Answers

Answers to Chapter 1

1-01. $100 - (25 + 25) = 50$. (25 study economics and $30 - 5$ study only political science, so there are 50 students in all who study neither.)

1-02. (a) True statement, since $A \cap B = \{2, 3, 4\}$.

(b) True statement.

(c) False statement, since $(A \setminus B) \cap C = \{5\} \cap \{1, 3, 6, 7\} = \emptyset$.

(d) True statement, since $A \cap C = \{3\} \subseteq B$.

1-03. (a) False implication. Put, for example, $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{1, 4\}$.

(b) True equality. Use a Venn diagram.

(c) True equality. Use a Venn diagram.

(d) Valid implication. Use a Venn diagram.

1-04. (a) False. For example, $(-1)^3 + 1^3 = -1 + 1 = 0$.

(b) Valid equivalence.

(c) Valid implication. (The reverse implication is false.)

(d) Valid implication. $2 \cdot 3 + 4 \cdot 5 = 26$.

1-05. (a) Can be reversed. If $x^3 = 27$, then $x = 3$ satisfies the equation and is the only solution because x^3 is strictly increasing.

(b) Can be reversed. Since $x^4 + 1$ is never 0, it follows that $x(x^2 + 1) = 0$ implies $x = 0$.

(c) The implication cannot be reversed since $(x + 2)^2(x - 3) = 0$ when $x = -2$.

(d) Can be reversed. If $\sqrt{1+x} = 5 - x$, then $1 + x = (5 - x)^2$, or $x^2 - 11x + 24 = 0$. This equation has the two solutions $x = 3$ and $x = 8$, but only $x = 3$ satisfies the given equation.

1-06. (a) Expresses the opposite implication.

(b) Expresses the same as the given statement. ($P \Rightarrow Q$ is equivalent to (not Q) $\Rightarrow$ (not P).)

(c) Expresses the same.

1-07. If $\sqrt{2} + 3$ were rational, we would have $\sqrt{2} + 3 = r$, with r rational. But then $\sqrt{2} = r - 3$ would also be rational, contradicting the fact the $\sqrt{2}$ is irrational.

1-08. The identity $\sqrt{a} - \sqrt{b} = \dfrac{a - b}{\sqrt{a} + \sqrt{b}}$ shows that $\sqrt{a} - \sqrt{b}$ is rational. But since $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are both rational, so is their sum. Thus, $2\sqrt{a}$ is rational, and then $\sqrt{a}$ is rational.

1-09. For $n = 3$, the inequality is satisfied because $7 < 8$. Suppose (∗) is valid for some value of n greater than or equal to 3, say for $n = k$, so (∗∗) $2k + 1 < 2^k$. We must prove that (∗) is valid also for $n = k + 1$. Using (∗∗), we have $2(k + 1) + 1 = 2k + 1 + 2 < 2^k + 2 < 2^k + 2^k = 2^{k+1}$. Thus, (∗) is valid for $n = k + 1$, and the proof is complete.

1-10. (a) For each natural number n let $S(n) = 1\cdot 2 + \cdots + n(n+1)$, and let $A(n)$ be the statement $S(n) = \frac{1}{3}n(n+1)(n+2)$. Direct calculation shows that $A(1)$ is true, so we have a starting point for the induction. Now for the induction step. Suppose $A(k)$ is true for some natural number k. We want to show that $A(k + 1)$ is true. We have:

$$\begin{aligned} S(k+1) &= \underbrace{1\cdot 2 + \cdots + k(k+1)}_{S(k)} + (k+1)(k+2) = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \\ &= \frac{1}{3}[k(k+1)(k+2) + 3(k+1)(k+2)] = \frac{1}{3}(k+1)(k+2)(k+3) \end{aligned}$$

which shows that $A(k+1)$ is true. It follows by induction that $A(n)$ is true for all natural numbers n.

(b) For each natural number n let $B(n)$ be the statement $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$. It is easy to check that $B(1)$ is true. If we can show that the induction step, $B(k) \Rightarrow B(k + 1)$, holds for every natural number k, it will follow that $B(n)$ is true for all natural numbers n. So suppose $B(k)$ is true. Then:

$$\begin{aligned} 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 = \frac{1}{4}(k+1)^2[k^2 + 4(k+1)] \\ &= \frac{1}{4}(k+1)^2[k^2 + 4k + 4] = \frac{1}{4}(k+1)^2(k+2)^2 \end{aligned}$$

which is exactly what $B(k + 1)$ claims. So $B(n)$ is indeed true for all natural numbers n.

1-11. $s_1 = 41$, $s_2 = 43$, $s_3 = 47$, $s_4 = 53$, $s_5 = 61$ are all primes. For $n = 41$, we have $n^2 - n + 41 = 41^2$, which is not a prime!

Answers to Chapter 2

2-01. (a) An integer. (b) A rational number equal to 123/100. (c) Rational. (d) Rational. (Periodic decimal fraction.) (e) Irrational. (There is no finite sequence of digits that repeats itself indefinitely, because one extra 1 is added between each successive pair of 3's.)

2-02. If $x = 0.151515\ldots$, then $100x = 15.151515\ldots$ and $100x - x = 15$, so $99x = 15$ and thus $x = 15/99 = 5/33$.

2-03. (a) $\frac{1}{2}x + 3$ (b) $\dfrac{a}{b-10}$ (c) $\frac{1}{3}(n + \frac{3}{7}p)$ (d) $4x - x = 5x + 1$ (e) $\frac{1}{10}a + 10b$

2-04. (a) $5^{2+x} = 5^7$, so $2 + x = 7$ and hence $x = 5$. (b) $x = 0$
(c) $10^x \div 10^5 = 10^{x-5} = 10^{-2}$, so $x = 3$. (d) $(25)^2 = 5^4 = 5^x$, so $x = 4$.
(e) $2^{10} = 2^2 \cdot 2^x = 2^{2+x}$, so $x = 8$. (f) $x = 0$.

2-05. (a) $(1^{-2} + 2^{-2} + 3^{-2})^{-1} = (1 + 1/4 + 1/9)^{-1} = \left(\dfrac{36+9+4}{36}\right)^{-1} = \left(\dfrac{49}{36}\right)^{-1} = \dfrac{36}{49}$.

(b) $\left(1+\dfrac{1}{n}\right)\left(1-\dfrac{1}{m}\right)=1+\dfrac{1}{n}-\dfrac{1}{m}-\dfrac{1}{nm}=1$ implies $\dfrac{1}{n}=\dfrac{1}{m}+\dfrac{1}{mn}$. Multiplying each side by mn gives $m = n + 1$.

2-06. (a) False, because $3^5 = 243$, $5^3 = 125$.

(b) False. The left-hand side is $(5^2)^3 = 5^6$, whereas the right-hand side is $5^{2^3} = 5^8$.

(c) True. By the rule for a power of a power, $(a^p)^q = a^{pq} = (a^q)^p$.

(d) False. $(5 + 7)^2 = 12^2 = 144$, $5^2 + 7^2 = 25 + 49 = 74$.

(e) False, because $(2x + 4)/2 = 2x/2 + 4/2 = x + 2$.

(f) True. Both sides are equal to $2x - 2y$.

2-07. (a) $40{,}000(1.025)^{10} \approx 51{,}203.38$ (b) $30{,}000(1.06)^{-8} \approx 18{,}822.37$.

2-08. $\left(\dfrac{1}{x}+\dfrac{1}{y}\right)^2=\dfrac{1}{x^2}+\dfrac{2}{xy}+\dfrac{1}{y^2}=\dfrac{1}{x^2}+\dfrac{1}{y^2}+2\dfrac{1}{xy}=A+\dfrac{2}{B}$.

2-09. (a) $\dfrac{5a-3}{25a^2-9}=\dfrac{5a-3}{(5a+3)(5a-3)}=\dfrac{1}{5a+3}$. (b) $\dfrac{4x^2yz}{2xy+2xyz}=\dfrac{4x^2yz}{2xy(1+z)}=\dfrac{2xz}{1+z}$.

(c) $\dfrac{t^4-16}{(t-2)(t^2+4)}=\dfrac{(t^2+4)(t^2-4)}{(t-2)(t^2+4)}=\dfrac{(t^2+4)(t+2)(t-2)}{(t-2)(t^2+4)}=t+2$.

2-10. $\dfrac{\frac{100}{p}}{(1+\frac{p}{100})^{-1}}-1=\dfrac{100}{p}\left(1+\dfrac{p}{100}\right)-1=\dfrac{100}{p}+1-1=\dfrac{100}{p}$.

2-11. (a) $\dfrac{556^2-555^2}{1111}=\dfrac{(556+555)(556-555)}{1111}=1$. (b) $\dfrac{125^{-2/3}}{5^{-3}}=\dfrac{(\sqrt[3]{125})^{-2}}{5^{-3}}=\dfrac{5^{-2}}{5^{-3}}=5$.

(c) $\left(\dfrac{2}{9}-\dfrac{1}{6}\right)^{-1}=\left(\dfrac{1}{18}\right)^{-1}=18$. (d) $y^{\beta} z^{2\gamma}$.

2-12. (a) $\dfrac{(896-1)897}{895}=897$. (b) $\dfrac{1}{1/2}+\dfrac{1}{3/4}+\dfrac{1}{3/2}=2+\dfrac{4}{3}+\dfrac{2}{3}=4$. (c) $\dfrac{p^{2\alpha}q^{-\beta}}{p^{\alpha}q^{-2\beta}}=p^{\alpha}q^{\beta}$.

2-13. (a) $\dfrac{9{,}986\cdot 9{,}987-9{,}987}{9{,}985}=\dfrac{(9{,}986-1)\cdot 9{,}987}{9{,}985}=\dfrac{9{,}985\cdot 9{,}987}{9{,}985}=9{,}987$.

(b) $\left(\dfrac{1}{r}\right)^{-3}\div r^2=r^3\div r^2=r$. (c) $125^{-2/3}=\dfrac{1}{(\sqrt[3]{125})^2}=\dfrac{1}{5^2}=\dfrac{1}{25}$.

2-14. (a) $3(\sqrt{a})^3-2a\sqrt{a}-(a^{1/4})^2/a^{-1}=3a^{3/2}-2a^{3/2}-a^{1/2}/a^{-1}=3a^{3/2}-2a^{3/2}-a^{3/2}=0$.

(b) $\dfrac{x^{2\beta}(x^2y^2)^{\gamma}}{x^{\beta+2\gamma}}=\dfrac{x^{2\beta}x^{2\gamma}y^{2\gamma}}{x^{\beta+2\gamma}}=x^{2\beta+2\gamma-\beta-2\gamma}y^{2\gamma}=x^{\beta}y^{2\gamma}$.

(c) $\sqrt[3]{-64x^6}=(-64x^6)^{1/3}=(-64)^{1/3}(x^6)^{1/3}=-4x^2$.

2-15. (a) $\sqrt{25^2 - 15^2} = \sqrt{625 - 225} = \sqrt{400} = 20$. (Note that $\sqrt{a^2 - b^2}$ is NOT equal to $a - b$.)

(b) $\dfrac{(-2a)^3 a^{-2/3}}{-32(2a)^{-2} a^{1/3}} = \dfrac{-8a^3 a^{-2/3}}{-32 \cdot \frac{1}{4} a^{-2} a^{1/3}} = a^{3-2/3+2-1/3} = a^4$

(c) $\sqrt[3]{\dfrac{5^{pq-q} \cdot 5^{2p}}{5^{pq-p} \cdot 5^{2q}}} = \sqrt[3]{5^{pq-q+2p-pq+p-2q}} = \sqrt[3]{5^{3p-3q}} = \left(5^{3p-3q}\right)^{1/3} = 5^{p-q}$.

(d) $\frac{1}{2}[(P+Q+R)^2 - P^2 - Q^2 - R^2] = \frac{1}{2}[P^2 + Q^2 + R^2 + 2PQ + 2PR + 2QR - P^2 - Q^2 - R^2]$
$= \frac{1}{2}[2PQ + 2PR + 2QR] = PQ + PR + QR.$

2-16. (a) $2^{10}(32)^{-9/5} = 2^{10}(\sqrt[5]{32})^{-9} = 2^{10}2^{-9} = 2.$ (b) $\sqrt{169} - 10 = 3.$ (c) $\dfrac{a^{-3c}a^{3c}}{a^{-5c}a^{4c}} = \dfrac{1}{a^{-c}} = a^c.$

(d) $\dfrac{(1-x)+1+(1+x)}{1-x^2} = \dfrac{3}{1-x^2}$.

2-17. (a) $1 - \frac{5}{2} + \frac{3}{2} + 1 = 1.$ (b) $64 \cdot 32^{-3/5} = 2^6 \cdot (\sqrt[5]{32})^{-3} = 2^6 \cdot 2^{-3} = 8.$ (c) $\dfrac{8 + 2(x-4) - 2x}{x(x-4)} = 0$.

2-18. (a) $V = (4/3)\pi r^3$ implies $r = (3V/4\pi)^{1/3}$, so
$S = 4\pi r^2 = 4\pi(3V/4\pi)^{2/3} = (4\pi)^{1/3}3^{2/3}V^{2/3} = \sqrt[3]{36\pi}V^{2/3} \approx 4.836V^{2/3}$.
(b) The surface area is $4.836(100)^{2/3}$ m^2 = $4.836 \cdot 21.544 \approx 104.19$ m^2. You need 104.19/5 litres, slightly less than 21 litres.

2-19. (a) We start by collecting all the terms on the left, and get:

$$-\frac{1}{2}(x-5) - 2x + 1 \le 0 \Leftrightarrow -\frac{5}{2}x + \frac{7}{2} \le 0 \Leftrightarrow -\frac{5}{2}\left(x - \frac{7}{5}\right) \le 0 \Leftrightarrow x - \frac{7}{5} \ge 0 \Leftrightarrow x \ge \frac{7}{5}$$

because $-5/2$ is negative. (b) $-2 < x < 3$ (c) $x < -4$, $-2 \le x < 0$, or $x \ge 2$. (You should test the answer by checking it for values of x in each of the intervals.)

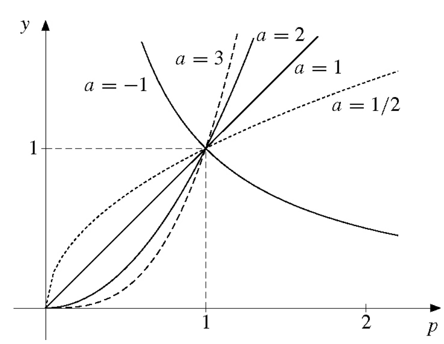

Figure A2-20

2-20. (a) Wrong. Look at $p = 1/4$ for example. Actually, $p < \sqrt{p}$ for all p in (0, 1).