Solution Manual

Section 1.1

1. first-order, linear
2. first-order, nonlinear
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4. third-order, linear
5. second-order, linear
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8. second-order, linear
9. second-order, nonlinear
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11. first-order, nonlinear
12. second-order, nonlinear
13. first-order, nonlinear
14. third-order, linear
15. second-order, nonlinear
16. third-order, nonlinear

Section 1.2

1. Because the differential equation can be rewritten $e^{-y}\,dy = x\,dx$, integration immediately gives $-e^{-y} = \frac{1}{2}x^2 - C$, or $y = -\ln(C - x^2/2)$.

2. Separating variables, we have that $dx/(1+x^2) = dy/(1+y^2)$. Integrating this equation, we find that $\tan^{-1}(x) - \tan^{-1}(y) = \tan(C)$, or $(x-y)/(1+xy) = C$.

3. Because the differential equation can be rewritten $\ln(x)dx/x = y\,dy$, integration immediately gives $\frac{1}{2}\ln^2(x) + C = \frac{1}{2}y^2$, or $y^2(x) - \ln^2(x) = 2C$.

4. Because the differential equation can be rewritten $y^2\,dy = (x + x^3)\,dx$, integration immediately gives $y^3(x)/3 = x^2/2 + x^4/4 + C$.

5. Because the differential equation can be rewritten $y\,dy/(2+y^2) = x\,dx/(1+x^2)$, integration immediately gives $\frac{1}{2}\ln(2+y^2) = \frac{1}{2}\ln(1+x^2) + \frac{1}{2}\ln(C)$, or $2 + y^2(x) = C(1+x^2)$.

6. Because the differential equation can be rewritten $dy/y^{1/3} = x^{1/3}\,dx$, integration immediately gives $\frac{3}{2}y^{2/3} = \frac{3}{4}x^{4/3} + \frac{3}{2}C$, or $y(x) = \left(\frac{1}{2}x^{4/3} + C\right)^{3/2}$.

7. Because the differential equation can be rewritten $e^{-y}\,dy = e^x\,dx$, integration immediately gives $-e^{-y} = e^x - C$, or $y(x) = -\ln(C - e^x)$.

8. Because the differential equation can be rewritten $dy/(y^2+1) = (x^3 + 5)\,dx$, integration immediately gives $\tan^{-1}(y) = \frac{1}{4}x^4 + 5x + C$, or $y(x) = \tan\bigl(\frac{1}{4}x^4 + 5x + C\bigr)$.

9. Because the differential equation can be rewritten $y^2\,dy/(b - ay^3) = dt$, integration immediately gives $\ln[b - ay^3]\big|_{y_0}^{y} = -3at$, or $(ay^3 - b)/(ay_0^3 - b) = e^{-3at}$.

10. Because the differential equation can be written $du/u = dx/x^2$, integration immediately gives $u = Ce^{-1/x}$ or $y(x) = x + Ce^{-1/x}$.

11. From the hydrostatic equation and ideal gas law, $dp/p = -g\,dz/(RT)$. Substituting for $T(z)$,

$$\frac{dp}{p} = -\frac{g}{R(T_0 - \Gamma z)}\,dz.$$

Integrating from 0 to z,

$$\ln\left[\frac{p(z)}{p_0}\right] = \frac{g}{R\Gamma}\ln\left(\frac{T_0 - \Gamma z}{T_0}\right), \quad \text{or} \quad \frac{p(z)}{p_0} = \left(\frac{T_0 - \Gamma z}{T_0}\right)^{g/(R\Gamma)}.$$

12. For $0 < z < H$, we simply use the previous problem. At $z = H$, the pressure is

$$p(H) = p_0\left(\frac{T_0 - \Gamma H}{T_0}\right)^{g/(R\Gamma)}.$$

Then we follow the example in the text for an isothermal atmosphere for $z \ge H$.

13. Separating variables, we find that

$$\frac{dV}{V + RV^2/S} = \frac{dV}{V} - \frac{R\,dV}{S(1 + RV/S)} = -\frac{dt}{RC}.$$

Integration yields

$$\ln\left(\frac{V}{1 + RV/S}\right) = -\frac{t}{RC} + \ln(C).$$

Upon applying the initial conditions,

$$V(t) = \frac{V_0}{1 + RV_0/S}e^{-t/(RC)} + \frac{RV_0/S}{1 + RV_0/S}e^{-t/(RC)}V(t).$$

Solving for $V(t)$, we obtain

$$V(t) = \frac{SV_0\, e^{-t/(RC)}}{S + RV_0\left[1 - e^{-t/(RC)}\right]}.$$

14. From the definition of γ, we can write the differential equation

$$\frac{A}{B}\frac{dT}{dt} + T^4 = \gamma^4,$$

or

$$\begin{aligned}\frac{B}{A}\,dt &= -\frac{dT}{T^4 - \gamma^4} = \frac{1}{2\gamma^2}\left[\frac{dT}{T^2+\gamma^2} - \frac{dT}{T^2-\gamma^2}\right] \\ &= \frac{1}{4\gamma^3}\left[\frac{2\gamma\, dT}{T^2+\gamma^2} - \frac{dT}{T-\gamma} + \frac{dT}{T+\gamma}\right].\end{aligned}$$

The final answer follows from direction integration.

15. Separating the variables yields

$$\frac{dN}{N\ln(K/N)} = b\,dt, \quad \text{or} \quad \frac{d[\ln(K/N)]}{\ln(K/N)} = -b\,dt.$$

Integration leads to

$$\ln\left[\ln(K/N)\right] - \ln\left\{\ln[K/N(0)]\right\} = -bt$$

or

$$\ln\left\{\ln(K/N)/\ln[K/N(0)]\right\} = -bt$$

or

$$\ln(K/N) = \ln[K/N(0)]e^{-bt}$$

or

$$\ln[N/N(0)] = \ln[K/N(0)]\left(1 - e^{-bt}\right)$$

or

$$N(t) = N(0)\exp\left\{\ln[K/N(0)]\left(1 - e^{-bt}\right)\right\}.$$

16. Separating the variables yields

$$\frac{dI}{I} - \frac{\beta}{\alpha}\frac{dI}{1 + \beta I/\alpha} = -\alpha\,dz.$$

Integration leads to

$$\ln\left[\frac{I(z)}{1+\beta I(z)/\alpha}\frac{1+\beta I(0)/\alpha}{I(0)}\right] = -\alpha z,$$

or

$$\frac{I(z)}{1+\beta I(z)/\alpha} = \frac{I(0)}{1+\beta I(0)/\alpha}e^{-\alpha z}, \quad \text{or} \quad I(z) = \frac{\alpha I(0) e^{-\alpha z}}{\alpha + \beta I(0)\left[1-e^{-\alpha z}\right]}.$$

17. Separating the variables yields

$$\frac{d[\mathrm{X}]}{([\mathrm{A}]_0-[\mathrm{X}])([\mathrm{B}]_0-[\mathrm{X}])([\mathrm{C}]_0-[\mathrm{X}])} = k\,dt$$

$$\frac{d[\mathrm{X}]}{([\mathrm{A}]_0-[\mathrm{B}]_0)([\mathrm{A}]_0-[\mathrm{C}]_0)([\mathrm{A}]_0-[\mathrm{X}])} + \frac{d[\mathrm{X}]}{([\mathrm{B}]_0-[\mathrm{A}]_0)([\mathrm{B}]_0-[\mathrm{C}]_0)([\mathrm{B}]_0-[\mathrm{X}])} + \frac{d[\mathrm{X}]}{([\mathrm{C}]_0-[\mathrm{A}]_0)([\mathrm{C}]_0-[\mathrm{B}]_0)([\mathrm{C}]_0-[\mathrm{X}])} = k\,dt$$

Integration yields

$$\frac{1}{([\mathrm{A}]_0-[\mathrm{B}]_0)([\mathrm{A}]_0-[\mathrm{C}]_0)}\ln\left(\frac{[\mathrm{A}]_0}{[\mathrm{A}]_0-[\mathrm{X}]}\right) + \frac{1}{([\mathrm{B}]_0-[\mathrm{A}]_0)([\mathrm{B}]_0-[\mathrm{C}]_0)}\ln\left(\frac{[\mathrm{B}]_0}{[\mathrm{B}]_0-[\mathrm{X}]}\right) + \frac{1}{([\mathrm{C}]_0-[\mathrm{A}]_0)([\mathrm{C}]_0-[\mathrm{B}]_0)}\ln\left(\frac{[\mathrm{C}]_0}{[\mathrm{C}]_0-[\mathrm{X}]}\right) = kt.$$

18. Separation of variables yields

$$\frac{d[\mathrm{X}]}{\alpha-[\mathrm{X}]} = (k_1+k_2)\,dt.$$

Integrating both sides,

$$\ln(\alpha-[\mathrm{X}]) - \ln(\alpha-[\mathrm{X}]_0) = -(k_1+k_2)t.$$

Because $[\mathrm{X}]_0 = 0$,

$$\alpha-[\mathrm{X}] = \alpha e^{-(k_1+k_2)t}, \quad \text{or} \quad [\mathrm{X}] = \alpha\left[1-e^{-(k_1+k_2)t}\right].$$

Section 1.3

1. Because $M(x,y) = -y$ and $N(x,y) = x+y$, we have that $M(tx,ty) = -ty = tM(x,y)$, and $N(tx,ty) = tx+ty = tN(x,y)$. Therefore, the differential equation is homogeneous.

Let $y = ux$. Substituting into the differential equation, $(ux + x)(u\,dx + x\,du) = ux\,dx$, or $-u^2x\,dx = (1 + u)x^2\,du$, or

$$-\frac{dx}{x} = \left(\frac{1}{u} + \frac{1}{u^2}\right) du.$$

Integrating this last equation,

$$-\ln|x| = \ln(u) - \frac{1}{u} - C, \quad \text{or} \quad \ln|y| - \frac{x}{y} = C.$$

2. Because $M(x, y) = y - x$ and $N(x, y) = x + y$, we have that $M(tx, ty) = ty - tx = tM(x, y)$, and $N(tx, ty) = tx + ty = tN(x, y)$. Therefore, the differential equation is homogeneous.

Let $y = ux$. Substituting into the differential equation, $(u-1)x\,dx + (u+1)x(u\,dx + x\,du) = 0$, or

$$(u^2 + 2u - 1)\,dx = -(u + 1)x\,du, \quad \text{or} \quad -\frac{dx}{x} = \frac{u+1}{u^2 + 2u - 1}\,du.$$

Integrating this last equation,

$$-\ln|x| = \tfrac{1}{2}\ln|u^2 + 2u - 1| + C, \quad \text{or} \quad x^2\left(\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right) = y^2 + 2xy - x^2 = C.$$

3. Because $M(x, y) = x^2 + y^2$ and $N(x, y) = 2xy$, we have that $M(tx, ty) = t^2x^2 + t^2y^2 = t^2(x^2 + y^2) = t^2M(x, y)$, and $N(tx, ty) = 2t^2xy = t^2N(x, y)$. Therefore, the differential equation is homogeneous.

Let $y = ux$. Substituting into the differential equation,

$$2x(ux)(u\,dx + x\,du) + (x^2 + x^2u^2)\,dx = 0$$

or

$$2xu\,du + (1 + 3u^2)\,dx = 0, \quad \text{or} \quad \frac{dx}{x} = -\frac{2u}{1 + 3u^2}\,du.$$

Integrating this last equation,

$$\ln|x| = -\tfrac{1}{3}\ln(1 + 3u^2) + \ln(C_1).$$

Inverting the logarithms,

$$|x|(1 + 3y^2/x^2)^{1/3} = C_1, \quad \text{or} \quad |x|(x^2 + 3y^2) = C.$$

4. Because $M(x, y) = y(y - x)$ and $N(x, y) = x(x + y)$, we have that $M(tx, ty) = ty(ty - tx) = t^2M(x, y)$, and $N(tx, ty) = tx(tx + ty) = t^2N(x, y)$. Therefore, the differential equation is homogeneous.

Let $y = ux$. Substituting into the differential equation,

$$x^2 u(u-1)\,dx + x^2(u+1)(u\,dx + x\,du) = 0$$

or

$$2u^2\,dx + (u+1)x\,du = 0, \quad \text{or} \quad 2\frac{dx}{x} = -\frac{u+1}{u^2}\,du.$$

Integrating this last equation,

$$\ln|x|^2 = -\ln|u| + \frac{1}{u^2} + C, \quad \text{or} \quad \ln|ux^2| = C - \frac{x}{y}, \quad \text{or} \quad \ln|xy| = C - \frac{x}{y}.$$

5. Because $M(x,y) = y + 2\sqrt{xy}$ and $N(x,y) = -x$, we have that $M(tx,ty) = ty + 2\sqrt{t^2xy} = ty + 2t\sqrt{xy} = tM(x,y)$, and $N(tx,ty) = -tx = tN(x,y)$. Therefore, the differential equation is homogeneous.

Let $y = ux$. Substituting into the differential equation,

$$x(u\,dx + x\,du) = (xu + 2x\sqrt{u}\,)\,dx, \quad \text{or} \quad \frac{du}{2\sqrt{u}} = \frac{dx}{x}.$$

Integrating this last equation,

$$u^{1/2} = \ln|x| + C, \quad \text{or} \quad y = x\left(\ln|x| + C\right)^2.$$

6. Because $M(x,y) = \sqrt{x^2+y^2} - y$ and $N(x,y) = x$, we have that $M(tx,ty) = \sqrt{t^2x^2+t^2y^2} - ty = t\left(\sqrt{x^2+y^2} - y\right) = tM(x,y)$, and $N(tx,ty) = tx = tN(x,y)$. Therefore, the differential equation is homogeneous.

Let $y = ux$. Substituting into the differential equation,

$$\left(\sqrt{x^2+x^2u^2} - ux\right)\,dx + x(x\,du + u\,dx) = 0,$$

or

$$x\sqrt{1+u^2}\,dx + x^2\,du = 0, \quad \text{or} \quad \frac{dx}{x} = -\frac{du}{\sqrt{1+u^2}}.$$

Integrating this last equation,

$$-\ln(x) = -\ln\left(u + \sqrt{1+u^2}\right) - \ln(C).$$

Inverting the logarithms,

$$ux + \sqrt{u^2x^2 + x^2} = C, \quad \text{or} \quad y + \sqrt{x^2+y^2} = C.$$

7. Because $M(x,y) = \sec(y/x) + y/x$ and $N(x,y) = -1$, we have that $M(tx,ty) = \sec[(ty)/(tx)] + (ty)/(tx) = \sec(y/x) + y/x = M(x,y)$, and

$N(tx, ty) = -1 = N(x, y)$. Therefore, the differential equation is homogeneous.

Let $y = ux$. Substituting into the differential equation,

$$u\,dx + x\,du = [\sec(u) + u]\,dx, \quad \text{or} \quad \cos(u)\,du = \frac{dx}{x}.$$

Integrating and substituting for u, the final answer is

$$\sin(y/x) - \ln|x| = C.$$

8. Because $M(x, y) = e^{y/x} + y/x$ and $N(x, y) = -1$, we have that $M(tx, ty) = e^{(ty)/(tx)} + (ty)/(tx) = e^{y/x} + y/x = M(x, y)$, and $N(tx, ty) = -1 = N(x, y)$. Therefore, the differential equation is homogeneous.

Let $y = ux$. Substituting into the differential equation,

$$u\,dx + x\,du = (e^u + u)\,dx, \quad \text{or} \quad e^{-u}\,du = \frac{dx}{x}.$$

Integrating and substituting for u, the final answer is

$$y(x) = -x\ln\left(C - \ln|x|\right).$$

Section 1.4

1. Since $M(x, y) = y^2 - x^2$, and $N(x, y) = 2xy$,

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = y^2 - x^2,$$

then $u(x, y) = xy^2 - \frac{1}{3}x^3 + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = 2xy + f'(y) = 2xy.$$

Therefore, $f'(y) = 0$, and $u(x, y) = xy^2 - \frac{1}{3}x^3 = C$.

2. Since $M(x, y) = y - x$, and $N(x, y) = x + y$,

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = y - x,$$

then $u(x,y) = xy - \frac{1}{2}x^2 + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = x + f'(y) = x + y.$$

Therefore, $f'(y) = y$, and $u(x,y) = xy + \frac{1}{2}y^2 - \frac{1}{2}x^2 = C$.

3. Since $M(x,y) = y^2 - 1$, and $N(x,y) = 2xy - \sin(y)$,

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = y^2 - 1,$$

then $u(x,y) = xy^2 - x + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = 2xy + f'(y) = 2xy - \sin(y).$$

Therefore, $f'(y) = -\sin(y)$, and $u(x,y) = xy^2 - x + \cos(y) = C$.

4. Since $M(x,y) = \sin(y) - 2xy + x^2$, and $N(x,y) = x\cos(y) - x^2$,

$$\frac{\partial M}{\partial y} = \cos(y) - 2x = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = \sin(y) - 2xy + x^2,$$

then $u(x,y) = x\sin(y) - x^2y + \frac{1}{3}x^3 + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = x\cos(y) - x^2 + f'(y) = x\cos(y) - x^2.$$

Therefore, $f'(y) = 0$, and $u(x,y) = x\sin(y) - x^2y + \frac{1}{3}x^3 = C$.

5. Since $M(x,y) = -y/x^2$, and $N(x,y) = 1/x + 1/y$,

$$\frac{\partial M}{\partial y} = -\frac{1}{x^2} = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = -\frac{y}{x^2},$$

then $u(x, y) = y/x + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = \frac{1}{x} + f'(y) = \frac{1}{x} + \frac{1}{y}.$$

Therefore, $f'(y) = 1/y$, and $u(x, y) = y/x + \ln(y) = C$.

6. Since $M(x, y) = 3x^2 - 6xy$, and $N(x, y) = -3x^2 - 2y$,

$$\frac{\partial M}{\partial y} = -6x = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = 3x^2 - 6xy,$$

then $u(x, y) = x^3 - 3x^2y + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = -3x^2 + f'(y) = -3x^2 - 2y.$$

Therefore, $f'(y) = -2y$, and $u(x, y) = x^3 - 3x^2y - y^2 = C$.

7. Since $M(x, y) = y\sin(xy)$, and $N(x, y) = x\sin(xy)$,

$$\frac{\partial M}{\partial y} = \sin(xy) + xy\cos(xy) = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = y\sin(xy),$$

then $u(x, y) = -\cos(xy) + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = x\sin(xy) + f'(y) = x\sin(xy).$$

Therefore, $f'(y) = 0$, and $u(x, y) = -\cos(xy) = C$.

8. Since $M(x, y) = 2xy^2 + 3x^2$, and $N(x, y) = 2x^2y$,

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = 2xy^2 + 3x^2,$$

then $u(x, y) = x^2y^2 + x^3 + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = 2x^2y + f'(y) = 2x^2y.$$

Therefore, $f'(y) = 0$, and $u(x, y) = x^2y^2 + x^3 = C$.

9. Since $M(x, y) = 2xy^3 + 5x^4y$, and $N(x, y) = 3x^2y^2 + x^5 + 1$,

$$\frac{\partial M}{\partial y} = 6xy^2 + 5x^4 = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = 2xy^3 + 5x^4y,$$

then $u(x, y) = x^2y^3 + x^5y + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = 3x^2y^2 + x^5 + f'(y) = 3x^2y^2 + x^5 + 1.$$

Therefore, $f'(y) = 1$, and $u(x, y) = x^2y^3 + x^5y + y = C$.

10. Since $M(x, y) = x^3 + y/x$, and $N(x, y) = y^2 + \ln(x)$,

$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = x^3 + \frac{y}{x},$$

then $u(x, y) = \frac{1}{4}x^4 + y\ln(x) + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = \ln(x) + f'(y) = y^2 + \ln(x).$$

Therefore, $f'(y) = y^2$, and $u(x, y) = \frac{1}{4}x^4 + y\ln(x) + \frac{1}{3}y^3 = C$.

11. $[x + e^{-y} + x\ln(y)]\,dy + [y\ln(y) + e^x]\,dx = 0$

Since $M(x,y) = y\ln(y) + e^x$, and $N(x,y) = x + e^{-y} + x\ln(y)$,

$$\frac{\partial M}{\partial y} = 1 + \ln(y) = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = y\ln(y) + e^x,$$

then $u(x,y) = xy\ln(y) + e^x + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = x[\ln(y) + 1] + f'(y) = x + e^{-y} + x\ln(y).$$

Therefore, $f'(y) = e^{-y}$, and $u(x,y) = xy\ln(y) + e^x - e^{-y} = C$.

12. Since $M(x,y) = \cos(4y^2)$, and $N(x,y) = -8xy\sin(4y^2)$,

$$\frac{\partial M}{\partial y} = -8y\sin(4y^2) = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = \cos(4y^2),$$

then $u(x,y) = x\cos(4y^2) + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = -8xy\sin(4y^2) + f'(y) = -8xy\sin(4y^2).$$

Therefore, $f'(y) = 0$, and $u(x,y) = x\cos(4y^2) = C$.

13. Since $M(x,y) = \sin^2(x+y)$, and $N(x,y) = -\cos^2(x+y)$,

$$\frac{\partial M}{\partial y} = 2\sin(x+y)\cos(x+y) = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = \sin^2(x+y) = \tfrac{1}{2}\left[1 - \cos(2x+2y)\right],$$

then $u(x,y) = x/2 - \sin(2x+2y)/4 + f(y)$. To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = -\tfrac{1}{2}\cos(2x+2y) + f'(y) = -\cos^2(x+y) = -\tfrac{1}{2}\left[1 + \cos(2x+2y)\right].$$

Therefore, $f'(y) = -\frac{1}{2}$, and $u(x,y) = y - x + \frac{1}{2}\sin(2x+2y) = C$.

14. After multiplying by the integrating factor,

$$M(x,y) = \alpha \frac{y^{a+1}}{(1-y)^{a+1}}, \quad \text{and} \quad N(x,y) = (x-y)\frac{y^a}{(1-y)^{a+2}}.$$

Checking the exactness criteria,

$$\frac{\partial M}{\partial y} = \frac{y^a}{(1-y)^{a+2}} = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = \alpha \frac{y^{a+1}}{(1-y)^{a+1}},$$

$$u(x,y) = \alpha x \frac{y^{a+1}}{(1-y)^{a+1}} + f(y) = C.$$

To find $f(y)$, we use

$$\frac{\partial u}{\partial y} = x\frac{y^a}{(1-y)^{a+2}} + f'(y) = (x-y)\frac{y^a}{(1-y)^{a+2}}.$$

Therefore,

$$f'(y) = -\frac{y^{a+1}}{(1-y)^{a+2}}.$$

Integrating, we find that

$$f(y) = -\int_0^y \frac{\xi^{a+1}}{(1-\xi)^{a+2}}\,d\xi.$$

and the final answer is

$$u(x,y) = \alpha x \frac{y^{a+1}}{(1-y)^{a+1}} - \int_0^y \frac{\xi^{a+1}}{(1-\xi)^{a+2}}\,d\xi = C.$$

Section 1.5

1. Since $P(x) = 1$, $\mu(x) = e^x$. Multiplying the differential equation by the integrating, we have that $e^x y' + e^x y = e^{2x}$, or $d\left(e^x y\right)/dx = e^{2x}$ or $e^x y = \frac{1}{2}e^{2x} + C$, or $y = \frac{1}{2}e^x + Ce^{-x}$. This general solution applies to any x on the interval $(-\infty, \infty)$.

2. Since $P(x) = 2x$, $\mu(x) = e^{x^2}$. Multiplying the differential equation by the integrating factor, we have that $e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}$, $d\left(e^{x^2}y\right)/dx = xe^{x^2}$, or $e^{x^2}y = \frac{1}{2}e^{x^2} + C$, or $y = \frac{1}{2} + Ce^{-x^2}$. This general solution applies to any x on the interval $(-\infty, \infty)$.

3. Since the canonical form of the differential equation is $y' + y/x = 1/x^2$, $P(x) = 1/x$, and $\mu(x) = x$. Multiplying the canonical differential equation by the integrating factor, we have that $xy' + y = x^{-1}$, or $d\,(xy)\,/dx = x^{-1}$, or $xy = \ln(x) + C$, or $y = \ln(x)/x + Cx^{-1}$. This general solution applies to any x as long as $x \neq 0$.

4. Since the canonical form of the differential equation is $y' - 2y/x = x$, $P(x) = -2/x$, and $\mu(x) = x^{-2}$. Multiplying the canonical differential equation by the integrating factor, we have that $x^{-2}y' - 2x^{-3}y = x^{-1}$, $d\,(y/x)\,/dx = 1/x$, or $y/x^2 = \ln(x) + C$, or $y = 2x\ln(x) + Cx^2$. This general solution applies to any x on the interval $(-\infty, \infty)$.

5. Directly from the differential equation, we have that $P(x) = -3/x$, and $\mu(x) = x^{-3}$. Multiplying the differential equation by the integrating factor, we have that $x^{-3}y' - 3x^{-4}y = 2x^{-1}$, $d\left(y/x^3\right) = 2/x$, or $y/x^3 = 2\ln(x) + C$, or $y = 2x^3\ln(x) + Cx^3$. This general solution applies to any x on the interval $(-\infty, \infty)$.

6. Since $P(x) = 2$, $\mu(x) = e^{2x}$. Multiplying the differential equation by the integrating, we have that $e^{2x}y' + 2e^{2x}y = 2e^{2x}\sin(x)$, or $d\left(e^{2x}y\right)/dx = 2e^{2x}\sin(2x)$, or $e^{2x}y = \frac{2}{5}e^{2x}\left[2\sin(x) - \cos(x)\right] + C$, or $y = \frac{4}{5}\sin(x) - \frac{2}{5}\cos(x) + Ce^{-2x}$. This general solution applies to any x on the interval $(-\infty, \infty)$.

7. Since the differential equation is already in canonical form, we immediately have $P(x) = 2\cos(2x)$, and $\mu(x) = \exp[\sin(2x)]$. Multiplying the differential equation by the integrating factor, we have that $e^{\sin(2x)}dy/dx + 2\cos(2x)e^{\sin(2x)}y = 0$, $d\left[e^{\sin(2x)}y\right]/dx = 0$, or $e^{\sin(2x)}y = C$. This general solution applies to any x on the interval $n\pi + \varphi < 2x < (n+1)\pi + \varphi$, where φ is any real and n is any integer.

8. Dividing through by x, we immediately have $P(x) = 1/x$, and $\mu(x) = x$. Multiplying the differential equation by the integrating factor, we have that $xdy/dx + y = \ln(x)$, or $d(xy)/dx = \ln(x)$, or $xy = C + x\ln(x) - x$, or $y = C/x + \ln(x) - 1$. This general solution applies to any x on the interval $(0, \infty)$.

9. Since the differential equation is already in canonical form, we immediately have $P(x) = 3$, and $\mu(x) = e^{3x}$. Multiplying the differential equation by the integrating factor, we have that $e^{3x}dy/dx + 3e^{3x}y = 4e^{3x}$, or $d\left(e^{3x}y\right)/dx =$

$4e^{3x}$, or $e^{3x}y(x) - y(0) = \frac{4}{3}\left(e^{3x} - 1\right)$, or $e^{3x}y(x) - 5 = \frac{4}{3}\left(e^{3x} - 1\right)$, or $y(x) = \frac{4}{3} + \frac{11}{3}e^{-3x}$. This particular solution applies to any x on the interval $(-\infty, \infty)$.

10. Since the differential equation is already in canonical form, we immediately have $P(x) = -1$, and $\mu(x) = e^{-x}$. Multiplying the differential equation by the integrating factor, we have that $e^{-x}dy/dx - e^{-x}y = 1/x$, or $d\left(e^{-x}y\right)/dx = 1/x$, or $e^{-x}y(x) - e^{-e}y(e) = \ln(x) - \ln(e)$, or $y(x) = e^{x}\left[\ln(x) - 1\right]$.

11. By inspection, we immediately have that $d\left[\sin(x)y\right]/dx = 1$, or $\sin(x)y = x + C$, or $y(x) = (x + C)/\sin(x)$.

12. Since the canonical form of the differential equation is

$$\frac{dy}{dx} + \frac{2\sin(x)}{1 - \cos(x)}y = \frac{\tan(x)}{1 - \cos(x)},$$

$P(x) = 2\sin(x)/[1-\cos(x)]$, and $\mu(x) = [1-\cos(x)]^2$. Multiplying the canonical differential equation by the integrating factor, we have that

$$\begin{aligned}[1 - \cos(x)]^2\frac{dy}{dx} + 2\sin(x)[1 - \cos(x)]y &= \tan(x)[1 - \cos(x)]\\ \frac{d}{dx}\left\{[1 - \cos(x)]^2 y\right\} &= \tan(x) - \sin(x)\\ [1 - \cos(x)]^2 y &= -\ln|\cos(x)| + \cos(x) + C.\end{aligned}$$

This general solution applies to any x on the interval $n\pi+\varphi < x < (n+1)\pi+\varphi$, where φ is any real and n is any integer.

13. Since the differential equation is already in canonical form, $P(x) = a\tan(x) + b\sec(x)$, and

$$\mu(x) = \frac{[\sec(x) + \tan(x)]^b}{\cos^a(x)}.$$

Multiplying by the integrating factor, we have that

$$\frac{d}{dx}\left\{\frac{[\sec(x) + \tan(x)]^b}{\cos^a(x)}y(x)\right\} = c\frac{\sec(x)[\sec(x) + \tan(x)]^b}{\cos^a(x)}.$$

Integrating both sides of this equation,

$$\frac{[\sec(x) + \tan(x)]^b}{\cos^a(x)}y(x) - y(0) = c\int_0^x \frac{[\sec(\xi) + \tan(\xi)]^b}{\cos^{a+1}(\xi)}\,d\xi$$

or

$$y(x) = \frac{\cos^a(x)\ y(0)}{[\sec(x) + \tan(x)]^b} + \frac{c\ \cos^a(x)}{[\sec(x) + \tan(x)]^b}\int_0^x \frac{[\sec(\xi) + \tan(\xi)]^b}{\cos^{a+1}(\xi)}\,d\xi.$$

14. Writing the differential equation in canonical form, we have

$$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right) y = \frac{1}{x}.$$

Therefore,

$$\mu(x) = \exp\left[\int^x \left(1 + \frac{1}{\xi}\right) d\xi\right] = \exp[x + \ln(x)] = xe^x.$$

Multiplying the differential equation by the integrating factor, we find

$$\begin{aligned} xe^x \frac{dy}{dx} + (x+1)e^x y &= e^x \\ \frac{d}{dx}[xe^x y] &= e^x \\ xe^x y &= e^x + C \\ y &= \frac{1}{x} + Ce^{-x}. \end{aligned}$$

15. Since the differential equation is already in canonical form, $P(x) = 2a$, and $\mu(x) = e^{2ax}$. Multiplying by the integrating factor, we have that

$$\frac{d}{dx}\left[e^{2ax} y(x)\right] = \frac{x}{2} e^{2ax} - \frac{\sin(2\omega x)}{4\omega} e^{2ax}.$$

Integrating both sides of this equation,

$$e^{2ax} y(x) - y(0) = \tfrac{1}{2} \int_0^x \xi e^{2a\xi}\, d\xi - \frac{1}{4\omega} \int_0^x \sin(2\omega\xi) e^{2a\xi}\, d\xi$$

or

$$e^{2ax} y(x) = \left.\frac{e^{2a\xi}}{8a^2}(2a\xi - 1)\right|_0^x - \left.\frac{e^{2a\xi}}{8\omega(a^2+\omega^2)}\left[a\sin(2\omega\xi) - \omega\cos(2\omega\xi)\right]\right|_0^x.$$

Solving for $y(x)$,

$$y(x) = \frac{2ax - 1}{8a^2} + \frac{\omega^2 e^{-2ax}}{8a^2(a^2+\omega^2)} - \frac{a\sin(2\omega x) - \omega\cos(2\omega x)}{8\omega(a^2+\omega^2)}.$$

16. Since the differential equation is already in canonical form, $P(x) = 2k/x^3$, and $\mu(x) = \exp(-k/x^2)$. Multiplying by the integrating factor, we have that

$$\frac{d}{dx}\left[e^{-k/x^2} y(x)\right] = \ln\left(\frac{x+1}{x}\right) e^{-k/x^2}.$$

Integrating both sides of this equation,

$$e^{-k/x^2}y(x) - e^{-k}y(1) = \int_1^x \ln\left(\frac{\xi+1}{\xi}\right)\exp\left(-\frac{k}{\xi^2}\right)\,d\xi.$$

Because $y(1) = 0$,

$$y(x) = \exp\left(\frac{k}{x^2}\right)\int_1^x \ln\left(\frac{\xi+1}{\xi}\right)\exp\left(-\frac{k}{\xi^2}\right)\,d\xi.$$

17. Substituting in the variable $p(x) = y^2(x)$, we have that

$$\frac{dp}{dx} - \frac{2}{kx}p = -\frac{2}{k}, \qquad p(1) = 0.$$

Then, multiplying through with the integrating factor, we find that

$$x^{-2/k}\frac{dp}{dx} - \frac{2}{k}x^{-2/k-1}p = -\frac{2}{k}x^{-2/k}$$
$$\frac{d}{dx}\left[x^{-2/k}p\right] = -\frac{2}{k}x^{-2/k}.$$

If $k \neq 2$, an integration yields

$$x^{-2/k}p(x) = -\frac{2/k}{1-2/k}x^{1-2/k} + C, \quad \text{or} \quad y^2(x) = -\frac{2}{k-2}x + Cx^{2/k}.$$

Applying the initial condition, the final answer is

$$y^2(x) = \frac{2}{2-k}\left(x - x^{2/k}\right),$$

provided $k \neq 2$. If $k = 2$, then we have that

$$\frac{p(x)}{x} = -\ln(x) + C.$$

Applying the initial condition, $C = 0$ and the final answer is

$$y^2(x) = x\ln(1/x).$$

18. We must solve

$$\frac{dx}{dt} - (1-N)rx = S, \qquad x(0) = 1.$$

Multiplying both side of the equation by $e^{-(1-N)rt}$, we have

$$\frac{d}{dt}\left[e^{-(1-N)rt}x(t)\right] = Se^{-(1-N)rt}.$$

Integrating both sides of this equation from 0 to t, we obtain

$$e^{-(1-N)rt}x(t) - 1 = \frac{S}{(1-N)r}\left[1 - e^{-(1-N)rt}\right].$$

Solving $x(t)$, we find that

$$x(t) = e^{(1-N)rt} + \frac{S}{(1-N)r}\left[e^{(1-N)rt} - 1\right].$$

19. From separation of variables,

$$\frac{d[\mathrm{A}]}{[\mathrm{A}]} = -k_1\, dt.$$

Integration yields

$$[\mathrm{A}] = [\mathrm{A}]_0 e^{-k_1 t}.$$

Substituting [A] into the equation for [B],

$$\frac{d[\mathrm{B}]}{dt} + k_2[\mathrm{B}] = k_1[\mathrm{A}]_0 e^{-k_1 t}.$$

Multiplying by the integrating factor, we have that

$$\frac{d}{dt}\left(e^{k_2 t}\,[\mathrm{B}]\right) = k_1[\mathrm{A}]_0 e^{(k_2-k_1)t}.$$

Integrating this equation, we find that

$$e^{k_2 t}[\mathrm{B}] - [\mathrm{B}]_0 = \frac{k_1[\mathrm{A}]_0}{k_2 - k_1}\left[e^{(k_2-k_1)t} - 1\right].$$

Because $[\mathrm{B}]_0 = 0$, we find that

$$[\mathrm{B}] = \frac{k_1[\mathrm{A}]_0}{k_2 - k_1}\left[e^{-k_1 t} - e^{-k_2 t}\right].$$

Finally, substituting for [B] into the [C] equation,

$$\frac{d[\mathrm{C}]}{dt} = \frac{k_1 k_2[\mathrm{A}]_0}{k_2 - k_1}\left[e^{-k_1 t} - e^{-k_2 t}\right].$$

Integrating this equation,

$$[\mathrm{C}] - [\mathrm{C}]_0 = \frac{k_2[\mathrm{A}]_0}{k_2 - k_1}\left(1 - e^{-k_1 t}\right) - \frac{k_1[\mathrm{A}]_0}{k_2 - k_1}\left(1 - e^{-k_2 t}\right)$$

or

$$[\mathrm{C}] = [\mathrm{A}]_0 \left(1 + \frac{k_1 e^{-k_2 t} - k_2 e^{-k_1 t}}{k_2 - k_1}\right).$$

since $[\mathrm{C}]_0 = 0$.

20. The differential equation is

$$L\frac{dI}{dt} + RI = E_0 \cos^2(\omega t), \qquad I(0) = 0.$$

Multiplying the integrating factor, we can rewrite this differential equation

$$\frac{d}{dt}\left[e^{Rt/L} I(t)\right] = \frac{E_0}{L} e^{Rt/L} \cos^2(\omega t).$$

Integrating both sides of the differential equation,

$$e^{Rt/L} I(t) - I(0) = \frac{E_0}{L}\int_0^t e^{R\tau/L} \cos^2(\omega\tau)\, d\tau.$$

Using the fact that $I(0) = 0$ and replacing $\cos^2(\omega t)$ with $\frac{1}{2}[1 + \cos(2\omega t)]$, we have that

$$e^{Rt/L} I(t) = \frac{E_0}{2L}\int_0^t \left[e^{R\tau/L} + e^{R\tau/L}\cos(2\omega\tau)\right]\, d\tau,$$

or

$$e^{Rt/L} I(t) = \frac{E_0\left(e^{R\tau/L} - 1\right)}{2R} + \frac{E_0\left\{e^{R\tau/L}\left[R\cos(2\omega\tau) + 2\omega L\sin(2\omega\tau)\right] - R\right\}}{2R^2 + 8\omega^2 L^2}.$$

Solving for $I(t)$,

$$I(t) = \frac{E_0\left(1 - e^{-R\tau/L}\right)}{2R} + \frac{E_0\left\{R\left[\cos(2\omega\tau) - e^{-Rt/L}\right] + 2\omega L\sin(2\omega\tau)\right\}}{2R^2 + 8\omega^2 L^2}.$$

21. Because $n = 2$, $z = y^{-1}$ and the linear ordinary differential equation is

$$\frac{dz}{dx} - \frac{z}{x} = 1, \quad \text{or} \quad \frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = \frac{d}{dx}\left(\frac{z}{x}\right) = \frac{1}{x}.$$

Integrating this equation yields

$$z(x) = Cx + x\ln(x).$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = \frac{1}{Cx + x\ln(x)}.$$

22. Because $n = 2$, $z = y^{-1}$ and the linear ordinary differential equation is

$$\frac{dz}{dx} + \frac{z}{x} = -\frac{1}{x^2}, \quad \text{or} \quad x\frac{dz}{dx} + z = \frac{d}{dx}(xz) = -\frac{1}{x}.$$

Integrating this equation yields

$$z(x) = [C - \ln(x)]/x.$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = \frac{x}{C - \ln(x)}.$$

23. Because $n = \frac{1}{2}$, $z = y^{1/2}$ and the linear ordinary differential equation is

$$\frac{dz}{dx} - \frac{2z}{x} = \frac{x}{2}, \quad \text{or} \quad \frac{1}{x^2}\frac{dz}{dx} - \frac{2z}{x^3} = \frac{d}{dx}\left(\frac{z}{x^2}\right) = \frac{1}{2x}.$$

Integrating this equation yields

$$z(x) = Cx^2 + \tfrac{1}{2}x^2\ln(x).$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = \left[Cx^2 + \tfrac{1}{2}x^2\ln(x)\right]^2.$$

24. Because $n = 2$, $z = y^{-1}$ and the linear ordinary differential equation is

$$\frac{dz}{dx} - \frac{z}{x} = x, \quad \text{or} \quad \frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = \frac{d}{dx}\left(\frac{z}{x}\right) = 1.$$

Integrating this equation yields

$$z(x) = Cx + x^2.$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = \frac{1}{Cx + x^2}.$$

25. Because $n = -1$, $z = y^2$ and the linear ordinary differential equation is

$$\frac{dz}{dx} - \frac{z}{x} = -1, \quad \text{or} \quad \frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = \frac{d}{dx}\left(\frac{z}{x}\right) = -\frac{1}{x}.$$

Integrating this equation yields

$$z(x) = Cx - x\ln(x).$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = [Cx - x\ln(x)]^{1/2}.$$

26. Because $n = 3$, $z = y^{-2}$ and the linear ordinary differential equation is

$$\frac{dz}{dx} - \frac{2z}{x} = -1$$

or

$$\frac{1}{x^2}\frac{dz}{dx} - \frac{2z}{x^3} = \frac{d}{dx}\left(\frac{z}{x^2}\right) = -\frac{1}{x^2}.$$

Integrating this equation yields

$$z(x) = Cx^2 + x.$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = \left[Cx^2 + x\right]^{-1/2}.$$

Section 1.6

5. The equilibrium points for this differential equation are $x = 0, \frac{1}{2}$, and 1. The right side is negative for $0 < x < \frac{1}{2}$ and positive for $\frac{1}{2} < x < 1$. Thus, the equilibrium point $x = \frac{1}{2}$ is unstable while the equilibrium points at $x = 0$ and 1 are stable.

6. The equilibrium points are $x = \pm 1$, and $x = \pm 2$. For $x < -2$, $-1 < x < 1$, and $x > 2$, $x' > 0$. For $-2 < x < -1$ and $1 < x < 2$, $x' < 0$. Therefore, the equilibrium points at $x = -2$ and $x = 1$ are stable while $x = -1$ and $x = 2$ are unstable.

7. There is only one equilibrium point, $x = 0$. For $x < 0$, $x' > 0$ while for $x > 0$, $x' < 0$. Therefore, this equilibrium point is stable.

8. The equilibrium points are $x = \pm 2$, and $x = 0$. For $x < -2$ and $0 < x < 2$, $x' > 0$. For $-2 < x < 0$ and $x > 2$, $x' < 0$. Therefore, the equilibrium point $x = 0$ is unstable while the points $x = \pm 2$ are stable.

Section 1.7

1. Because the differential equation can be written $x' - x = t$, we have that

$$\frac{d}{dt}\left(e^{-t}x\right) = -te^{-t}.$$

Integration yields $x(t) = Ce^t + t + 1$. Applying the initial condition, we find that $C = 1$. Therefore, the final answer is $x(t) = e^t + t + 1$.

2. Because the differential equation can be written $dx/x = t\,dt$, $x(t) = Ce^{t^2/2}$. Applying the initial condition, $C = 1$. Therefore, the final solution is $x(t) = e^{t^2/2}$.

3. Because the differential equation can be written $dx/x^2 = dt/(t+1)$, integration yields $1/x = C - \ln(t+1)$. Applying the initial condition, $C = 1$. Therefore, the final answer is $x(t) = [1 - \ln(t+1)]^{-1}$.

4. Because the differential equation can be written $x' - x = e^{-t}$, we have that

$$\frac{d}{dt}\left(e^{-t}x\right) = e^{-2t}.$$

Integration yields $x(t) = Ce^t - \frac{1}{2}e^{-t}$. Applying the initial condition, we find that $C = e^{-2}/2$. Therefore, the final answer is $x(t) = \frac{1}{2}\left(e^{t-2} - e^{-t}\right)$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%        MATLAB Code for Differential-Integral Equation
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% initialize parameters
clear; beta = 0.4; deltat = 0.01; K = 1000;

% vary value of b

for n = 1:4

b = 0.2 * (n-1);
```

```
% take the first time step

x(1) = 0; t(1) = 0; t(2) = deltat;
x(2) = x(1) + deltat - b * deltat * sign(x(1)) * abs(x(1))^beta;
sum = x(1) + x(2);

% take the remaining time steps

  for k = 2:K
    t(k) = t(k-1) + deltat;
    x(k) = x(k-1) + deltat ...
       - b * deltat * sign(x(k-1)) * abs(x(k-1))^beta ...
       - deltat * deltat * sum;
    sum = sum + x(k);
  end

% plot the results

subplot(2,2,n), plot(t,x); xlabel('time','Fontsize',20);
ylabel('x(t)','Fontsize',20); legend(['B = ',num2str(b)])

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Section 2.0

1. Since the second solution is $y_2(x) = u(x)$, $y_2'(x) = u'(x)$ and $y_2''(x) = u''(x)$. Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $xu'' + 2u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = -\frac{2}{x}, \qquad u'(x) = Cx^{-2}.$$

Thus, $u(x) = A/x$, and the second solution is $y_2(x) = A/x$.

2. Since the second solution is $y_2(x) = u(x)e^x$,

$$y_2'(x) = u'(x)e^x + u(x)e^x, \quad \text{and} \quad y_2''(x) = u''(x)e^x + 2u'(x)e^x + u(x)e^x.$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $u'' + 3u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = -3, \qquad u'(x) = Ce^{-3x}.$$

Thus, $u(x) = Ae^{-3x}$, and the second solution is $y_2(x) = Ae^{-2x}$.

3. Since the second solution is $y_2(x) = xu(x)$,

$$y_2'(x) = xu'(x) + u(x), \quad \text{and} \quad y_2''(x) = xu''(x) + 2u'(x).$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $xu'' + 6u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = -\frac{6}{x}, \qquad u'(x) = Cx^{-6}.$$

Thus, $u(x) = Ax^{-5}$, and the second solution is $y_2(x) = Ax^{-4}$.

4. Since the second solution is $y_2(x) = e^x u(x)$,

$$y_2'(x) = e^x u(x) + e^x u'(x), \quad \text{and} \quad y_2''(x) = e^x u(x) + 2e^x u'(x) + e^x u''(x).$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $xu'' + (x-1)u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = \frac{1}{x} - 1, \qquad u'(x) = -Axe^{-x}.$$

Thus, $u(x) = A(x+1)e^{-x}$, and the second solution is $y_2(x) = A(x+1)$.

5. Since the second solution is $y_2(x) = (x-1)u(x)$,

$$y_2'(x) = (x-1)u'(x) + u(x), \quad \text{and} \quad y_2''(x) = (x-1)u''(x) + 2u'(x).$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $x(2-x)(x-1)u'' + 2u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = \frac{2}{x(x-2)(x-1)} = \frac{1}{x} + \frac{1}{x-2} - \frac{2}{x-1},$$

and

$$u'(x) = A\frac{x(x-2)}{(x-1)^2} = A\left[1 - \frac{1}{(x-1)^2}\right].$$

Thus, $u(x) = Ax + A/(x-1)$, and the second solution is $y_2(x) = A(x^2 - x + 1)$.

6. Since the second solution is $y_2(x) = u(x)\sin^3(x)$,

$$y_2'(x) = 3\sin^2(x)\cos(x)u(x) + \sin^3(x)u'(x),$$

and

$$y_2''(x) = 6\sin(x)\cos^2(x)u(x) - 3\sin^3(x)u(x) + 6\sin^2(x)\cos(x)u'(x) + \sin^3(x)u''(x).$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that

$$\sin(x)\cos(x)u'' + [6\cos^2(x) + \sin^2(x)]u' = 0.$$

Therefore,

$$\frac{u''(x)}{u'(x)} = -6\cot(x) - \tan(x), \qquad u'(x) = -\frac{A\cos(x)}{5\sin^6(x)}.$$

Thus, $u(x) = A/\sin^5(x)$, and the second solution is $y_2(x) = A/\sin^2(x)$.

7. Since the second solution is $y_2(x) = u(x)\cos(x)/\sqrt{x}$,

$$y_2'(x) = -\frac{\sin(x)}{\sqrt{x}}u(x) - \frac{\cos(x)}{2x\sqrt{x}}u(x) + \frac{\cos(x)}{\sqrt{x}}u'(x)$$

and

$$y_2''(x) = -\frac{\cos(x)}{\sqrt{x}}u(x) + \frac{3\cos(x)}{4x^2\sqrt{x}}u(x) + \frac{\sin(x)}{x\sqrt{x}}u(x) - \frac{\cos(x)}{x\sqrt{x}}u'(x) - \frac{2\sin(x)}{\sqrt{x}}u'(x) + \frac{\cos(x)}{\sqrt{x}}u''(x).$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $\cos(x)u'' - 2\sin(x)u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = \frac{2\sin(x)}{\cos(x)}, \qquad u'(x) = A\sec^2(x).$$

Thus, $u(x) = A\tan(x)$, and the second solution is $y_2(x) = A\sin(x)/\sqrt{x}$.

8. Since the second solution is $y_2(x) = u(x)e^{-bx^2/2}$

$$y_2'(x) = -bxe^{-bx^2/2}u(x) + e^{-bx^2/2}u'(x),$$

and

$$y_2''(x) = (b^2x^2 - b)e^{-bx^2/2}u(x) - 2bxe^{-bx^2/2}u'(x) + u''(x)e^{-bx^2/2}.$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $u'' + (a - 2bx)u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = 2bx - a.$$

Thus, $u(x) = \int^x e^{b\xi^2 - a\xi}\, d\xi$, and the second solution is

$$y_2(x) = e^{-bx^2/2} \int^x e^{b\xi^2 - a\xi}\, d\xi.$$

9. Letting $v = y'$, we can rewrite the differential equation $y(dv/dy) = v^2$. Assuming $v \neq 0$, we can integrate this equation to give $v = C_1 y$. Therefore, $dy/dx = C_1 y$. Integrating this, we obtain the final answer $y(x) = C_2 e^{C_1 x}$.

10. Letting $v = y'$, we can rewrite the differential equation $dv/dy = 2y$. Assuming $v \neq 0$, we can integrate this equation to give $v = y^2 + C_1$. Because $v(1) = 1$, $C_1 = 0$ and $dy/dx = y^2$. Integrating this equation, $1/y = C_2 - x$. Again, using the initial conditions, $y(x) = 1/(1 - x)$.

11. Letting $v = y'$, we can rewrite the differential equation $yv(dv/dy) = v + v^2$. Assuming $v \neq 0$, we can integrate this equation to give $1 + v = C_1 y$. Substituting for v, $dy/dx = C_1 y - 1$. Integrating this equation, $y = \left(1 + C_2 e^{C_1 x}\right)/C_1$.

12. Letting $v = y'$, we can rewrite the differential equation $2yv(dv/dy) = 1 + v^2$. Assuming $v \neq 0$, we can integrate this equation to give $1 + v^2 = C_1 y$. Substituting for v, $dy/dx = \sqrt{C_1 y - 1}$. Integrating this equation, $y = \left[1 + (C_1 x + C_2)^2/4\right]/C_1$.

13. Letting $v = y'$, we can rewrite the differential equation $v(dv/dy) = e^{2y}$ with $v(0) = 1$. We can integrate this equation and find $v = e^y$. Substituting for v, $dy/dx = e^y$. Integrating this equation, $e^{-y} = C - x$, or $y = -\ln|1 - x|$.

14. We begin by integrating once and using the initial conditions, $y'' = \frac{3}{2}y^2$. Letting $v = y'$, we can rewrite this differential equation $v(dv/dy) = \frac{3}{2}y^2$. Integrating this equation, we have $v = y^{3/2}$. Substituting for v, $dy/dx = y^{3/2}$. Integrating this equation, $y = 4/(2 - x)^2 = 4/(x - 2)^2$.

15. If we define $z = 1/v$, the Bernoulli equation becomes $z' + z/x = -\frac{1}{2}$. Its solution is $z = -A^2/x - x/4$. Therefore, $y' = v = -4x/(x^2 + 4A^2)$. Integration yields the final answer $y(x) = B - 2\ln(x^2 + 4A^2)$.

16. First we compute

$$y'(x) = u'(x) \exp\left[-\frac{1}{2}\int^x \frac{a_1(\xi)}{a_2(\xi)}\, d\xi\right] - \frac{a_1}{2a_2} u(x) \exp\left[-\frac{1}{2}\int^x \frac{a_1(\xi)}{a_2(\xi)}\, d\xi\right],$$

and

$$y''(x) = u''(x) \exp\left[-\frac{1}{2}\int^x \frac{a_1(\xi)}{a_2(\xi)}\, d\xi\right] - \frac{1}{2}\frac{d}{dx}\left(\frac{a_1}{a_2}\right) u(x) \exp\left[-\frac{1}{2}\int^x \frac{a_1(\xi)}{a_2(\xi)}\, d\xi\right]$$
$$- \frac{a_1}{a_2} u'(x) \exp\left[-\frac{1}{2}\int^x \frac{a_1(\xi)}{a_2(\xi)}\, d\xi\right] + \frac{a_1^2}{4a_2^2} u(x) \exp\left[-\frac{1}{2}\int^x \frac{a_1(\xi)}{a_2(\xi)}\, d\xi\right].$$

Substituting $y(x)$, $y'(x)$, and $y''(x)$ into the original ordinary differential equation yields the final answer.

Section 2.1

1. The characteristic equation is $m^2+6m+5 = (m+1)(m+5) = 0$. Therefore, the general solution is $y(x) = C_1e^{-x} + C_2e^{-5x}$.

2. The characteristic equation is $m^2 - 6m + 10 = (m - 3 + i)(m - 3 - i) = 0$. Therefore, the general solution is $y(x) = C_1e^{3x}\cos(x) + C_2e^{3x}\sin(x)$.

3. The characteristic equation is $m^2 - 2m + 1 = (m - 1)^2 = 0$. Therefore, the general solution is $y(x) = C_1e^x + C_2xe^x$.

4. The characteristic equation is $m^2-3m+2 = (m-1)(m-2) = 0$. Therefore, the general solution is $y(x) = C_1e^{2x} + C_2e^x$.

5. The characteristic equation is $m^2 - 4m + 8 = (m - 2)^2 + 4 = 0$. Therefore, the general solution is $y(x) = C_1e^{2x}\cos(2x) + C_2e^{2x}\sin(2x)$.

6. The characteristic equation is $m^2 + 6m + 9 = (m + 3)^2 = 0$. Therefore, the general solution is $y(x) = C_1e^{-3x} + C_2xe^{-3x}$.

7. The characteristic equation is $m^2 + 6m - 40 = (m + 10)(m - 4) = 0$. Therefore, the general solution is $y(x) = C_1e^{-10x} + C_2e^{4x}$.

8. The characteristic equation is $m^2 + 4m + 5 = (m + 2)^2 + 1 = 0$. Therefore, the general solution is $y(x) = C_1e^{-2x}\cos(x) + C_2e^{-2x}\sin(x)$.

9. The characteristic equation is $m^2+8m+25 = (m+4)^2+9 = 0$. Therefore, the general solution is $y(x) = e^{-4x}[C_1\cos(3x) + C_2\sin(3x)]$.

10. The characteristic equation is $4m^2 - 12m + 9 = (2m-3)^2 = 0$. Therefore, the general solution is $y(x) = e^{3x/2}(C_1 + C_2x)$.

11. The characteristic equation is $m^2 + 8m + 16 = (m + 4)^2 = 0$. Therefore, the general solution is $y(x) = C_1e^{-4x} + C_2xe^{-4x}$.

12. The characteristic equation is $m^3 + 4m^2 = m^2(m + 4) = 0$. Therefore, the general solution is $y(x) = C_1 + C_2x + C_3e^{-4x}$.

13. The characteristic equation is $m^4 + 4m^2 = m^2(m^2 + 4) = 0$. Therefore, the general solution is $y(x) = C_1 + C_2x + C_3\cos(2x) + C_4\sin(2x)$.

14. The characteristic equation is $m^4 + 2m^3 + m^2 = m^2(m + 1)^2 = 0$. Therefore, the general solution is $y(x) = C_1 + C_2x + C_3e^{-x} + C_4xe^{-x}$.