = 1000\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\Rightarrow \quad \begin{bmatrix} F_1 = ? \\ F_2 = 0 \\ F_3 = 500 \end{bmatrix} = 1000\begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{0} \\ \cancel{-1} & 2 & -1 \\ \cancel{0} & -1 & 1 \end{bmatrix}\begin{Bmatrix} \cancel{u_1 = 0} \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$\Rightarrow \quad 0 = 2000\, u_2 - 1000\, u_3 \qquad (1)$$

$$500 = -1000\, u_2 + 1000\, u_3 \qquad (2)$$

From (1)

$$u_2 = \frac{1000}{2000}\, u_3 \Rightarrow u_2 = 0.5\, u_3 \qquad (3)$$

Substituting (3) into (2)

$$\Rightarrow \quad 500 = -1000\,(0.5\, u_3) + 1000\, u_3$$

$$\Rightarrow \quad 500 = 500\, u_3$$

$$\Rightarrow \quad u_3 = 1 \text{ in.}$$

$$\Rightarrow \quad u_2 = (0.5)\,(1 \text{ in.}) \Rightarrow u_2 = 0.5 \text{ in.}$$

Element 1–2

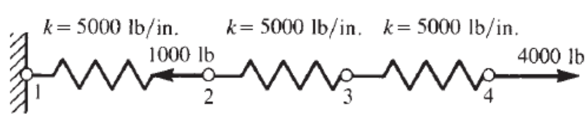
$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 1000\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \text{ in.} \\ 0.5 \text{ in.} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -500 \text{ lb} \\ f_{2x}^{(1)} = 500 \text{ lb} \end{matrix}$$

Element 2–3

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 1000\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.5 \text{ in.} \\ 1 \text{ in.} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -500 \text{ lb} \\ f_{3x}^{(2)} = 500 \text{ lb} \end{matrix}$$

$$F_{1x} = 500\,[1 \;\; -1 \;\; 0]\begin{bmatrix} 0 \\ 0.5 \text{ in.} \\ 1 \text{ in.} \end{bmatrix} \Rightarrow F_{1x} = -500 \text{ lb}$$

2.9



$$[k^{(1)}] = \begin{array}{c} \begin{matrix} (1) & \quad (2) \end{matrix} \\ \begin{bmatrix} 5000 & -5000 \\ -5000 & 5000 \end{bmatrix} \\ \begin{matrix} (2) & \quad (3) \end{matrix} \end{array}$$

$$[k^{(2)}] = \begin{bmatrix} 5000 & -5000 \\ -5000 & 5000 \end{bmatrix}$$

$$[k^{(3)}] = \begin{array}{c} \begin{array}{cc} (3) & (4) \end{array} \\ \begin{bmatrix} 5000 & -5000 \\ -5000 & 5000 \end{bmatrix} \end{array}$$

$$[K] = \begin{array}{c} \begin{array}{cccc} (1) & (2) & (3) & (4) \end{array} \\ \begin{bmatrix} 5000 & -5000 & 0 & 0 \\ -5000 & 10000 & -5000 & 0 \\ 0 & -5000 & 10000 & -5000 \\ 0 & 0 & -5000 & 5000 \end{bmatrix} \end{array}$$

$$\begin{Bmatrix} \cancel{F_{1x} = ?} \\ F_{2x} = -1000 \\ F_{3x} = 0 \\ F_{4x} = 4000 \end{Bmatrix} = \begin{bmatrix} \cancel{5000} & \cancel{-5000} & \cancel{0} & \cancel{0} \\ \cancel{-5000} & 10000 & -5000 & 0 \\ \cancel{0} & -5000 & 10000 & -5000 \\ \cancel{0} & 0 & -5000 & 5000 \end{bmatrix} \begin{Bmatrix} \cancel{u_1 = 0} \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$\Rightarrow$ $u_1 = 0$ in.
$u_2 = 0.6$ in.
$u_3 = 1.4$ in.
$u_4 = 2.2$ in.

Reactions

$$F_{1x} = [5000 \quad -5000 \quad 0 \quad 0] \begin{Bmatrix} u_1 = 0 \\ u_2 = 0.6 \\ u_3 = 1.4 \\ u_4 = 2.2 \end{Bmatrix} \Rightarrow F_{1x} = -3000 \text{ lb}$$

Element forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} 5000 & -5000 \\ -5000 & 5000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.6 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{1x}^{(1)} = -3000 \text{ lb} \\ f_{2x}^{(1)} = 3000 \text{ lb} \end{array}$$

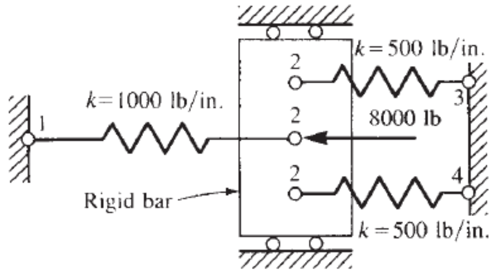
Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} 5000 & -5000 \\ -5000 & 5000 \end{bmatrix} \begin{Bmatrix} 0.6 \\ 1.4 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{2x}^{(2)} = -4000 \text{ lb} \\ f_{3x}^{(2)} = 4000 \text{ lb} \end{array}$$

Element (3)

$$\begin{Bmatrix} f_{3x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = \begin{bmatrix} 5000 & -5000 \\ -5000 & 5000 \end{bmatrix} \begin{Bmatrix} 1.4 \\ 2.2 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{3x}^{(3)} = -4000 \text{ lb} \\ f_{4x}^{(3)} = 4000 \text{ lb} \end{array}$$

2.10



$$[k^{(1)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = -8000 \\ F_{3x} = ? \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \qquad u_2 = \frac{-8000}{2000} = -4 \text{ in.}$$

Reactions

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ -4 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 4000 \\ -8000 \\ 2000 \\ 2000 \end{Bmatrix} \text{lb}$$

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ -4 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 4000 \\ -4000 \end{Bmatrix} \text{lb}$$

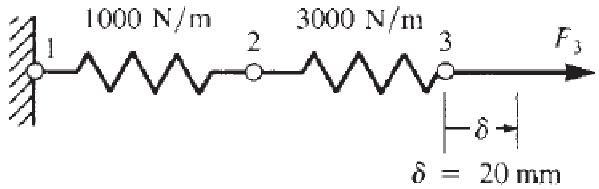
Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} -4 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -2000 \\ 2000 \end{Bmatrix} \text{lb}$$

Element (3)

$$\begin{Bmatrix} f_{2x}{}^{(3)} \\ f_{4x}{}^{(3)} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} -4 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}{}^{(3)} \\ f_{4x}{}^{(3)} \end{Bmatrix} \begin{Bmatrix} -2000 \\ 2000 \end{Bmatrix} \text{lb}$$

2.11



$$[k^{(1)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}; \quad [k^{(2)}] = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = ? \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 4000 & -3000 \\ 0 & -3000 & 3000 \end{bmatrix} \begin{bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0.02 \text{ m} \end{bmatrix}$$

$$\Rightarrow \quad u_2 = 0.015 \text{ m}$$

Reactions

$$F_{1x} = (-1000)(0.015) \Rightarrow F_{1x} = -15 \text{ N}$$

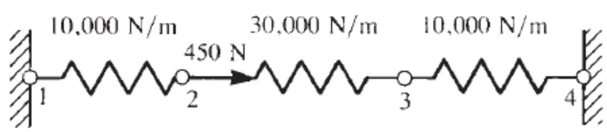
Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.015 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} -15 \\ 15 \end{Bmatrix} \text{ N}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{Bmatrix} 0.015 \\ 0.02 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{Bmatrix} -15 \\ 15 \end{Bmatrix} \text{ N}$$

2.12



$$[k^{(1)}] = [k^{(3)}] = 10000 \begin{Bmatrix} 1 & -1 \\ -1 & 1 \end{Bmatrix}$$

$$[k^{(2)}] = 10000 \begin{Bmatrix} 3 & -3 \\ -3 & 3 \end{Bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 450 \text{ N} \\ F_{3x} = 0 \\ F_{4x} = ? \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 4 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$0 = -3\,u_2 + 4\,u_3 \Rightarrow u_2 = \frac{4}{3}\,u_3 \Rightarrow u_2 = 1.33\,u_3$$

$$450 \text{ N} = 40000\,(1.33\,u_3) - 30000\,u_3$$

$$\Rightarrow \quad 450 \text{ N} = (23200\ \frac{\text{N}}{\text{m}})\,u_3 \Rightarrow u_3 = 1.93 \times 10^{-2}\text{ m}$$

$$\Rightarrow \quad u_2 = 1.5\,(1.94 \times 10^{-2}) \Rightarrow u_2 = 2.57 \times 10^{-2}\text{ m}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 2.57 \times 10^{-2} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -257 \text{ N} \\ f_{2x}^{(1)} = 257 \text{ N} \end{matrix}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 30000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 2.57 \times 10^{-2} \\ 1.93 \times 10^{-2} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = 193 \text{ N} \\ f_{3x}^{(2)} = -193 \text{ N} \end{matrix}$$

Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.93 \times 10^{-2} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x}^{(3)} = 193 \text{ N} \\ f_{4x}^{(3)} = -193 \text{ N} \end{matrix}$$

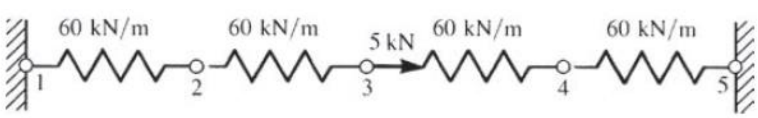
Reactions

$$\{F_{1x}\} = (10000\ \frac{\text{N}}{\text{m}})\ [1 - 1] \begin{Bmatrix} 0 \\ 2.57 \times 10^{-2} \end{Bmatrix} \Rightarrow F_{1x} = -257 \text{ N}$$

$$\{F_{4x}\} = (10000\ \frac{\text{N}}{\text{m}})\ [-1 \quad 1] \begin{Bmatrix} 1.93 \times 10^{-2} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad F_{4x} = -193 \text{ N}$$

2.13



$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 5\text{ kN} \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = 0 \end{Bmatrix}$$

$$\left.\begin{array}{l} 0 = 2u_2 - u_3 \Rightarrow u_2 = 0.5\,u_3 \\ 0 = -u_3 + 2u_4 \Rightarrow u_4 = 0.5\,u_3 \end{array}\right\} \Rightarrow u_2 = u_4$$

$\Rightarrow \quad 5\text{ kN} = -60\,u_2 + 120\,(2\,u_2) - 60\,u_2$

$\Rightarrow \quad 5 = 120\,u_2 \Rightarrow u_2 = 0.042\text{ m}$

$\Rightarrow \quad u_4 = 0.042\text{ m}$

$\Rightarrow \quad u_3 = 2(0.042) \Rightarrow u_3 = 0.084\text{ m}$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.042 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{1x}^{(1)} = -2.5\text{ kN} \\ f_{2x}^{(1)} = 2.5\text{ kN} \end{array}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.042 \\ 0.084 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{2x}^{(2)} = -2.5\text{ kN} \\ f_{3x}^{(2)} = 2.5\text{ kN} \end{array}$$

Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.084 \\ 0.042 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{3x}^{(3)} = 2.5\text{ kN} \\ f_{4x}^{(3)} = -2.5\text{ kN} \end{array}$$

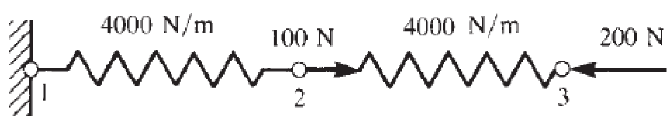
Element (4)

$$\begin{Bmatrix} f_{4x} \\ f_{5x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.042 \\ 0 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{4x}^{(4)} = 2.5\text{ kN} \\ f_{5x}^{(4)} = -2.5\text{ kN} \end{array}$$

$$F_{1x} = 60\,[1 \quad -1] \begin{Bmatrix} 0 \\ 0.042 \end{Bmatrix} \Rightarrow F_{1x} = -2.5\text{ kN}$$

$$F_{5x} = 60\,[-1 \quad 1] \begin{Bmatrix} 0.042 \\ 0 \end{Bmatrix} \Rightarrow F_{5x} = -2.5\text{ kN}$$

2.14



$$[k^{(1)}] = [k^{(2)}] = 4000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$