# Chapter 1

# A PRELUDE TO CONTROL THEORY

## 1.5 Exercises and projects

The exercises in this chapter are meant to stimulate student thinking about interconnected systems. Students should be encouraged to use their background knowledge in physics, signals and systems, electronics, logic design, and other systems-level courses to piece together the concepts of time constants, transport delays, logical elements, time-frequency relationships, and such to be able to formulate a high-level system description. Given this breadth for problem solving, students are expected to develop the hierarchical structure of any system.

The fact is that there is a diversity of answers, and the advice is to let students brainstorm their ideas. These exercises are meant to expand student thinking about sensors and possible feedback architectures that will improve overall controllability and observability of systems.

**Exercise 1** Hero's control system is a classic system for beginning engineers to formulate what an open-loop control structure looks like. While there is no problem in terms of stability, the exercise demonstrates that in this, and in a multitude of other open-loop systems, humans do indeed become the feedback provider. In Hero's system, the fire on the altar is the control input. The human is the one who through the observations can control the altar fire either to open or close the temple doors.

To obtain a processing structure one needs to consider each element of the process as a processing block. The input and output of each block needs to be specified. If there are time delays, then such delay elements must be identified and marked as such. Where there are pulleys and spindles there must be appropriate conversion of forces that translate linear to rotational motion, and vice versa. Appropriate coupling equations must be shown that couple the dynamics of water discharged from the kettle to increase in mass of the bucket.

The notion of time constant in the input-output behavior of subsystems must be exploited.

A diagram showing the sequential processing of various elements of the control system results in a block diagram.

**Exercise 2** In this exercise the student is expected to recognize the shortfall in the response of open-loop systems for various open-loop examples of the text. Once the shortfalls are recognized, the question asked is how one converts such systems into more reliable systems that have feedback. This question is meant to force the student towards critical thinking about how to obtain a desired response.

Obtaining a desired response requires knowing what the plant is producing and comparing the response with some desired quantity. The resulting error is naturally a difference between desired and actual response. Say in the case of a toaster oven the desired level of "toasting" can be specified by how dark or light the toast should be, then an appropriate sensor, say a camera that can take the picture of the toast and compare with a desired toast level can be used to yield a more "controlled" response.

# Chapter 2

# MATHEMATICAL MODELS IN CONTROL

## 2.10 Exercises and projects

**Exercise 1** To show that a system is or is not linear we must prove that the principle of superposition does or does not apply.

a. Suppose $v(t) = v_1(t) + v_2(t)$. The given system is linear if and only if

$$y(t) = v_1(t)\frac{d}{dt}(v_1(t)) + v_2(t)\frac{d}{dt}(v_2(t))$$

However, given $v(t) = v_1(t) + v_2(t)$,

$$\begin{aligned} y(t) &= (v_1(t) + v_2(t))\frac{d}{dt}(v_1(t) + v_2(t)) \\ &= v_1(t)\frac{d}{dt}(v_1(t) + v_2(t)) + v_2(t)\frac{d}{dt}(v_1(t) + v_2(t)) \\ &= v_1(t)\frac{d}{dt}(v_1(t)) + v_1(t)\frac{d}{dt}(v_2(t)) \\ &\quad + v_2(t)\frac{d}{dt}(v_1(t)) + v_2(t)\frac{d}{dt}(v_2(t)) \end{aligned}$$

Hence the system is not linear.

b. For the cascade connected system, the time-domain output $q(t) = N_2 * N_1 * (v_1(t) + v_2(t))$, where "$*$" represents the convolution operator. The output is

$$y(t) = N_2 * q(t) = N_2 * N_1 * v(t)$$

Suppose the input is $v(t) = v_1(t) + v_2(t)$. Then

$$\begin{aligned} y(t) &= N_2 * N_1 * (v_1(t) + v_2(t)) \\ &= N_2 * N_1 * v_1(t) + N_2 * N_1 * v_2(t) \end{aligned}$$

Hence the system is linear.

**Exercise 2**

a. Consider a complex sinusoidal input signal $v(t) = Ae^{j(\varpi t+\phi)}$ in which the term $Ae^{j\phi}$ represents the phasor value of $v(t)$, and $Ae^{j(\varpi t)}$ is the sinusoidal carrier of $v(t)$. Expressing the complex sinusoid in terms of real and imaginary quantities yields

$$v(t) = \mathrm{Re}\, A\cos(\varpi t + \phi) + \mathrm{Im}\, A\sin(\varpi t + \phi)$$

For a linear system the input $v(t)$ produces an output $y(t)$. Let $y(t) = Be^{j(\varpi t+\theta)}$. Expressing the output in terms of real and imaginary quantities yields

$$y(t) = \mathrm{Re}\, B\cos(\varpi t + \theta) + \mathrm{Im}\, B\sin(\varpi t + \theta)$$

We can conclude therefore that an input $\mathrm{Re}\, v(t) = \mathrm{Re}\, A\cos(\varpi t + \phi)$ produces an output $\mathrm{Re}\, y(t) = \mathrm{Re}\, B\cos(\varpi t + \theta)$.

b. The response to an input $v(t)$ is $y(t)$. For a linear system this can be expressed as $y(t) = Av(t)$, where $A$ is a linear operator. Therefore

$$\frac{dy(t)}{dt} = A\frac{dv(t)}{dt}$$

**Exercise 3**

($i$) For the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we have the matrices

$$A = \left[ \begin{array}{cc} 1 & 0 \\ 2 & 2 \end{array} \right] \quad B = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \quad C = \left[ \begin{array}{cc} 2 & 1 \end{array} \right]$$

(a) For this system the controllability matrix is

$$C_0 = [B \quad AB] = \left[ \begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array} \right]$$

A system is fully controllable if and only if this matrix is of full rank, in this case rank 2 since the system is of second-order. A matrix has full rank if and only if its determinant is not zero. We see that

$$\det C_0 = \det \left[ \begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array} \right] = 2 \neq 0$$

and hence the system is fully controllable.

(b) For this second-order system, the observability matrix is

$$O_0 = [C \quad CA]^T = \begin{bmatrix} C^T & A^T C^T \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

A system is fully observable if and only if the observability matrix is of full rank, in this case 2 since the system is of second-order. This matrix has full rank if and only if the determinant is non-zero. We can either compute the determinant or rank, or both. Observing that

$$\det O_0 = \det \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = 0$$

or that $Rank\,[O_0] = 1 < 2$, proves the system is not fully observable. Note: Finding the rank of the matrix does not by itself explain *why* the system is not observable. To understand the reason why, we need to examine the transfer function as requested in (c).

(c) The transfer function of a linear time-invariant system is given by

$$Y(s) = [C]\,[sI - A]^{-1}\,B + D$$

where $A$, $B$, $C$, and $D$ are the matrices obtained from the state and output equations, and $I$ is the identity matrix. In this system, as well as in the others, $D = 0$.

$$\begin{aligned}
Y(s) &= \begin{bmatrix} 2 & 1 \end{bmatrix} \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \\
&= \begin{bmatrix} 2 & 1 \end{bmatrix} \left\{ \begin{bmatrix} s-1 & 0 \\ -2 & s-2 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \end{bmatrix} \left\{ \begin{bmatrix} \dfrac{(s-2)}{(s-1)(s-2)} & 0 \\ \dfrac{2}{(s-1)(s-2)} & \dfrac{(s-1)}{(s-1)(s-2)} \end{bmatrix} \right\} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \end{bmatrix} \left\{ \begin{bmatrix} \dfrac{(s-2)}{(s-1)(s-2)} \\ \dfrac{2}{(s-1)(s-2)} \end{bmatrix} \right\} \\
&= \frac{2(s-2)}{(s-1)(s-2)} + \frac{2}{(s-1)(s-2)} = \frac{2(s-2)+2}{(s-1)(s-2)} \\
&= \frac{2(s-1)}{(s-1)(s-2)} = \frac{2}{(s-2)}
\end{aligned}$$

The pole-zero cancellation makes the system clearly unobservable for the pole located at $s = 1$.

(d) To determine whether or not the system is stable, we need to examine the roots of the characteristic polynomial. If all the roots lie in the left-half $s$-plane, the system is stable. If not, the system is unstable. It is important to note that if even one root of the characteristic polynomial lies in the right-half $s$-plane, the system is unstable.

To find the characteristic polynomial, we look at the equation

$$\det[\lambda I - A] = 0$$

Now

$$\begin{aligned} \det[\lambda I - A] &= \det\left\{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}\right\} \\ &= \det\left\{\begin{bmatrix} \lambda - 1 & 0 \\ -2 & \lambda - 2 \end{bmatrix}\right\} = (\lambda - 1)(\lambda - 2) \end{aligned}$$

which gives us the equation

$$(\lambda - 1)(\lambda - 2) = 0$$

Since $\lambda = 1$, and $\lambda = 2$, both roots lie in the right-half $s$-plane showing that the system is unstable.

($ii$) For the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

we have the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \qquad D = 0$$

For third- and higher-order systems it is convenient to use a computer algebra system to perform the computations. The following MATLAB code can be used to obtain the solutions.

```
% Define the A, B, C, and D matrices
A=[0 1 0;0 0 1;0 -2 -3];
B=[0;1;1];
C=[1 0 1];
D=[0];
% Set up the controllability matrix
CO=[B A*B (A^2)*B];
% Test for Full Controllability
% You can either use the determinant or directly compute the rank
tCO1=det(CO);
```

```
tCO2=rank(CO);
% Set up the observability matrix
OB=[C' A'*C' (A'^2)*C'];
% Test for full observability
% You can either use the determinant or directly compute the rank
tOB1=det(OB);
tOB2=rank(OB);
% Obtain the numerator and denominator coefficients of the transfer function
[num,den]=ss2tf(A,B,C,D,1);
% To test the stability compute the eigenvalues.
% Eigenvalues are the roots of the characteristic equation
e=eig(A);
```

For this system the answers are as follows:

(a) Controllability matrix $C_0 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -5 \\ 1 & -5 & 13 \end{bmatrix}$, $\det C_0 = -24\ (\neq 0)$, $\operatorname{rank} C_0 = 3$. Therefore the system is fully controllable.

(b) Observability matrix $O_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 6 \\ 1 & -3 & 8 \end{bmatrix}$; $\det O_0 = 10\ (\neq 0)$; rank$)_o = 3$. Therefore the system is fully observable.

(c) Transfer function $Y(s) = \dfrac{s^2 - s + 4}{s^3 + 3s^2 + 2s} = \dfrac{s^2 - s + 4}{s\,(s+1)\,(s+2)}$.

(d) The eigenvalues are $\lambda = 0$, $\lambda = -1$, $\lambda = -2$, and therefore the system is stable.

$(iii)$ For the linear time-invariant system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

we have the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \qquad D = 0$$

(a) The controllability matrix $C_0 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$ has determinant $-2\ (\neq 0)$ and rank 3; therefore the system is fully controllable.