Chapter 1

Introduction to Differential Equations

1.1 Definitions and Terminology

3. Fourth order; linear

6. Second order; nonlinear because of R^2

9. Writing the differential equation in the form $x(dy/dx) + y^2 = 1$, we see that it is nonlinear in y because of y^2. However, writing it in the form $(y^2 - 1)(dx/dy) + x = 0$, we see that it is linear in x.

12. From $y = \frac{6}{5} - \frac{6}{5}e^{-20t}$ we obtain $dy/dt = 24e^{-20t}$ so that

$$\frac{dy}{dt} + 20y = 24e^{-20t} + 20\left(\frac{6}{5} - \frac{6}{5}e^{-20t}\right) = 24.$$

15. The domain of the function, found by solving $x+2 \geq 0$, is $[-2, \infty)$. From $y' = 1+2(x+2)^{-1/2}$ we have

$$\begin{aligned}(y-x)y' &= (y-x)[1+2(x+2)^{-1/2}]\\ &= y - x + 2(y-x)(x+2)^{-1/2}\\ &= y - x + 2[x + 4(x+2)^{1/2} - x](x+2)^{-1/2}\\ &= y - x + 8(x+2)^{1/2}(x+2)^{-1/2} = y - x + 8.\end{aligned}$$

An interval of definition for the solution of the differential equation is $(-2, \infty)$ because y' is not defined at $x = -2$.

18. The function is $y = 1/\sqrt{1 - \sin x}$, whose domain is obtained from $1 - \sin x \neq 0$ or $\sin x \neq 1$. Thus, the domain is $\{x \mid x \neq \pi/2 + 2n\pi\}$. From $y' = -\frac{1}{2}(1 - \sin x)^{-3/2}(-\cos x)$ we have

$$2y' = (1 - \sin x)^{-3/2}\cos x = [(1 - \sin x)^{-1/2}]^3 \cos x = y^3 \cos x.$$

An interval of definition for the solution of the differential equation is $(\pi/2, 5\pi/2)$. Another one is $(5\pi/2, 9\pi/2)$ and so on.

21. Differentiating $P = c_1e^t/\left(1 + c_1e^t\right)$ we obtain

$$\frac{dP}{dt} = \frac{\left(1 + c_1e^t\right)c_1e^t - c_1e^t \cdot c_1e^t}{(1 + c_1e^t)^2} = \frac{c_1e^t}{1 + c_1e^t}\,\frac{\left[\left(1 + c_1e^t\right) - c_1e^t\right]}{1 + c_1e^t}$$

$$= \frac{c_1e^t}{1 + c_1e^t}\left[1 - \frac{c_1e^t}{1 + c_1e^t}\right] = P(1 - P).$$

24. From $y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$ we obtain

$$\frac{dy}{dx} = -c_1x^{-2} + c_2 + c_3 + c_3\ln x + 8x,$$

$$\frac{d^2y}{dx^2} = 2c_1x^{-3} + c_3x^{-1} + 8,$$

and

$$\frac{d^3y}{dx^3} = -6c_1x^{-4} - c_3x^{-2}$$

so that

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (-6c_1 + 4c_1 + c_1 + c_1)x^{-1} + (-c_3 + 2c_3 - c_2 - c_3 + c_2)x$$

$$+ (-c_3 + c_3)x\ln x + (16 - 8 + 4)x^2$$

$$= 12x^2.$$

In Problems 25–28, we use the Product Rule and the derivative of an integral ((12) of this section): $\dfrac{d}{dx}\displaystyle\int_a^x g(t)\,dt = g(x)$.

27. Differentiating $y = \dfrac{5}{x} + \dfrac{10}{x}\displaystyle\int_1^x \frac{\sin t}{t}\,dt$ we obtain $\dfrac{dy}{dx} = -\dfrac{5}{x^2} - \dfrac{10}{x^2}\displaystyle\int_1^x \frac{\sin t}{t}\,dt + \frac{\sin x}{x}\cdot\frac{10}{x}$ or $\dfrac{dy}{dx} = -\dfrac{5}{x^2} - \dfrac{10}{x^2}\displaystyle\int_1^x \frac{\sin t}{t}\,dt + \frac{10\sin x}{x^2}$ so that

$$x^2\frac{dy}{dx} + xy = x^2\left(-\frac{5}{x^2} - \frac{10}{x^2}\int_1^x \frac{\sin t}{t}\,dt + \frac{10\sin x}{x^2}\right) + x\left(\frac{5}{x} + \frac{10}{x}\int_1^x \frac{\sin t}{t}\,dt\right)$$

$$= -5 - 10\int_1^x \frac{\sin t}{t}\,dt + 10\sin x + 5 + 10\int_1^x \frac{\sin t}{t}\,dt = 10\sin x.$$

30. The function $y(x)$ is not continuous at $x = 0$ since $\lim\limits_{x\to 0^-} y(x) = 5$ and $\lim\limits_{x\to 0^+} y(x) = -5$. Thus, $y'(x)$ does not exist at $x = 0$.

33. Force the function $y = e^{mx}$ into the equation $y'' - 5y' + 6y = 0$ to get

$$(e^{mx})'' - 5(e^{mx})' + 6(e^{mx}) = 0$$

$$m^2e^{mx} - 5me^{mx} + 6e^{mx} = 0$$

$$e^{mx}(m^2 - 5m + 6) = 0$$

$$e^{mx}(m - 2)(m - 3) = 0.$$

Now, since $e^{mx} > 0$ for all values of x, we must have $m = 2$ and $m = 3$; therefore, $y = e^{2x}$ and $y = e^{3x}$ are solutions.

36. Force the function $y = x^m$ into the equation $x^2y'' - 7xy' + 15y = 0$ to get

$$\begin{aligned} x^2 \cdot (x^m)'' - 7x \cdot (x^m)' + 15(x^m) &= 0 \\ x^2 \cdot m(m-1)x^{m-2} - 7x \cdot mx^{m-1} + 15x^m &= 0 \\ (m^2 - m)x^m - 7mx^m + 15x^m &= 0 \\ x^m[m^2 - 8m + 15] &= 0 \\ x^m[(m-3)(m-5)] &= 0. \end{aligned}$$

The last line implies that $m = 3$ and $m = 5$; therefore, $y = x^3$ and $y = x^5$ are solutions.

In Problems 37–40, we substitute $y = c$ into the differential equations and use $y' = 0$ and $y'' = 0$.

39. Since $1/(c-1) = 0$ has no solutions, the differential equation has no constant solutions.

42. From $x = \cos 2t + \sin 2t + \frac{1}{5}e^t$ and $y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$ we obtain

$$\frac{dx}{dt} = -2\sin 2t + 2\cos 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{dy}{dt} = 2\sin 2t - 2\cos 2t - \frac{1}{5}e^t$$

and

$$\frac{d^2x}{dt^2} = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{d^2y}{dt^2} = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t.$$

Then

$$4y + e^t = 4(-\cos 2t - \sin 2t - \frac{1}{5}e^t) + e^t = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t = \frac{d^2x}{dt^2}$$

and

$$4x - e^t = 4(\cos 2t + \sin 2t + \frac{1}{5}e^t) - e^t = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t = \frac{d^2y}{dt^2}.$$

45. The first derivative of $f(x) = e^x$ is e^x. The first derivative of $f(x) = e^{kx}$ is ke^{kx}. The differential equations are $y' = y$ and $y' = ky$, respectively.

48. Since the first and second derivatives of $\sin t$ and $\cos t$ involve $\sin t$ and $\cos t$, it is plausible that a linear combination of these functions, $A\sin t + B\cos t$, could be a solution of the differential equation. Using $y' = A\cos t - B\sin t$ and $y'' = -A\sin t - B\cos t$ and substituting into the differential equation we get

$$\begin{aligned} y'' + 2y' + 4y &= -A\sin t - B\cos t + 2A\cos t - 2B\sin t + 4A\sin t + 4B\cos t \\ &= (3A - 2B)\sin t + (2A + 3B)\cos t = 5\sin t. \end{aligned}$$

Thus, $3A - 2B = 5$ and $2A + 3B = 0$. Solving these simultaneous equations we find $A = \frac{15}{13}$ and $B = -\frac{10}{13}$. A particular solution is $y = \frac{15}{13}\sin t - \frac{10}{13}\cos t$.

51. Differentiating $(x^3+y^3)/xy = 3c$ we obtain

$$\frac{xy(3x^2+3y^2y')-(x^3+y^3)(xy'+y)}{x^2y^2}=0$$

$$3x^3y+3xy^3y'-x^4y'-x^3y-xy^3y'-y^4=0$$

$$(3xy^3-x^4-xy^3)y'=-3x^3y+x^3y+y^4$$

$$y'=\frac{y^4-2x^3y}{2xy^3-x^4}=\frac{y(y^3-2x^3)}{x(2y^3-x^3)}\,.$$

54. To determine if a solution curve passes through $(0,3)$ we let $t=0$ and $P=3$ in the equation $P=c_1e^t/(1+c_1e^t)$. This gives $3=c_1/(1+c_1)$ or $c_1=-\frac{3}{2}$. Thus, the solution curve

$$P=\frac{(-3/2)e^t}{1-(3/2)e^t}=\frac{-3e^t}{2-3e^t}$$

passes through the point $(0,3)$. Similarly, letting $t=0$ and $P=1$ in the equation for the one-parameter family of solutions gives $1=c_1/(1+c_1)$ or $c_1=1+c_1$. Since this equation has no solution, no solution curve passes through $(0,1)$.

57. The differential equation $yy'-xy=0$ has normal form $dy/dx=x$. These are not equivalent because $y=0$ is a solution of the first differential equation but not a solution of the second.

60. **(a)** The derivative of a constant solution $y=c$ is 0, so solving $5-c=0$ we see that $c=5$ and so $y=5$ is a constant solution.

(b) A solution is increasing where $dy/dx=5-y>0$ or $y<5$. A solution is decreasing where $dy/dx=5-y<0$ or $y>5$.

63. In *Mathematica* use

```
Clear[y]
y[x_]:= x Exp[5x] Cos[2x]
y[x]
y''''[x] - 20y'''[x] + 158y''[x] - 580y'[x] + 841y[x]//Simplify
```

The output will show $y(x)=e^{5x}x\cos 2x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

1.2 Initial-Value Problems

3. Letting $x=2$ and solving $1/3=1/(4+c)$ we get $c=-1$. The solution is $y=1/(x^2-1)$. This solution is defined on the interval $(1,\infty)$.

6. Letting $x = 1/2$ and solving $-4 = 1/(1/4 + c)$ we get $c = -1/2$. The solution is $y = 1/(x^2 - 1/2) = 2/(2x^2 - 1)$. This solution is defined on the interval $(-1/\sqrt{2}\,, 1/\sqrt{2}\,)$.

In Problems 7–10, we use $x = c_1 \cos t + c_2 \sin t$ and $x' = -c_1 \sin t + c_2 \cos t$ to obtain a system of two equations in the two unknowns c_1 and c_2.

9. From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} - \frac{1}{2}c_2 + \frac{\sqrt{3}}{2} = 0.$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4)\cos t + (1/4)\sin t.$$

In Problems 11–14, we use $y = c_1 e^x + c_2 e^{-x}$ and $y' = c_1 e^x - c_2 e^{-x}$ to obtain a system of two equations in the two unknowns c_1 and c_2.

12. From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$

$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is

$$y = \frac{1}{2}e^x - \frac{1}{2}e^2 e^{-x} = \frac{1}{2}e^x - \frac{1}{2}e^{2-x}.$$

15. Two solutions are $y = 0$ and $y = x^3$.

18. For $f(x, y) = \sqrt{xy}$ we have $\partial f/\partial y = \frac{1}{2}\sqrt{x/y}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

21. For $f(x, y) = x^2/(4 - y^2)$ we have $\partial f/\partial y = 2x^2 y/(4 - y^2)^2$. Thus, the differential equation will have a unique solution in any region where $y < -2$, $-2 < y < 2$, or $y > 2$.

24. For $f(x, y) = (y + x)/(y - x)$ we have $\partial f/\partial y = -2x/(y - x)^2$. Thus, the differential equation will have a unique solution in any region where $y < x$ or $y > x$.

In Problems 25–28, we identify $f(x, y) = \sqrt{y^2 - 9}$ and $\partial f/\partial y = y/\sqrt{y^2 - 9}$. We see that f and $\partial f/\partial y$ are both continuous in the regions of the plane determined by $y < -3$ and $y > 3$ with no restrictions on x.

27. Since $(2, -3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(2, -3)$.

30. **(a)** Since $\dfrac{d}{dx}\tan(x + c) = \sec^2(x + c) = 1 + \tan^2(x + c)$, we see that $y = \tan(x + c)$ satisfies the differential equation.

(b) Solving $y(0) = \tan c = 0$ we obtain $c = 0$ and $y = \tan x$. Since $\tan x$ is discontinuous at $x = \pm\pi/2$, the solution is not defined on $(-2, 2)$ because it contains $\pm\pi/2$.

(c) The largest interval on which the solution can exist is $(-\pi/2, \pi/2)$.

33. (a) Differentiating $3x^2 - y^2 = c$ we get $6x - 2yy' = 0$ or $yy' = 3x$.

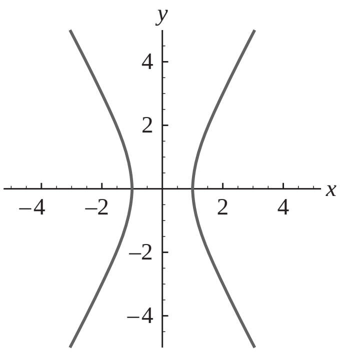
(b) Solving $3x^2 - y^2 = 3$ for y we get

$$y = \phi_1(x) = \sqrt{3(x^2-1)}, \qquad 1 < x < \infty,$$

$$y = \phi_2(x) = -\sqrt{3(x^2-1)}, \qquad 1 < x < \infty,$$

$$y = \phi_3(x) = \sqrt{3(x^2-1)}, \qquad -\infty < x < -1,$$

$$y = \phi_4(x) = -\sqrt{3(x^2-1)}, \qquad -\infty < x < -1.$$



(c) Only $y = \phi_3(x)$ satisfies $y(-2) = 3$.

In Problems 35–38, we consider the points on the graphs with x-coordinates $x_0 = -1$, $x_0 = 0$, and $x_0 = 1$. The slopes of the tangent lines at these points are compared with the slopes given by $y'(x_0)$ in (a) through (f).

36. The graph satisfies the conditions in (e).

39. Using the function $y = c_1 \cos 3x + c_2 \sin 3x$ and the first boundary condition, we get

$$y(0) = c_1 \cos 0 + c_2 \sin 0 = 0.$$

Therefore, $c_1 = 0$. Similarly for the second boundary condition we get

$$y(\pi/6) = c_2 \sin 3\,(\pi/6) = -1.$$

Therefore, $c_2 = -1$. The solution to the boundary value problem is $y(x) = -\sin 3x$.

42. The derivative of the function $y = c_1 \cos 3x + c_2 \sin 3x$ is $y' = -3c_1 \sin 3x + 3c_2 \cos 3x$, and using the two boundary conditions we get

$$y(0) = c_1 + 0 = 1.$$

Therefore, $c_1 = 1$. In addition

$$y'(\pi) = 0 - 3c_2 = 5.$$

Therefore, $c_2 = -5/3$. The solution to this boundary value problem is $y(x) = \cos 3x - \dfrac{5}{3} \sin 3x$.

45. Integrating $y' = 8e^{2x} + 6x$ we obtain

$$y = \int (8e^{2x} + 6x)\,dx = 4e^{2x} + 3x^2 + c.$$

Setting $x = 0$ and $y = 9$ we have $9 = 4 + c$ so $c = 5$ and $y = 4e^{2x} + 3x^2 + 5$.

48. If the solution is tangent to the x-axis at $(x_0, 0)$, then $y' = 0$ when $x = x_0$ and $y = 0$. Substituting these values into $y' + 2y = 3x - 6$ we get $0 + 0 = 3x_0 - 6$ or $x_0 = 2$.

51. We note that the initial condition $y(0) = 0$,

$$0 = \int_0^y \frac{1}{\sqrt{t^3+1}}\,dt$$

is satisfied only when $y = 0$. For any $y > 0$, necessarily

$$\int_0^y \frac{1}{\sqrt{t^3+1}}\,dt > 0$$

because the integrand is positive on the interval of integration. Then from (12) of Section 1.1 and the Chain Rule we have

$$\frac{d}{dx}x = \frac{d}{dx}\int_0^y \frac{1}{\sqrt{t^3+1}}\,dt \qquad \text{and} \qquad \frac{dy}{dx} = \sqrt{y^3+1}$$

$$1 = \frac{1}{\sqrt{y^3+1}}\frac{dy}{dx} \qquad\qquad y'(0) = \left.\frac{dy}{dx}\right|_{x=0} = \sqrt{(y(0))^3+1} = \sqrt{0+1} = 1.$$

Computing the second derivative, we see that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\sqrt{y^3+1} = \frac{3y^2}{2\sqrt{y^3+1}}\frac{dy}{dx} = \frac{3y^2}{2\sqrt{y^3+1}} \cdot \sqrt{y^3+1} = \frac{3}{2}y^2$$

$$\frac{d^2y}{dx^2} = \frac{3}{2}y^2.$$

This is equivalent to $2\dfrac{d^2y}{dx^2} - 3y^2 = 0$.

1.3 Differential Equations as Mathematical Models

3. Let b be the rate of births and d the rate of deaths. Then $b = k_1P$ and $d = k_2P^2$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1P - k_2P^2$.

6. By inspecting the graph in the text we take T_m to be $T_m(t) = 80 - 30\cos(\pi t/12)$. Then the temperature of the body at time t is determined by the differential equation

$$\frac{dT}{dt} = k\left[T - \left(80 - 30\cos\left(\frac{\pi}{12}t\right)\right)\right], \quad t > 0.$$

9. The rate at which salt is leaving the tank is

$$R_{out}\ (3\ \text{gal/min})\cdot\left(\frac{A}{300}\ \text{lb/gal}\right)=\frac{A}{100}\ \text{lb/min}.$$

Thus, $dA/dt = A/100$. The initial amount is $A(0) = 50$.

12. The rate at which salt is entering the tank is

$$R_{in}=(c_{in}\ \text{lb/gal})\cdot(r_{in}\ \text{gal/min})=c_{in}r_{in}\ \text{lb/min}.$$

Now, let $A(t)$ denote the number of pounds of salt and $N(t)$ the number of gallons of brine in the tank at time t. The concentration of salt in the tank as well as in the outflow is $c(t) = x(t)/N(t)$. But the number of gallons of brine in the tank remains steady, is increased, or is decreased depending on whether $r_{in} = r_{out}$, $r_{in} > r_{out}$, or $r_{in} < r_{out}$. In any case, the number of gallons of brine in the tank at time t is $N(t) = N_0 + (r_{in} - r_{out})t$. The output rate of salt is then

$$R_{out}=\left(\frac{A}{N_0+(r_{in}-r_{out})t}\ \text{lb/gal}\right)\cdot(r_{out}\ \text{gal/min})=r_{out}\ \frac{A}{N_0+(r_{in}-r_{out})t}\ \text{lb/min}.$$

The differential equation for the amount of salt, $dA/dt = R_{in} - R_{out}$, is

$$\frac{dA}{dt}=c_{in}r_{in}-r_{out}\ \frac{A}{N_0+(r_{in}-r_{out})t}\quad\text{or}\quad\frac{dA}{dt}+\frac{r_{out}}{N_0+(r_{in}-r_{out})t}\ A=c_{in}r_{in}.$$

15. Since $i = dq/dt$ and $L\,d^2q/dt^2 + R\,dq/dt = E(t)$, we obtain $L\,di/dt + Ri = E(t)$.

18. Since the barrel in Figure 1.3.17(b) in the text is submerged an additional y feet below its equilibrium position, the number of cubic feet in the additional submerged portion is the volume of the circular cylinder: $\pi\times(\text{radius})^2\times$ height or $\pi(s/2)^2y$. Then we have from Archimedes' principle

$$\begin{aligned}\text{upward force of water on barrel} &= \text{weight of water displaced}\\ &= (62.4)\times(\text{volume of water displaced})\\ &= (62.4)\pi(s/2)^2y = 15.6\pi s^2y.\end{aligned}$$

It then follows from Newton's second law that

$$\frac{w}{g}\frac{d^2y}{dt^2}=-15.6\pi s^2y\qquad\text{or}\qquad\frac{d^2y}{dt^2}+\frac{15.6\pi s^2g}{w}\ y=0,$$

where $g = 32$ and w is the weight of the barrel in pounds.

21. As the rocket climbs (in the positive direction), it spends its amount of fuel, and therefore, the mass of the fuel changes with time. The air resistance acts in the opposite direction of the motion, and the upward thrust R works in the same direction. Using Newton's second law we get

$$\frac{d}{dt}(mv) = -mg - kv + R.$$

Now, because the mass is variable, we must use the product rule to expand the left side of the equation. Doing so gives us the following:

$$\frac{d}{dt}(mv) = -mg - kv + R$$

$$v \times \frac{dm}{dt} + m \times \frac{dv}{dt} = -mg - kv + R.$$

The last line is the differential equation we wanted to find.

24. The gravitational force on m is $F = -kM_r m/r^2$. Since $M_r = 4\pi\delta r^3/3$ and $M = 4\pi\delta R^3/3$ we have $M_r = r^3 M/R^3$ and

$$F = -k\,\frac{M_r m}{r^2} = -k\,\frac{r^3 Mm/R^3}{r^2} = -k\,\frac{mM}{R^3}\,r.$$

Now, from $F = ma = d^2r/dt^2$ we have

$$m\,\frac{d^2r}{dt^2} = -k\,\frac{mM}{R^3}\,r \quad \text{or} \quad \frac{d^2r}{dt^2} = -\frac{kM}{R^3}\,r.$$

27. The differential equation is $x'(t) = r - kx(t)$, where $k > 0$.

30. The differential equation is $dP/dt = kP$, so from Problem 41 in Exercises 1.1, a one-parameter family of solutions is $P = ce^{kt}$.

33. This differential equation could describe a population that undergoes periodic fluctuations.

36. **(a)** If ρ is the mass density of the raindrop, then $m = \rho V$ and

$$\frac{dm}{dt} = \rho\,\frac{dV}{dt} = \rho\,\frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = \rho\left(4\pi r^2\frac{dr}{dt}\right) = \rho S\,\frac{dr}{dt}\,.$$

If dr/dt is a constant, then $dm/dt = kS$, where $\rho\,dr/dt = k$ or $dr/dt = k/\rho$. Since the radius is decreasing, $k < 0$. Solving $dr/dt = k/\rho$ we get $r = (k/\rho)t + c_0$. Since $r(0) = r_0$, $c_0 = r_0$ and $r = kt/\rho + r_0$.

(b) From Newton's second law, $\dfrac{d}{dt}[mv] = mg$, where v is the velocity of the raindrop. Then

$$m\,\frac{dv}{dt} + v\,\frac{dm}{dt} = mg \qquad \text{or} \qquad \rho\left(\frac{4}{3}\pi r^3\right)\frac{dv}{dt} + v(k4\pi r^2) = \rho\left(\frac{4}{3}\pi r^3\right)g.$$

Dividing by $4\rho\pi r^3/3$ we get

$$\frac{dv}{dt} + \frac{3k}{\rho r}v = g \qquad \text{or} \qquad \frac{dv}{dt} + \frac{3k/\rho}{kt/\rho + r_0}v = g, \quad k < 0.$$

39. At time t, when the population is 2 million cells, the differential equation $P'(t) = 0.15P(t)$ gives the rate of increase at time t. Thus, when $P(t) = 2$ (million cells), the rate of increase is $P'(t) = 0.15(2) = 0.3$ million, or 300,000, cells per hour.

Chapter 1 in Review

3. $\dfrac{d}{dx}(c_1 \cos kx + c_2 \sin kx) = -kc_1 \sin kx + kc_2 \cos kx;$

$$\frac{d^2}{dx^2}(c_1 \cos kx + c_2 \sin kx) = -k^2 c_1 \cos kx - k^2 c_2 \sin kx = -k^2 \overbrace{(c_1 \cos kx + c_2 \sin kx)}^{y};$$

$$\frac{d^2y}{dx^2} = -k^2 y \quad \text{or} \quad \frac{d^2y}{dx^2} + k^2 y = 0$$

6. $y' = -c_1 e^x \sin x + c_1 e^x \cos x + c_2 e^x \cos x + c_2 e^x \sin x;$

$$y'' = -c_1 e^x \cos x - c_1 e^x \sin x - c_1 e^x \sin x + c_1 e^x \cos x - c_2 e^x \sin x + c_2 e^x \cos x + c_2 e^x \cos x + c_2 e^x \sin x$$

$$= -2c_1 e^x \sin x + 2c_2 e^x \cos x;$$

$$y'' - 2y' = -2c_1 e^x \cos x - 2c_2 e^x \sin x = -2y; \qquad y'' - 2y' + 2y = 0$$

9. b **12.** a, b, d

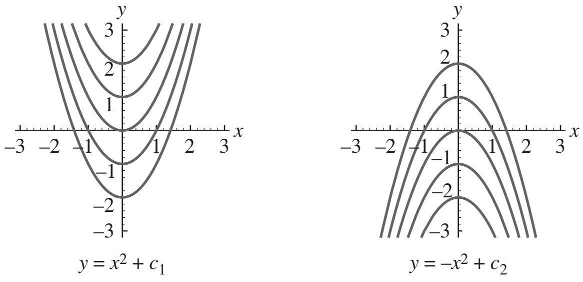
15. The slope of the tangent line at (x, y) is y', so the differential equation is $y' = x^2 + y^2$.

18. **(a)** Differentiating $y^2 - 2y = x^2 - x + c$ we obtain $2yy' - 2y' = 2x - 1$ or $(2y - 2)y' = 2x - 1$.

(b) Setting $x = 0$ and $y = 1$ in the solution we have $1 - 2 = 0 - 0 + c$ or $c = -1$. Thus, a solution of the initial-value problem is $y^2 - 2y = x^2 - x - 1$.

(c) Solving the equation $y^2 - 2y - (x^2 - x - 1) = 0$ by the quadratic formula we get $y = (2 \pm \sqrt{4 + 4(x^2 - x - 1)}\,)/2 = 1 \pm \sqrt{x^2 - x} = 1 \pm \sqrt{x(x-1)}$. Since $x(x-1) \geq 0$ for $x \leq 0$ or $x \geq 1$, we see that neither $y = 1 + \sqrt{x(x-1)}$ nor $y = 1 - \sqrt{x(x-1)}$ is differentiable at $x = 0$. Thus, both functions are solutions of the differential equation, but neither is a solution of the initial-value problem.

21. **(a)**



(b) When $y = x^2 + c_1$, $y' = 2x$ and $(y')^2 = 4x^2$. When $y = -x^2 + c_2$, $y' = -2x$ and $(y')^2 = 4x^2$.

(c) Pasting together x^2, $x \geq 0$ and $-x^2$, $x \leq 0$, we get $y = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x > 0\,. \end{cases}$

24. Differentiating $y = x\sin x + (\cos x)\ln(\cos x)$ we get

$$\begin{aligned} y' &= x\cos x + \sin x + \cos x\left(\frac{-\sin x}{\cos x}\right) - (\sin x)\ln(\cos x) \\ &= x\cos x + \sin x - \sin x - (\sin x)\ln(\cos x) \\ &= x\cos x - (\sin x)\ln(\cos x) \end{aligned}$$

and

$$\begin{aligned} y'' &= -x\sin x + \cos x - \sin x\left(\frac{-\sin x}{\cos x}\right) - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \frac{\sin^2 x}{\cos x} - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \frac{1-\cos^2 x}{\cos x} - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \sec x - \cos x - (\cos x)\ln(\cos x) \\ &= -x\sin x + \sec x - (\cos x)\ln(\cos x). \end{aligned}$$

Thus,

$$y'' + y = -x\sin x + \sec x - (\cos x)\ln(\cos x) + x\sin x + (\cos x)\ln(\cos x) = \sec x.$$

To obtain an interval of definition we note that the domain of $\ln x$ is $(0,\infty)$, so we must have $\cos x > 0$. Thus, an interval of definition is $(-\pi/2, \pi/2)$.

In Problems 27–30 we use (12) of Section 1.1 and the Product Rule.

27.

$$\begin{aligned} y &= e^{\cos x}\int_0^x te^{-\cos t}\,dt \\ \frac{dy}{dx} &= e^{\cos x}\left(xe^{-\cos x}\right) - \sin x e^{\cos x}\int_0^x te^{-\cos t}\,dt \\ \frac{dy}{dx} + (\sin x)\,y &= e^{\cos x}xe^{-\cos x} - \sin x e^{\cos x}\int_0^x te^{-\cos t}\,dt + \sin x\left(e^{\cos x}\int_0^x te^{-\cos t}\,dt\right) \\ &= x - \sin x e^{\cos x}\int_0^x te^{-\cos t}\,dt + \sin x e^{\cos x}\int_0^x te^{-\cos t}\,dt = x \end{aligned}$$

30.

$$\begin{aligned} y &= \sin x\int_0^x e^{t^2}\cos t\,dt - \cos x\int_0^x e^{t^2}\sin t\,dt \\ y' &= \sin x\left(e^{x^2}\cos x\right) + \cos x\int_0^x e^{t^2}\cos t\,dt - \cos x\left(e^{x^2}\sin x\right) + \sin x\int_0^x e^{t^2}\sin t\,dt \\ &= \cos x\int_0^x e^{t^2}\cos t\,dt + \sin x\int_0^x e^{t^2}\sin t\,dt \end{aligned}$$

$$\begin{aligned}
y'' &= \cos x\left(e^{x^2}\cos x\right) - \sin x\int_0^x e^{t^2}\cos t\,dt + \sin x\left(e^{x^2}\sin x\right) + \cos x\int_0^x e^{t^2}\sin t\,dt \\
&= e^{x^2}\left(\cos^2 x + \sin^2 x\right) - \left(\overbrace{\sin x\int_0^x e^{t^2}\cos t\,dt - \cos x\int_0^x e^{t^2}\sin t\,dt}^{y}\right) \\
&= e^{x^2} - y
\end{aligned}$$

$$y'' + y = e^{x^2} - y + y = e^{x^2}$$

33. Using implicit differentiation we get

$$\begin{aligned}
y^3 + 3y &= 2 - 3x \\
3y^2y' + 3y' &= -3 \\
y^2y' + y' &= -1 \\
(y^2+1)y' &= -1 \\
y' &= \frac{-1}{y^2+1}.
\end{aligned}$$

Differentiating the last line and remembering to use the quotient rule on the right side leads to

$$y'' = \frac{2yy'}{(y^2+1)^2}.$$

Now, since $y' = -1\big/\left(y^2+1\right)$ we can write the last equation as

$$y'' = \frac{2y}{(y^2+1)^2}y' = \frac{2y}{(y^2+1)^2}\frac{-1}{(y^2+1)} = 2y\left(\frac{-1}{y^2+1}\right)^3 = 2y(y')^3,$$

which is what we wanted to show.

In Problems 35–38, $y = c_1e^{-3x} + c_2e^x + 4x$ is given as a two-parameter family of solutions of the second-order differential equation $y'' + 2y' - 3y = -12x + 8$.

36. If $y(0) = 5$ and $y'(0) = -11$, then

$$\begin{aligned}
c_1 + c_2 &= 5 \\
-3c_1 + c_2 &= -15.
\end{aligned}$$

Subtracting the second equation from the first gives us $4c_1 = 20$ or $c_1 = 5$, and thus $c_2 = 0$. Therefore, $y = 5e^{-3x} + 4x$.

39. We are to use the Leibniz's rule.

$$\frac{d}{dx}\int_{u(x)}^{v(x)} F(x,t)\,dt = F(x,v(x))\frac{dv}{dx} - F(x,u(x))\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x}F(x,t)\,dt$$

Since $y(x) = \dfrac{1}{3}\displaystyle\int_0^x f(t)\sin(3x-3t)\,dt$, take $F(x,t) = f(t)\sin(3x-3t)$, $u(x) = 0$, and $v(x) = x$ to get

$$\begin{aligned}
\frac{d}{dx}y(x) &= \frac{d}{dx}\frac{1}{3}\int_0^x f(t)\sin(3x-3t)\,dt \\
&= \frac{1}{3}\left[F(x,x)\cdot 1 - F(x,0)\cdot 0 + \int_0^x \frac{\partial}{\partial x} f(t)\sin(3x-3t)\,dt\right] \\
&= \frac{1}{3}\left[f(x)\sin(3x-3x) + \int_0^x 3f(t)\cos(3x-3t)\,dt\right] \\
&= \int_0^x f(t)\cos(3x-3t)\,dt.
\end{aligned}$$

Apply the Leibniz's rule a second time to $y' = \displaystyle\int_0^x f(t)\cos(3x-3t)\,dt$ by taking $F(x,t) = f(t)\cos(3x-3t)$, $u(x) = 0$, and $v(x) = x$ to get

$$\begin{aligned}
\frac{d}{dx}y'(x) &= \frac{d}{dx}\int_0^x f(t)\cos(3x-3t)\,dt \\
&= F(x,x)\cdot 1 - F(x,0)\cdot 0 + \int_0^x \frac{\partial}{\partial x} f(t)\cos(3x-3t)\,dt \\
&= f(x)\cos(3x-3x) + \int_0^x -3f(t)\sin(3x-3t)\,dt \\
&= f(x) - 3\int_0^x f(t)\sin(3x-3t)\,dt.
\end{aligned}$$

Therefore, $y'' = f(x) - 3\displaystyle\int_0^x f(t)\sin(3x-3t)\,dt$. Now, substituting y and y'' into the differential equation we get

$$y'' + 9y = f(x) - 3\int_0^x f(t)\sin(3x-3t)\,dt + 9\cdot\frac{1}{3}\int_0^x f(t)\sin(3x-3t)\,dt = f(x).$$

42. The differential equation is

$$\frac{dh}{dt} = -\frac{cA_0}{A_w}\sqrt{2gh}\,.$$

Using $A_0 = \pi(1/24)^2 = \pi/576$, $A_w = \pi(2)^2 = 4\pi$, and $g = 32$, this becomes

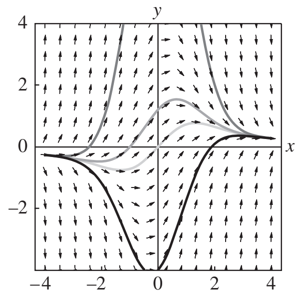
$$\frac{dh}{dt} = -\frac{c\pi/576}{4\pi}\sqrt{64h} = -\frac{c}{288}\sqrt{h}\,.$$

Chapter 2

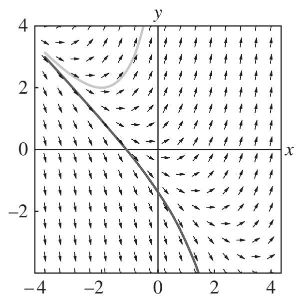
First-Order Differential Equations

2.1 Solution Curves Without a Solution

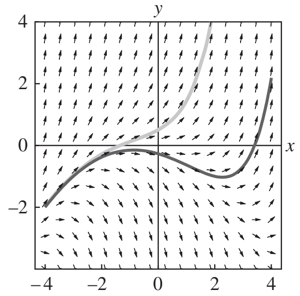
3.



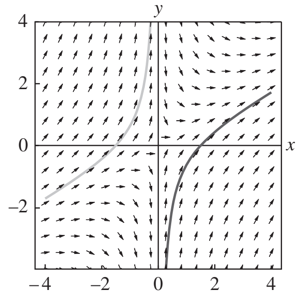
6.



9.



12.



15. (a) The isoclines have the form $y = -x + c$, which are straight lines with slope -1.