

Problem 1.1

Given points $P_1 (1, -1, 2)$ and $P_2 (1, 0, -3)$

(a) Computation of the unit vector in the direction of $\overrightarrow{P_1P_2}$

The position vectors of points P_1 and P_2 are

$$\overrightarrow{OP_1} = i_1 - i_2 + 2i_3$$

$$\overrightarrow{OP_2} = i_1 - 3i_3$$

According to the parallelogram rule the vector $\overrightarrow{P_1P_2}$ is obtained as follows

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = i_2 - 5i_3$$

The unit vector in the direction of $\overrightarrow{P_1P_2}$ is

$$i_{12} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{i_2 - 5i_3}{\sqrt{26}}$$

(b) Computation of the angle $\angle P_1OP_2$

The dot product of the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ is

$$(\overrightarrow{OP_1})(\overrightarrow{OP_2}) = 1 - 6 = -5$$

We have also that

$$(\overrightarrow{OP_1})(\overrightarrow{OP_2}) = |\overrightarrow{OP_1}| |\overrightarrow{OP_2}| \cos \theta = (\sqrt{6})(\sqrt{10}) \cos \theta$$

and

$$\cos \theta = \frac{-5}{\sqrt{60}} = -0.6454$$

or

$$\theta = 130.2^\circ$$

Computation of the angle $\angle OP_1P_2$

The vectors $\overrightarrow{P_1O}$ and $\overrightarrow{P_1P_2}$ are

$$\overrightarrow{P_1O} = -i_1 + i_2 - 2i_3$$

$$\overrightarrow{P_1P_2} = i_2 - 5i_3$$

Their dot product is

$$(\overrightarrow{P_1O})(\overrightarrow{P_1P_2}) = 1 + 10 = 11$$

We have also that

$$(\overrightarrow{P_1O})(\overrightarrow{P_1P_2}) = |\overrightarrow{P_1O}| |\overrightarrow{P_1P_2}| \cos \phi = (\sqrt{6})(\sqrt{26}) \cos \phi$$

and

$$\cos \phi = \frac{11}{(\sqrt{6})(\sqrt{26})} = 0.8807$$

or

$$\phi = 28.3^\circ$$

2 Chapter 1

(c) Computation of the unit vector \hat{i}_n normal to the plane specified by the points O, P_1 and P_2 .

$$\vec{OP}_1 = \hat{i}_1 - \hat{i}_2 + 2\hat{i}_3$$

$$\vec{OP}_2 = \hat{i}_1 - 3\hat{i}_3$$

The cross product of these vectors is

$$(\vec{OP}_1) \times (\vec{OP}_2) = (3+0)\hat{i}_1 + (2+3)\hat{i}_2 + (0+1)\hat{i}_3 = 3\hat{i}_1 + 5\hat{i}_2 + \hat{i}_3$$

and

$$\hat{i}_n = \pm \frac{3\hat{i}_1 + 5\hat{i}_2 + \hat{i}_3}{\sqrt{35}}$$

(d) Computation of the volume of the parallelepiped whose edges are OP_1 , OP_2 and OP_3 .

$$\vec{OP}_1 = \hat{i}_1 - \hat{i}_2 + 2\hat{i}_3$$

$$\vec{OP}_2 = \hat{i}_1 - 3\hat{i}_3$$

$$\vec{OP}_3 = -\hat{i}_1 + 2\hat{i}_2 + \hat{i}_3$$

As shown above

$$(\vec{OP}_1) \times (\vec{OP}_2) = 3\hat{i}_1 + 5\hat{i}_2 + \hat{i}_3$$

And the volume V of the parallelepiped is

$$V = (\vec{OP}_3) \cdot [(\vec{OP}_1) \times (\vec{OP}_2)] = -3 + 10 + 1 = 8 \text{ m}^3$$

Problem 1.2

Referring to relation (1.31) we have

$$a'_i = \sum_{j=1}^3 \lambda_{ij} a_j = \lambda_{i1} a_1 + \lambda_{i2} a_2$$

Thus,

$$a'_1 = \lambda_{11} a_1 + \lambda_{12} a_2 = a_1 = 4$$

$$a'_2 = \lambda_{21} a_1 + \lambda_{22} a_2 = \frac{a_2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$a'_3 = \lambda_{31} a_1 + \lambda_{32} a_2 = \frac{-a_2}{\sqrt{2}} = \frac{-3}{\sqrt{2}}$$

or

$$a = 4\hat{i}_1 + \frac{3}{\sqrt{2}}\hat{i}_2 - \frac{3}{\sqrt{2}}\hat{i}_3 = 4\hat{i}_1 + \left(\frac{3\sqrt{2}}{2}\right)\hat{i}_2 - \left(\frac{3\sqrt{2}}{2}\right)\hat{i}_3$$

Another way of solving this problem is referring to relation (1.33a)

$$\begin{Bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} 4 \\ 3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 4 \\ \frac{3}{\sqrt{2}} \\ -\frac{3}{\sqrt{2}} \end{Bmatrix}$$

Problem 1.3

(a) Calculation of the angle between the vectors \underline{a} and \underline{b} . We denote this angle by Θ

$$\cos \Theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|} = \frac{12+8-12}{\sqrt{29} \sqrt{56}} = \frac{8}{\sqrt{(29)(56)}} = 0.1985$$

or

$$\Theta = 78.55^\circ$$

(b) The component of the vector \underline{b} in the direction of vector \underline{a} is given by the following dot product

$$\underline{b} \cdot \frac{\underline{a}}{|\underline{a}|} = \frac{8}{\sqrt{29}} = 1.49 \text{ m}$$

(c) We calculate the cross product of the vectors \underline{a} and \underline{b} referring to relation (1.19)

$$\underline{a} \times \underline{b} = (24+4)\underline{i}_1 + (8+18)\underline{i}_2 + (-6+16)\underline{i}_3 = 28\underline{i}_1 + 26\underline{i}_2 + 10\underline{i}_3$$

The area of the parallelogram whose sides are the vectors \underline{a} and \underline{b} is equal to the magnitude of the vector $(\underline{a} \times \underline{b})$, thus

$$|(\underline{a} \times \underline{b})| = \sqrt{28^2 + 26^2 + 10^2} = 39.5 \text{ m}^2$$

(d) The volume of the parallelepiped specified by the vectors \underline{a} , \underline{b} and \underline{c} is equal to the absolute value of the dot product of the vectors \underline{c} and $(\underline{a} \times \underline{b})$, thus

$$|\underline{c} \cdot (\underline{a} \times \underline{b})| = |(\underline{i}_1 - 6\underline{i}_2 + \underline{i}_3) \cdot (28\underline{i}_1 + 26\underline{i}_2 + 10\underline{i}_3)| = |28 - 156 + 10| = |-118| = 118 \text{ m}^3$$

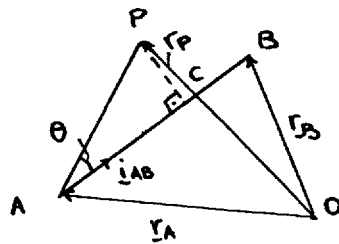
Problem 1.4

Figure a

The position vectors of points A, B and P are

$$\underline{r}_A = -2\underline{i}_1 + \underline{i}_2 + 3\underline{i}_3$$

$$\underline{r}_B = -\underline{i}_1 + 2\underline{i}_2 - \underline{i}_3$$

$$\underline{r}_P = -\underline{i}_1 + 2\underline{i}_2$$

4 Chapter 1

And the vector \vec{AB} is

$$\vec{AB} = \vec{r}_B - \vec{r}_A = \hat{i}_1 + \hat{i}_2 - 4\hat{i}_3$$

The unit vector in the direction of the vector \vec{AB} is

$$\hat{i}_{AB} = \frac{\hat{i}_1 + \hat{i}_2 - 4\hat{i}_3}{\sqrt{18}}$$

Referring to Fig. a the shortest distance from point P to the line passing through points A and B is equal to the magnitude of the vector PC . Thus,

$$|\vec{PC}| = |\vec{AP}| \sin \Theta \quad (a)$$

where

$$\vec{AP} = \vec{r}_P - \vec{r}_A = -\hat{i}_1 + 2\hat{i}_2 + 2\hat{i}_1 - \hat{i}_2 - 3\hat{i}_3 = \hat{i}_1 + \hat{i}_2 - 3\hat{i}_3$$

We have also that

$$|\vec{PC}| = |\hat{i}_{AB} \times \vec{AP}| \quad (b)$$

Thus,

$$\begin{aligned} \hat{i}_{AB} \times \vec{AP} &= \frac{(\hat{i}_1 + \hat{i}_2 - 4\hat{i}_3)}{\sqrt{18}} \times (\hat{i}_1 + \hat{i}_2 - 3\hat{i}_3) = \frac{1}{\sqrt{18}} [(-3+4)\hat{i}_1 + (-4+3)\hat{i}_2 + (1-1)\hat{i}_3] \\ &= \frac{1}{\sqrt{18}} (\hat{i}_1 - \hat{i}_2) \end{aligned}$$

And the magnitude of the vector $(\hat{i}_{AB} \times AP)$ is

$$|\hat{i}_{AB} \times \vec{AP}| = \sqrt{\frac{2}{18}} = \frac{1}{3} \quad (c)$$

Substituting relation (c) into (b) we have

$$|\hat{i}_{AB} \times \vec{AP}| = |\vec{PC}| = \frac{1}{3}$$

Problem 1.5

(a) Referring to relations (1.25a) we have

$$\begin{aligned} \lambda_{11}\lambda_{13} + \lambda_{21}\lambda_{23} + \lambda_{31}\lambda_{33} &= 0 \\ \lambda_{12}\lambda_{13} + \lambda_{22}\lambda_{23} + \lambda_{32}\lambda_{33} &= 0 \\ \lambda_{13}^2 + \lambda_{23}^2 + \lambda_{33}^2 &= 1 \end{aligned} \quad (a)$$

From the data of the problem we have

$$\begin{aligned} \lambda_{11} &= 0 & \lambda_{12} &= -\frac{4}{5} & \lambda_{13} &= \frac{3}{5} \\ \lambda_{21} &= 1 & \lambda_{22} &= 0 & \lambda_{23} &= 0 \end{aligned} \quad (b)$$

Substituting (b) into (a) we obtain

$$\begin{aligned}\lambda_{31}\lambda_{33} &= 0 \\ \left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) + \lambda_{32}\lambda_{33} &= 0 \\ \left(\frac{3}{5}\right)^2 + \lambda_{33}^2 &= 1\end{aligned}$$

or

$$\lambda_{33} = \pm \frac{4}{5}, \quad \lambda_{31} = 0, \quad \lambda_{32} = \pm \frac{3}{5}$$

(b) We denote the given tensor by $[A]$. Referring to relation (1.37a) we have

$$\begin{aligned}[A'] &= [A][\Lambda][\Lambda]^T = \\ &= \begin{bmatrix} 0 & -\frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{117}{25} & 0 & \frac{56}{25} \\ 0 & 2 & 0 \\ -\frac{69}{25} & 0 & -\frac{17}{25} \end{bmatrix} = \begin{bmatrix} 4.68 & 0 & 2.24 \\ 0 & 2 & 0 \\ -2.76 & 0 & -0.68 \end{bmatrix}\end{aligned}$$

Problem 1.6(a) The coordinates of point P with respect to the system of axes x_1', x_2', x_3' are

$$\begin{Bmatrix} x_1' \\ x_2' \\ x_3' \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{3}{5\sqrt{2}} & -\frac{4}{5\sqrt{2}} \\ 0 & \frac{4}{5} & -\frac{3}{5} \\ \frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} & \frac{4}{5\sqrt{2}} \end{bmatrix} \begin{Bmatrix} -2 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\frac{13}{5\sqrt{2}} \\ \frac{4}{5} \\ -\frac{7}{5\sqrt{2}} \end{Bmatrix} = \begin{Bmatrix} -1.84 \\ 0.8 \\ -0.99 \end{Bmatrix}$$

(b) Expanding relation (1.70) we obtain

$$\begin{aligned}A'_{ij} &= \sum_{k=1}^3 \sum_{m=1}^3 \lambda_{ik} \lambda_{jm} A_{km} = \sum_{k=1}^3 (\lambda_{ik} \lambda_{j1} A_{k1} + \lambda_{ik} \lambda_{j2} A_{k2} + \lambda_{ik} \lambda_{j3} A_{k3}) \\ &= \lambda_{i1} \lambda_{j2} A_{12} + \lambda_{i2} \lambda_{j2} A_{22} + \lambda_{i3} \lambda_{j2} A_{32}\end{aligned}$$

Thus substituting the given values in this equation we have

$$\begin{aligned}A'_{11} &= (\lambda_{11} \lambda_{12}) A_{12} + \lambda_{12}^2 A_{22} + (\lambda_{13} \lambda_{12}) A_{32} \\ &= -\frac{3}{5} - \frac{9}{50} - \frac{24}{50} = -\frac{63}{50} = -1.26\end{aligned}$$

$$A'_{12} = (\lambda_{11} \lambda_{22}) A_{12} + (\lambda_{12} \lambda_{22}) A_{22} + (\lambda_{13} \lambda_{22}) A_{32}$$

$$= \frac{8}{5\sqrt{2}} + \frac{12}{25\sqrt{2}} + \frac{32}{25\sqrt{2}} = \frac{84}{25\sqrt{2}} = 2.376$$

$$A'_{13} = (\lambda_{11} \lambda_{32}) A_{12} + (\lambda_{12} \lambda_{32}) A_{22} + (\lambda_{13} \lambda_{32}) A_{32}$$

$$= \frac{3}{5} + \frac{9}{50} + \frac{24}{50} = \frac{63}{50} = 1.260$$

$$A'_{21} = (\lambda_{22} \lambda_{12}) A_{22} + (\lambda_{23} \lambda_{12}) A_{32}$$

$$= \frac{12}{25\sqrt{2}} - \frac{18}{25\sqrt{2}} = \frac{-6}{25\sqrt{2}} = -0.170$$

$$A'_{22} = \lambda_{22}^2 A_{22} + (\lambda_{23} \lambda_{22}) A_{32}$$

$$= \frac{-16}{25} + \frac{24}{25} = \frac{8}{25} = 0.320$$

$$A'_{23} = (\lambda_{22} \lambda_{32}) A_{22} + (\lambda_{23} \lambda_{32}) A_{32}$$

$$= \frac{-12}{25\sqrt{2}} + \frac{18}{25\sqrt{2}} = \frac{6}{25\sqrt{2}} = 0.170$$

$$A'_{31} = (\lambda_{31} \lambda_{12}) A_{12} + (\lambda_{32} \lambda_{12}) A_{22} + (\lambda_{33} \lambda_{12}) A_{32}$$

$$= \frac{-3}{5} + \frac{9}{50} + \frac{24}{50} = \frac{3}{50} = 0.06$$

$$A'_{32} = (\lambda_{31} \lambda_{22}) A_{12} + (\lambda_{32} \lambda_{22}) A_{22} + (\lambda_{33} \lambda_{22}) A_{32}$$

$$= \frac{8}{5\sqrt{2}} - \frac{12}{25\sqrt{2}} - \frac{32}{25\sqrt{2}} = \frac{-4}{25\sqrt{2}} = -0.113$$

$$A'_{33} = (\lambda_{31} \lambda_{32}) A_{12} + (\lambda_{32} \lambda_{32}) A_{22} + (\lambda_{33} \lambda_{32}) A_{32}$$

$$= \frac{3}{5} - \frac{9}{50} - \frac{24}{50} = \frac{-3}{50} = -0.06$$

The same results may be obtained using matrix multiplication. That is

$$[A'] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{5\sqrt{2}} & \frac{-4}{5\sqrt{2}} \\ 0 & \frac{4}{5} & \frac{-3}{5} \\ \frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} & \frac{4}{5\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-3}{5\sqrt{2}} & \frac{4}{5} & \frac{3}{5\sqrt{2}} \\ \frac{-4}{5\sqrt{2}} & \frac{-3}{5} & \frac{4}{5\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-63}{50} & \frac{84}{25\sqrt{2}} & \frac{63}{50} \\ \frac{-6}{25\sqrt{2}} & \frac{8}{25} & \frac{6}{25\sqrt{2}} \\ \frac{3}{50} & \frac{-4}{25\sqrt{2}} & \frac{-3}{50} \end{bmatrix} = \begin{bmatrix} -1.26 & 2.376 & 1.26 \\ -0.17 & 0.32 & 0.17 \\ 0.06 & -0.113 & -0.06 \end{bmatrix}$$

(c) The equation of the plane $x_1 - x_2 + 2x_3 = 1$ can be rewritten as

$$\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 1 \quad (\text{a})$$

Referring to the equation (1.34a) we have

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [\Lambda_s]^T \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{3}{5\sqrt{2}} & \frac{4}{5} & \frac{3}{5\sqrt{2}} \\ -\frac{4}{5\sqrt{2}} & -\frac{3}{5} & \frac{4}{5\sqrt{2}} \end{bmatrix} \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} \quad (b)$$

Substituting equation (b) into (a) we have

$$\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{3}{5\sqrt{2}} & \frac{4}{5} & \frac{3}{5\sqrt{2}} \\ -\frac{4}{5\sqrt{2}} & -\frac{3}{5} & \frac{4}{5\sqrt{2}} \end{bmatrix} \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = 1$$

or

$$-2x'_2 + \sqrt{2}x'_3 = 1$$

Problem 1.7

(a) Referring to relation (1.33a) the coordinates of point P in the system of axes x'_i are

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = [\Lambda_s] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3\sqrt{2}+8}{\sqrt{6}} \\ \frac{3\sqrt{2}+2\sqrt{3}-4}{\sqrt{6}} \\ \frac{3\sqrt{2}-2\sqrt{3}-4}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 4.998 \\ 1.513 \\ -1.315 \end{bmatrix}$$

(b) Referring to equation (1.73b) we have

$$[A'] = [\Lambda_s] [A] [\Lambda_s]^T$$

Thus,

$$[A'] = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} -0.471 & -1.288 & 0.345 \\ 1.390 & -0.658 & -0.025 \\ -0.919 & -0.503 & 2.130 \end{bmatrix}$$

(c) The equation of the plane in the system of axes x_1, x_2, x_3 is

$$2x_1 + 3x_2 - x_3 = 1$$

or

$$\begin{bmatrix} 2 & 3 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 1 \quad (\text{a})$$

From relation (1.34a) we have

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [\Lambda]^T \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} \quad (\text{b})$$

Substituting equation (b) into (a) we have

$$\begin{bmatrix} 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = 1$$

Or

$$1.9712 x'_1 + 2.8678 x'_2 - 1.3749 x'_3 = 1$$

Problem 1.8A

Given the symmetric tensor of the second rank

$$[A] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Referring to relation (1.124) we have the following stationary values for the tensor $[A]$

$$\tilde{A}_1 = \frac{1}{2} (A_{11} + A_{22}) + \sqrt{\left(\frac{A_{11} - A_{22}}{2}\right)^2 + A_{12}^2} = 3 + \sqrt{2}$$

$$\tilde{A}_2 = \frac{1}{2} (A_{11} + A_{22}) - \sqrt{\left(\frac{A_{11} - A_{22}}{2}\right)^2 + A_{12}^2} = 3 - \sqrt{2}$$

$$\tilde{A}_3 = 3$$

From relation (1.122) we have

$$\tan 2\tilde{\phi}_{11} = \frac{2A_{12}}{A_{11} - A_{22}} = -1 \quad \text{and} \quad 2\tilde{\phi}_{11} = 135^\circ$$

Or

$$\tilde{\phi}_{11} = 67.5^\circ \quad \text{counter-clockwise}$$

In Figure a. we show the principal axes \tilde{x}_1, \tilde{x}_2

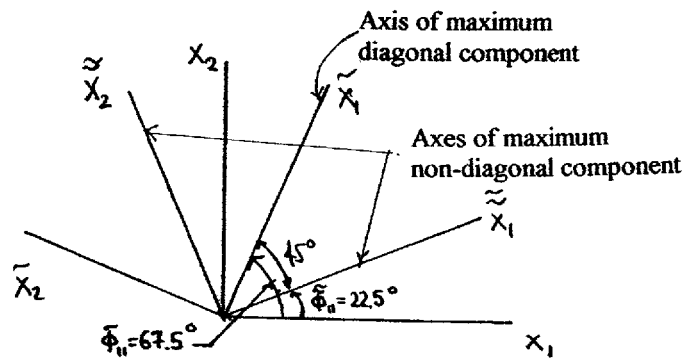


Figure a

(b) Referring to relation (1.129) the maximum value of the non-diagonal components of the tensor $[A]$ is

$$(A'_{12})_{\max} = \left[\left(\frac{A_{11} - A_{22}}{2} \right)^2 + A_{12}^2 \right]^{1/2} = \left[\left(\frac{2-4}{2} \right)^2 + 1^2 \right]^{1/2} = \sqrt{2}$$

From relation (1.127) we have

$$\tan 2\tilde{\phi}_{11} = -\frac{A_{11} - A_{22}}{2A_{12}} = -\frac{2-4}{2} = 1 \quad \text{and} \quad 2\tilde{\phi}_{11} = 45^\circ$$

Or

$$\tilde{\phi}_{11} = 22.5^\circ \quad \text{counter-clockwise}$$

In Fig. a we also show the axes $\tilde{\tilde{x}}_1, \tilde{\tilde{x}}_2$ with respect of which this maximum value of the non-diagonal components occurs.

(c) Referring to relation (1.61) the diagonal component of the tensor $[A]$ in the direction of the unit vector \mathbf{i}_n is

$$A_{nn} = \{n\}^T [A] \{n\} = \frac{1}{\sqrt{3}} [1 \ 1 \ -1] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{3}} = \frac{11}{3}$$

Problem 1.8B

Given the symmetric tensor of the second rank

$$[B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(a) Referring to relation (1.124) we have the following stationary values for the tensor $[B]$

$$\begin{aligned}\tilde{B}_1 &= 0 \\ \tilde{B}_2 &= \frac{1}{2}(B_{22} + B_{33}) + \sqrt{\left(\frac{B_{22} - B_{33}}{2}\right)^2 + B_{23}^2} = \frac{5 + \sqrt{5}}{2} \\ \tilde{B}_3 &= \frac{1}{2}(B_{22} + B_{33}) - \sqrt{\left(\frac{B_{22} - B_{33}}{2}\right)^2 + B_{23}^2} = \frac{5 - \sqrt{5}}{2}\end{aligned}$$

From relation (1.122) we have

$$\tan 2\tilde{\phi}_{22} = \frac{2B_{23}}{B_{22} - B_{33}} = \frac{-2}{3-2} = -2 \quad \text{and} \quad 2\tilde{\phi}_{22} = -63.43^\circ$$

Or

$$\tilde{\phi}_{22} = 31.72^\circ \quad \text{clockwise}$$

In Figure a we show the principal axes \tilde{x}_2, \tilde{x}_3

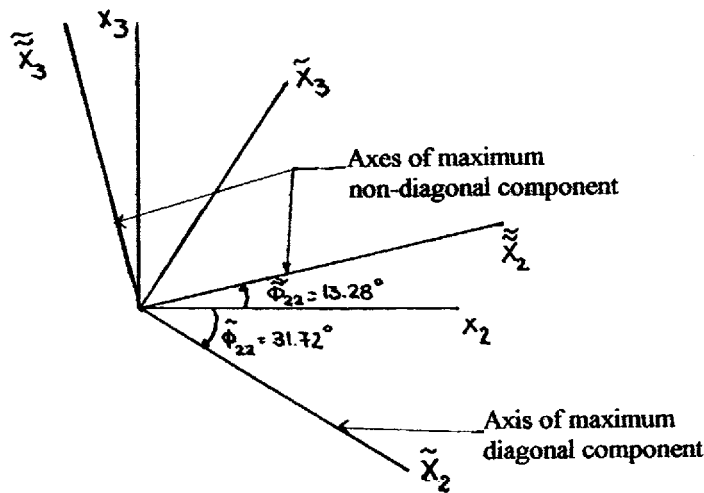


Figure a

(b) Referring to relation (1.129) the maximum value of the non-diagonal components of the tensor $[B]$ is

$$(B'_{23})_{\max} = \left[\left(\frac{B_{22} - B_{33}}{2} \right)^2 + B_{23}^2 \right]^{1/2} = \left[\left(\frac{3-2}{2} \right)^2 + (-1)^2 \right]^{1/2} = \frac{\sqrt{5}}{2}$$

From relation (1.127) we have

$$\tan 2\tilde{\phi}'_{22} = -\frac{B_{22} - B_{33}}{2B_{23}} = -\frac{3-2}{(-2)} = \frac{1}{2} \quad \text{and} \quad 2\tilde{\phi}'_{22} = 26.56^\circ$$

Or

$$\tilde{\phi}'_{22} = 13.28^\circ \quad \text{counter-clockwise}$$

In Fig. a we also show the axes \tilde{x}_2 and \tilde{x}_3 with respect of which this maximum value of the non-diagonal components occurs.

(c) Referring to relation (1.61) the diagonal component of the tensor [B] in the direction of the unit vector \mathbf{i}_n is

$$B_{nn} = \{\eta\}^T [B] \{\eta\} = \frac{1}{\sqrt{3}} [1 \ 1 \ -1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{3}} = \frac{7}{3}$$

Problem 1.8C

Given the symmetric tensor of the second rank

$$[C] = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 4 \end{bmatrix}$$

(a) Referring to relation (1.124) we have the following stationary values for the tensor C

$$\tilde{C}_3 = \frac{1}{2} (C_{33} + C_{11}) + \sqrt{\left(\frac{C_{33} - C_{11}}{2}\right)^2 + C_{13}^2} = 3 + \sqrt{10}$$

$$\tilde{C}_1 = \frac{1}{2} (C_{33} + C_{11}) - \sqrt{\left(\frac{C_{33} - C_{11}}{2}\right)^2 + C_{13}^2} = 3 - \sqrt{10}$$

$$\tilde{C}_2 = 0$$

From relation (1.122) we have

$$\tan 2\tilde{\phi}_{33} = \frac{2C_{13}}{C_{33} - C_{11}} = \frac{-6}{4 - 2} = -3 \quad \text{and} \quad 2\tilde{\phi}_{33} = -71.56^\circ$$

Or

$$\tilde{\phi}_{33} = 35.78^\circ \quad \text{clockwise}$$

In Figure a, we show the principal axes \tilde{x}_3, \tilde{x}_1

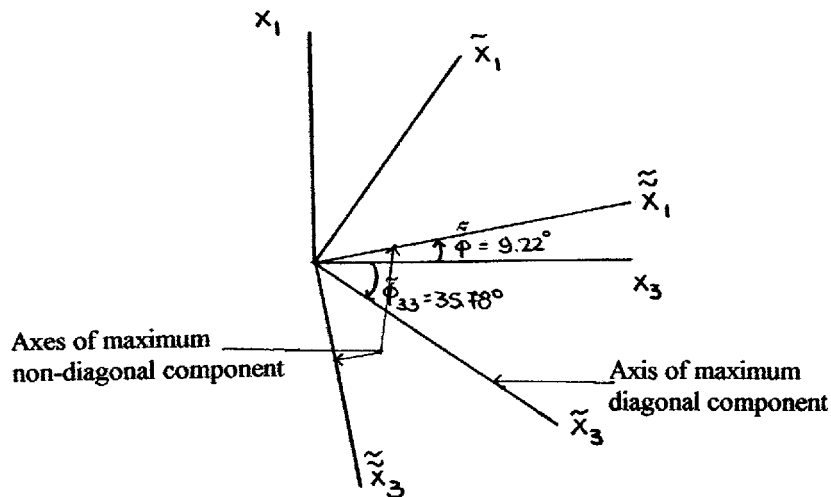


Figure a

(b) Referring to relation (1.129) the maximum value of the non-diagonal components of the tensor [C] is

$$(C'_{13})_{\max} = \left[\left(\frac{C_{33} - C_{11}}{2} \right)^2 + C_{13}^2 \right]^{1/2} = [1^2 + (-3)^2]^{1/2} = \sqrt{10}$$

From relation (1.127) we have

$$\tan 2\tilde{\Phi}_{33} = -\frac{C_{33} - C_{11}}{2C_{13}} = \frac{1}{3} \quad \text{and} \quad 2\tilde{\Phi}_{33} = 18.43^\circ$$

Or

$$\tilde{\Phi}_{33} = 9.22^\circ \quad \text{counter-clockwise}$$

In Fig. a we show also the axes \tilde{x}_3, \tilde{x}_1 with respect of which this maximum value of the non-diagonal components occurs.

(c) Referring to relation (1.61) the diagonal component of the tensor [C] in the direction of the unit vector \mathbf{i}_n is

$$C_{nn} = \{n\}^T [C] \{n\} = \frac{1}{\sqrt{3}} [1 \quad 1 \quad -1] \begin{bmatrix} 2 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{3}} = \frac{12}{3} = 4$$

Problem 1.9 A

Given the symmetric tensor of the second rank

$$[A] = \begin{bmatrix} -3 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) We plot Mohr's circle using the two points on the same diameter $X_1(-3, -2)$ and $X_2(2, 2)$. Referring to Figure a, we see that the center, C, of the circle is located on the A'_{11} axis at a distance $1/2(A_{11} + A_{22}) = -0.5$ from the origin O.

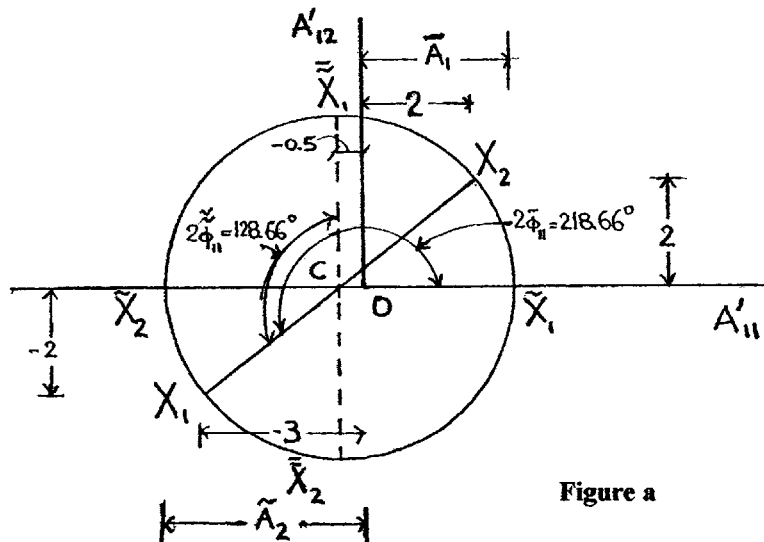


Figure a

From geometric considerations the radius, R , of the circle is

$$R = [(2.5)^2 + 2^2]^{1/2} = 3.2$$

Moreover as shown in Fig. a the maximum value of the diagonal components of the tensor $[A]$ is the abscissa of point \tilde{X}_1 , while the minimum value is the abscissa of point \tilde{X}_2 . Thus

$$(\tilde{A}_{11})_{\max} = \tilde{A}_1 = \overline{C\tilde{X}_1} - \overline{OC} = 3.2 - 0.5 = 2.7$$

$$(\tilde{A}_{11})_{\min} = \tilde{A}_2 = -(\overline{OC} + \overline{C\tilde{X}_2}) = -3.2 - 0.5 = -3.7$$

Referring to Fig. a. we have

$$\tan 2\tilde{\varphi}_{11} = \tan \angle X_1 C \tilde{X}_1 = 2/2.5$$

Hence

$$2\tilde{\varphi}_{11} = 218.66^\circ \quad \text{or} \quad \tilde{\varphi}_{11} = 109.33^\circ \text{ counter-clockwise.}$$

One of the principal directions is the i_3 , whereas the other two lie in the $x_1 x_2$ plane. So we see in Fig. a that point \tilde{X}_1 is located 218.66° clockwise from point X_1 . Consequently, the principal axis \tilde{x}_1 associated with the principal value \tilde{A}_1 of the diagonal component of the tensor is located $\tilde{\varphi}_{11} = 109.33^\circ$ counter-clockwise from the x_1 axis. In Fig. b. we show the principal axes of the tensor \tilde{x}_1, \tilde{x}_2 .

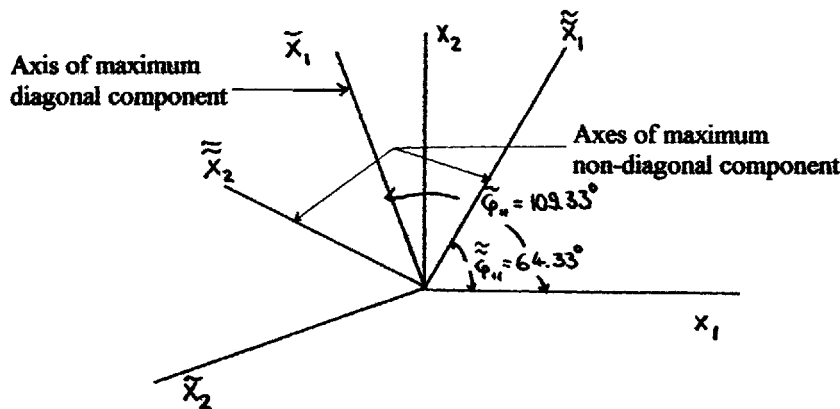


Figure b

(b) Referring to Fig. a the maximum non-diagonal component of the tensor is the ordinate of point \tilde{X}_1 . Thus,

$$(\tilde{A}_{12})_{\max} = 3.2$$

From Fig. a we can see also that the point \tilde{X}_1 is located 128.66° clockwise from point X_1 . Consequently, the axes \tilde{x}_1 and \tilde{x}_2 , where the maximum value of the non-diagonal components of tensor $[A]$ occur, are located as shown in Fig. b.

(c) From relation (1.61) the diagonal component of the tensor [A] in the direction of the unit vector \mathbf{i}_n is

$$A_{nn} = \{\eta\}^T [A] \{\eta\} = \frac{1}{\sqrt{3}} [1 \ 1 \ -1] \begin{bmatrix} -3 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{3}} = -\frac{4}{3}$$

Problem 1.9B

Given the symmetric tensor of the second rank

$$[B] = \begin{bmatrix} 6 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & -2 \end{bmatrix}$$

(a) We plot Mohr's circle using the two points on the same diameter $X_3 (-2, -3)$ and $X_1 (6, 3)$. Referring to Figure a. we see that the center, C, of the circle is located on the B'_{33} axis at a distance $1/2 (B_{11} + B_{22}) = 2$ from the origin O.

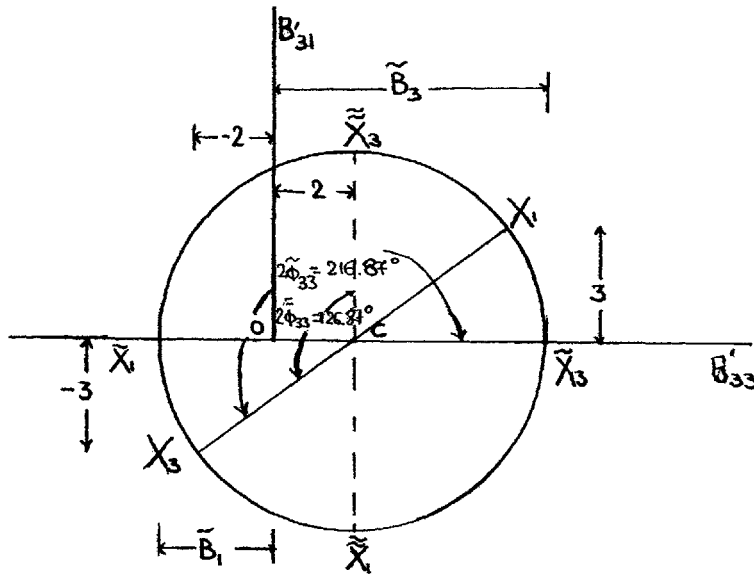


Figure a

From geometric considerations the radius, R, of the circle is

$$R = (3^2 + 4^2)^{1/2} = 5$$

Moreover as shown in Fig. a the maximum value of the diagonal components of the tensor [B] is the abscissa of point \tilde{X}_3 , while the minimum value is the abscissa of point \tilde{X}_1 . Thus,

$$(B'_{33})_{max} = \tilde{B}_3 = \overline{CX}_3 + \overline{OC} = 5 + 2 = 7.0$$

$$(B'_{11})_{min} = \tilde{B}_1 = -\overline{CX}_1 + \overline{OC} = -5 + 2 = -3.0$$

Referring to Fig. a. we have

$$\tan 2\tilde{\varphi}_{33} = \tan \angle X_3\tilde{C}X_3 = 3/4$$

Hence

$$2\tilde{\varphi}_{33} = 216.87^\circ \quad \text{or} \quad \tilde{\varphi}_{33} = 108.43^\circ \text{ counter-clockwise.}$$

One of the principal directions is the i_2 whereas the other two lie in the x_3x_1 plane. So we see in Fig. a that point \tilde{X}_3 is located 216.87° clockwise from point X_3 . Consequently, the principal axis \tilde{x}_3 associated with the principal value \tilde{B}_3 of the diagonal component of the tensor is located $\tilde{\varphi}_{33} = 108.43^\circ$ counter-clockwise from the x_3 axis. In Fig. b. we show the principal axes of the tensor \tilde{x}_3, \tilde{x}_1 .

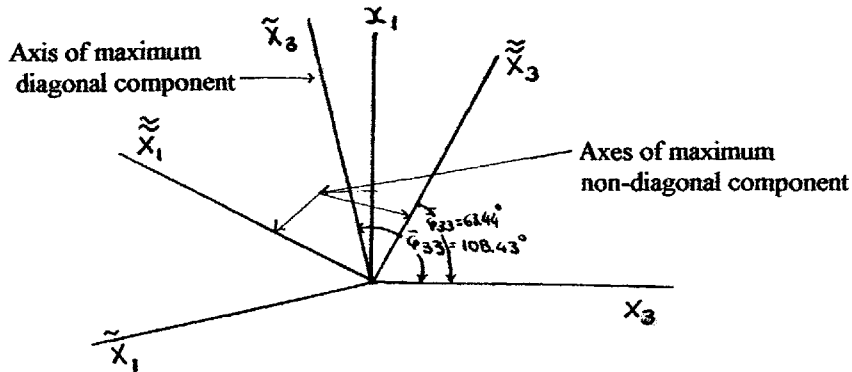


Figure b

(b) Referring to Fig. a the maximum non-diagonal component of the tensor is the ordinate of point X_3 . Thus,

$$(B'_{31})_{max} = 5$$

From Fig. a we can also see that point \tilde{X}_3 is located 126.87° clockwise from point X_3 . Consequently, the axes \tilde{x}_3 and \tilde{x}_1 , where the maximum value of the non-diagonal components of tensor $[B]$ occur, are located as shown in Fig. b.

(c) From relation (1.61) the diagonal component of the tensor $[B]$ in the direction of the unit vector i_n is

$$B_{nn} = \{n\}^T [B] \{n\} = \frac{1}{\sqrt{3}} [1 \ 1 \ -1] \begin{bmatrix} 6 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{3}} = \frac{10}{3}$$

Problem 1.9 C

Given the symmetric tensor of the second rank

$$[C] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & 6 \end{bmatrix}$$

(a) We plot Mohr's circle using the two points on the same diameter $X_2(-2, 4)$ and $X_3(6, -4)$. Referring to Figure a, we see that the center, C, of the circle is located on the C'_{22} axis at a distance $1/2 (C_{22} + C_{33}) = 2$ from the origin O.

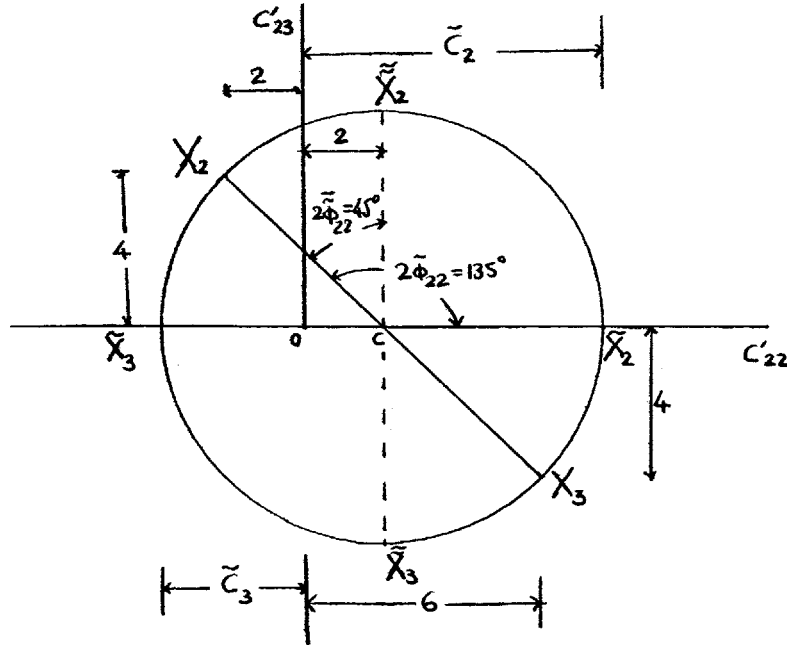


Figure a

From geometric considerations the radius, R, of the circle is

$$R = (4^2 + 4^2)^{1/2} = 5.657$$

Moreover as shown in Fig. a the maximum value of the diagonal components of the tensor [C] is the abscissa of point \tilde{X}_2 , while the minimum value is the abscissa of point \tilde{X}_3 . Thus

$$(C'_{22})_{\max} = \tilde{C}_2 = \overline{OC} + \overline{CX_2} = 5.657 + 2 = 7.657$$

$$(C'_{22})_{\min} = \tilde{C}_3 = -\overline{CX_3} + \overline{OC} = -5.657 + 2 = -3.657$$

Referring to Fig. a, we have

$$2\tilde{\phi}_{22} = \text{angle } X_2CX_2$$

Hence

$$2\tilde{\varphi}_{22} = 135^\circ \quad \text{or} \quad \tilde{\varphi}_{22} = 67.5^\circ \text{ counter-clockwise.}$$

One of the principal directions is the \mathbf{i}_1 whereas the other two lie in the x_2x_3 plane. So we see in Fig. a that point \tilde{X}_2 is located 135° clockwise from point X_2 . Consequently, the principal axis \tilde{x}_2 associated with the principal value \tilde{C}_2 of the diagonal component of the tensor is located $\tilde{\varphi}_{22} = 67.5^\circ$ counter-clockwise from the x_2 axis. In Fig. b. we show the principal axes of the tensor \tilde{x}_2, \tilde{x}_3 .

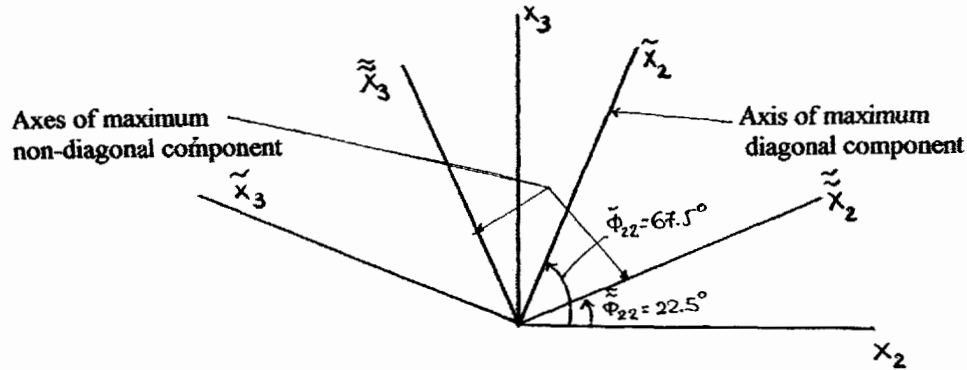


Figure b

(b) Referring to Fig. a the maximum non-diagonal component of the tensor is the ordinate of point X_2 . Thus,

$$(C_{23})_{\max} = 5.657$$

From Fig. a we can also see that point \tilde{X}_2 is located 45° clockwise from point X_2 . Consequently, the axes \tilde{x}_2 and \tilde{x}_3 , where the maximum value of the non-diagonal components of tensor $[C]$ occur, are located as shown in Fig. b.

(c) From relation (1.61) the diagonal component of the tensor $[C]$ in the direction of the unit vector \mathbf{i}_n is

$$C_{nn} = \{\eta\}^T [C] \{\eta\} = \frac{1}{\sqrt{3}} [1 \ 1 \ -1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{3}} = -\frac{4}{3}$$

Problem 1.9D

Given the symmetric tensor of the second rank

$$[D] = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) We plot Mohr's circle using the two points on the same diameter $X_1(-4, 2)$ and $X_2(-8, -2)$. Referring to Figure a. we see that the center, C, of the circle is located on the D'_{11} axis at a distance $1/2 (D'_{11} + D'_{22}) = -6$ from the origin O.

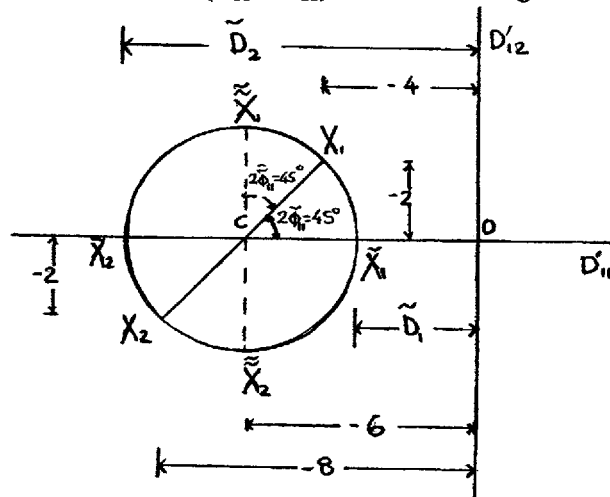


Figure a

From geometric considerations the radius, R, of the circle is

$$R = [2^2 + (-2)^2]^{1/2} = 2.828$$

Moreover as shown in Fig. a the maximum value of the diagonal components of the tensor [D] is the abscissa of point X_1 , while the minimum value is the abscissa of point X_2 . Thus

$$(D'_{11})_{\max} = \tilde{D}_1 = -\bar{CO} + \bar{CX}_1 = -6 + 2.828 = -3.172$$

$$(D'_{11})_{\min} = \tilde{D}_2 = -\bar{CO} - \bar{CX}_2 = -6 - 2.828 = -8.828$$

Referring to Fig. a. we have

$$\tan 2\tilde{\varphi}_{11} = \tan \angle X_1 C \tilde{X}_1 = 1$$

Hence

$$2\tilde{\varphi}_{11} = 45^\circ \quad \text{or} \quad \tilde{\varphi}_{11} = 22.5^\circ \text{ counter clockwise.}$$

One of the principal directions is the \tilde{i}_3 whereas the other two lie in the x_1x_2 plane. So we see in Fig. a that point \tilde{X}_1 is located 45° clockwise from point X_1 . Consequently, the principal axis \tilde{x}_1 associated with the principal value \tilde{D}_1 of the diagonal component of the tensor is located $\tilde{\varphi}_{11} = 22.5^\circ$ counter-clockwise from the x_1 axis. In Fig. b we show the principal axes of the tensor \tilde{x}_1, \tilde{x}_2 .

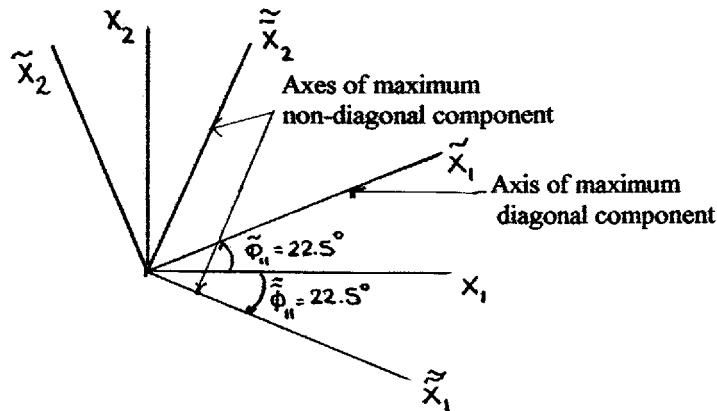


Figure b

(b) Referring to Fig. a the maximum non-diagonal component of the tensor is the ordinate of point X_1 . Thus,

$$(D'_{12})_{\max} = 2.828$$

From Fig. a we can see also that the point $\tilde{\tilde{X}}_1$ is located 45° clockwise from point X_1 . Consequently, the axes $\tilde{\tilde{x}}_1$ and $\tilde{\tilde{x}}_2$, where the maximum value of the non-diagonal components of tensor $[D]$ occur, are located as shown in Fig. b.

(c) From relation (1.61) the diagonal component of the tensor $[D]$ in the direction of the unit vector i_n is

$$D_{\eta\eta} = \{ \eta \}^T [D] \{ \eta \} = \frac{1}{\sqrt{3}} [1 \ 1 \ -1] \begin{bmatrix} -4 & 2 & 0 \\ 2 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{3}} = -\frac{8}{3}$$

Problem 1.9E

Given the symmetric tensor of the second rank

$$E = \begin{bmatrix} 0 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) We plot Mohr's circle using the two points on the same diameter $X_1(0, 3)$ and $X_2(-9, -3)$. Referring to Figure a, we see that the center, C , of the circle is located on the E'_{11} axis at a distance $1/2 (E_{11} + E_{22}) = -4.5$ from the origin O . From geometric considerations the radius, R , of the circle is

$$R = [3^2 + (-4.5)^2]^{1/2} = 5.41$$

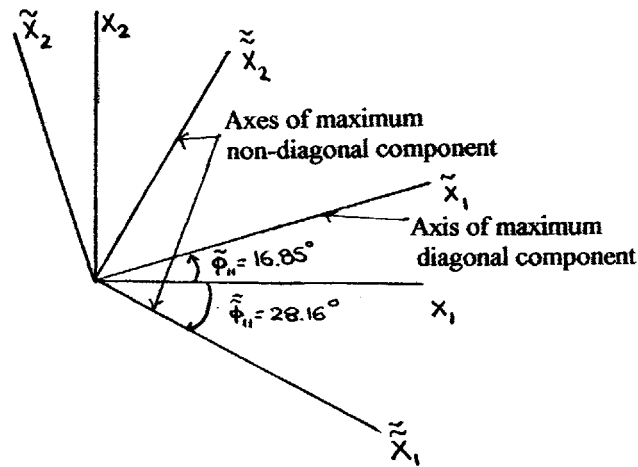


Figure b

(c) From relation (1.61) the diagonal component of the tensor $[E]$ in the direction of the unit vector \tilde{i}_n is

$$E_{nn} = \{\tilde{n}\}^T [E] \{\tilde{n}\} = \frac{1}{\sqrt{3}} [1 \ 1 \ -1] \begin{bmatrix} 0 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -\frac{1}{3}$$

Problem 1.9F

Given the symmetric tensor of the second rank

$$[F] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & -3 \\ 0 & -3 & -12 \end{bmatrix}$$

(a) We plot Mohr's circle using the two points on the same diameter X_2 $(-4, -3)$ and X_3 $(-12, 3)$. Referring to Figure a we see that the center, C , of the circle is located on the F'_{22} axis at a distance $1/2 (F_{22} + F_{33}) = -8$ from the origin O .

From geometric considerations the radius, R , of the circle is

$$R = [(-3)^2 + (-4)^2]^{1/2} = 5$$

Moreover as shown in Fig. a the maximum value of the diagonal components of the tensor $[F]$ is the abscissa of point \tilde{X}_2 , while the minimum value is the abscissa of point \tilde{X}_3 . Thus

$$(F'_{22})_{\max} = \tilde{F}_2 = -\overline{CO} + \overline{C\tilde{X}_2} = -8 + 5 = -3$$

$$(F'_{22})_{\min} = \tilde{F}_3 = -\overline{C\tilde{X}_3} - \overline{CO} = -5 - 8 = -13$$

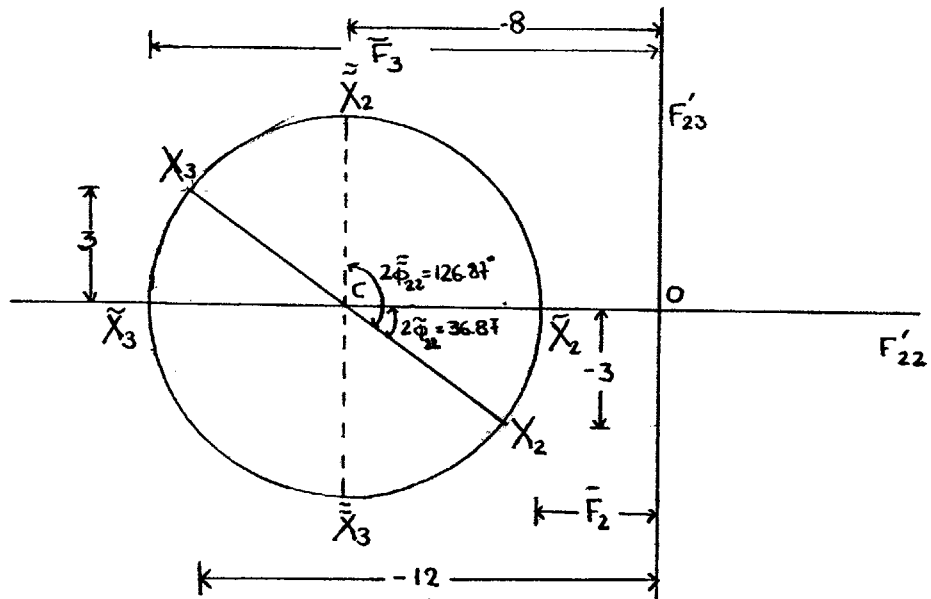


Figure a

Referring to Fig. a. we have

$$\tan 2\tilde{\varphi}_{22} = \tan \angle X_2 C \tilde{X}_2 = 3/4$$

Hence

$$2\tilde{\varphi}_{22} = 36.87^\circ \quad \text{or} \quad \tilde{\varphi}_{22} = 18.43^\circ \text{ clockwise.}$$

One of the principal directions is the i_1 whereas the other two lie in the x_2x_3 plane. So we see in Fig. a that point \tilde{X}_2 is located 36.87° counter-clockwise from point X_2 . Consequently, the principal axis \tilde{x}_2 associated with the principal value \tilde{F}_2 of the diagonal component of the tensor is located $\tilde{\varphi}_{22} = 18.43^\circ$ clockwise from the x_2 axis. In Fig. b. we show the principal axes of the tensor \tilde{x}_2, \tilde{x}_3 .

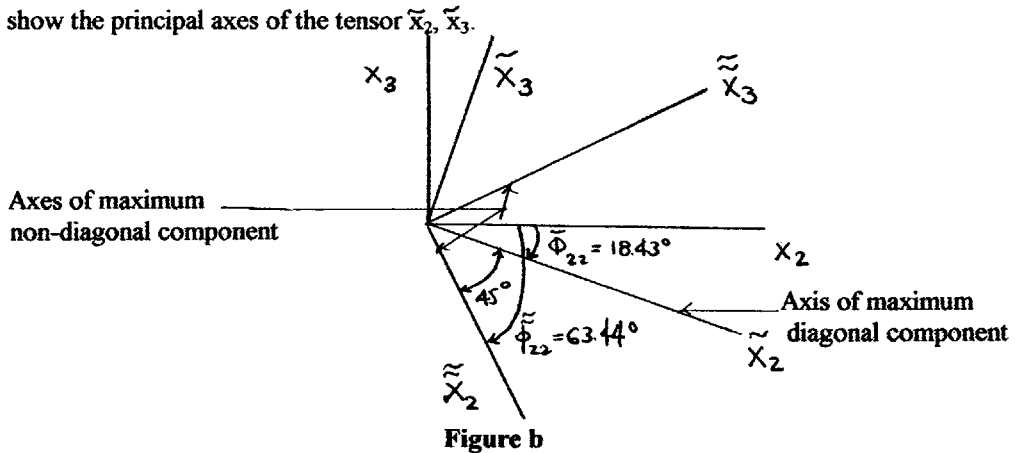


Figure b

(b) Referring to Fig. a the maximum non-diagonal component of the tensor is the ordinate of point \tilde{X}_2 . Thus,

$$(F'_{23})_{\max} = 5.0$$

From Fig. a we can also see that point \bar{X}_2 is located 126.87° counter-clockwise from point X_2 . Consequently, the axes \tilde{x}_2 and \tilde{x}_3 , where the maximum value of the non-diagonal components of tensor [C] occur, are located as shown in Fig. b.

(c) From relation (1.61) the diagonal component of the tensor [F] in the direction of the unit vector i_n is

$$F_{nn} = \{n\}^T [F] \{n\} = \frac{1}{\sqrt{3}} [1 \ 1 \ -1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & -3 \\ 0 & -3 & -12 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{3}} = -\frac{10}{3}$$

Problem 1.10A

Referring to the solution of problem 1.9A we plot Mohr's circle in Fig. a. Moreover we plot line $X'_1CX'_2$ making an angle of 60° clockwise with line X_1CX_2 . Referring to Fig. a we have that

angle $ACX'_1 = 21.34^\circ$ and

$$A'_{11} = -\bar{OC} - \bar{CA} = -0.5 - R \cos(21.34^\circ) = -3.481$$

$$A'_{12} = \bar{AX}'_1 = R \sin(21.34^\circ) = 3.2 \sin(21.34^\circ) = 1.164$$

$$A'_{22} = \bar{OB} = \bar{CB} - \bar{CO} = R \cos(21.34^\circ) - 0.5 = 2.481$$

Or

$$[A'] = \begin{bmatrix} -3.481 & 1.164 & 0 \\ 1.164 & 2.481 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

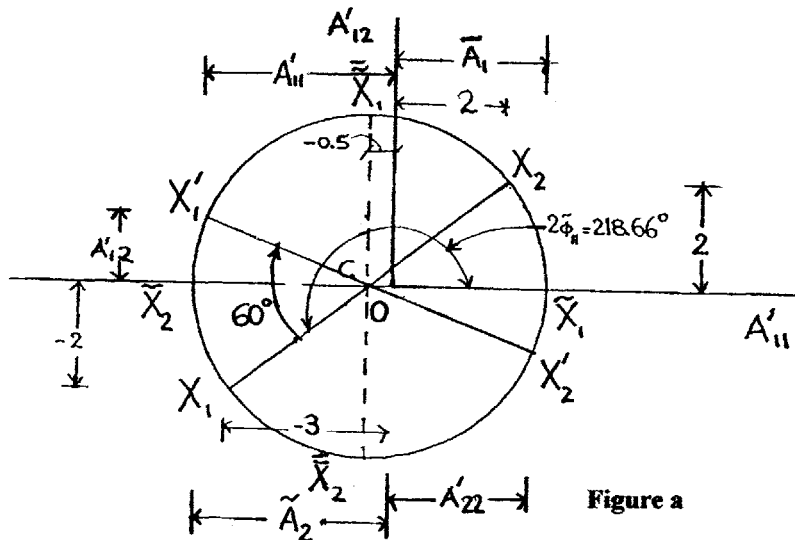


Figure a

Problem 1.10B

Referring to the solution of problem 1.9B we plot Mohr's circle in Fig. a. Moreover we plot line X'_3CX_1 making an angle of 60° clockwise with line X_3CX_1 . Referring to Fig. a we have that

$$\text{angle } X'_3CA = 23.13^\circ \text{ and}$$

$$B'_{33} = -\bar{CA} + \bar{CO} = -R \cos(23.13^\circ) + 2 = -2.598$$

$$B'_{31} = \bar{AX}' = R \sin(23.13^\circ) = 5 \sin(23.13^\circ) = 1.964$$

$$B'_{11} = \bar{OB} = \bar{OC} + \bar{CB} = 2 + R \cos(23.13^\circ) = 6.598$$

Or

$$[B'] = \begin{bmatrix} 6.598 & 0 & 1.964 \\ 0 & 0 & 0 \\ 1.964 & 0 & -2.598 \end{bmatrix}$$

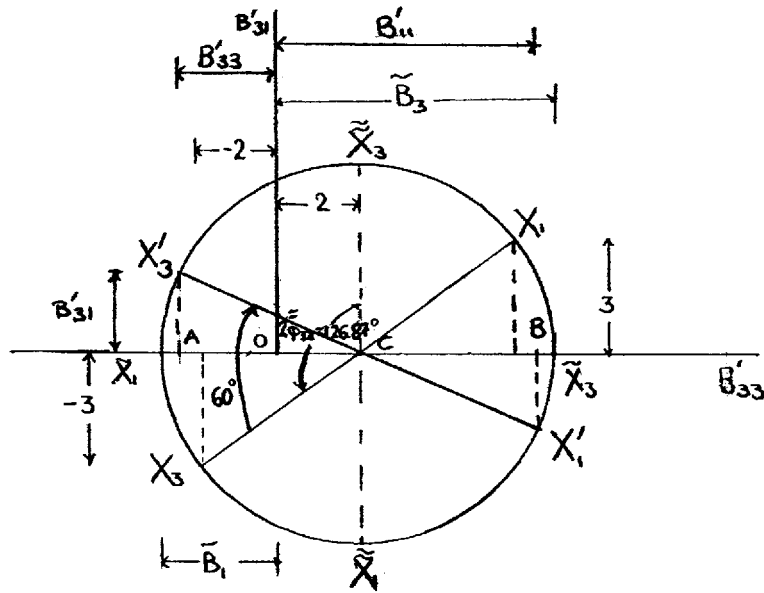


Figure a

Problem 1.10D

Referring to the solution of problem 1.9D we plot Mohr's circle in Fig. a. Moreover we plot line X'_1CX_2 making an angle of 60° clockwise with line X_1CX_2 . Referring to Fig. a we have that

$$\text{angle } X'_1CA = 15^\circ \text{ and}$$

$$D'_{11} = -\bar{CO} + R \cos(15^\circ) = -6 + 2.828 \cos(15^\circ) = -3.268$$

$$D'_{12} = -\bar{AX}' = -R \sin(15^\circ) = -2.828 \sin(15^\circ) = -0.732$$

$$D'_{22} = -\bar{BD} = -(\bar{CO} + \bar{CB}) = -6 - R \cos(15^\circ) = -8.732$$

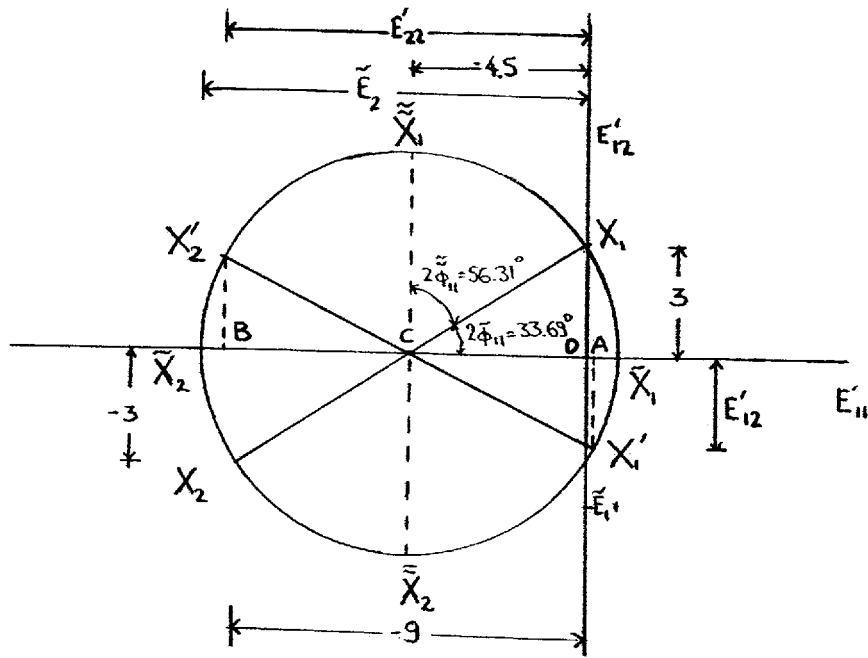


Figure a

Problem 1.11A

Given the symmetric tensor of the second rank

$$[A] = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Referring to relation (1.73a) we have

$$[A'] = [\Lambda_S][A][\Lambda_S]^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -12 & -4\sqrt{3} & 4 \\ -4\sqrt{3} & 4 & -\sqrt{3} \\ 4 & -\sqrt{3} & 5 \end{bmatrix}$$

Problem 1.11E

Given the symmetric tensor of the second rank

$$[\mathbf{E}] = \begin{bmatrix} 0 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Referring to relation (1.73a) we have

$$\begin{aligned} [\mathbf{E}'] &= [\mathbf{A}_s][\mathbf{E}][\mathbf{A}_s]^T \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 0 & 6\sqrt{3} & -6 \\ 6\sqrt{3} & -25 & 11\sqrt{3} \\ -6 & 11\sqrt{3} & -3 \end{bmatrix} \end{aligned}$$

Problem 1.12

The state of stress of the particle of Fig. 1P12 is

$$[\boldsymbol{\tau}] = \begin{bmatrix} 60 & 0 & 30 \\ 0 & 0 & 0 \\ 30 & 0 & -20 \end{bmatrix}$$

(a) The unit vector \mathbf{i}_s normal to \mathbf{i}_n , which lies in the x_1, x_3 plane, is

$$\mathbf{i}_s = \frac{4}{5} \mathbf{i}_1 - \frac{3}{5} \mathbf{i}_3$$

Thus referring to relation (1.61) we have

$$\tau_{nn} = \{\mathbf{n}\}^T [\boldsymbol{\tau}] \{\mathbf{n}\} = \frac{1}{5} [3 \ 0 \ 4] \begin{bmatrix} 60 & 0 & 30 \\ 0 & 0 & 0 \\ 30 & 0 & -20 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \frac{1}{5} = 37.6 \text{ MPa}$$

Referring to relation (1.64a) we have

$$\tau_{ns} = \{\mathbf{n}\}^T [\boldsymbol{\tau}] \{\mathbf{s}\} = \frac{1}{5} [3 \ 0 \ 4] \begin{bmatrix} 60 & 0 & 30 \\ 0 & 0 & 0 \\ 30 & 0 & -20 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} \frac{1}{5} = 46.80 \text{ MPa}$$

The positive sign indicates that τ_{ns} is in the direction of the unit vector \mathbf{i}_s , as we show in Fig. a.

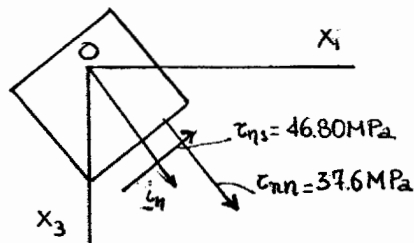


Figure a Results

Another way of solving this problem is using Mohr's circle. We plot the circle using two points $X_3(-20, 30)$ and $X_1(60, -30)$. These points are located on the same diameter of the circle. Referring to Fig. b we see that the center, C of the circle is located on the τ'_{33} axis at a distance $\overline{OC} = 1/2 (\tau_{33} + \tau_{11}) = 20$ from the origin O . From geometric considerations we have also that the radius of the circle is

$$R = (40^2 + 30^2)^{1/2} = 50 \quad \text{and} \quad \overline{AC} = \overline{AO} + \overline{OC} = 20 + 20 = 40$$

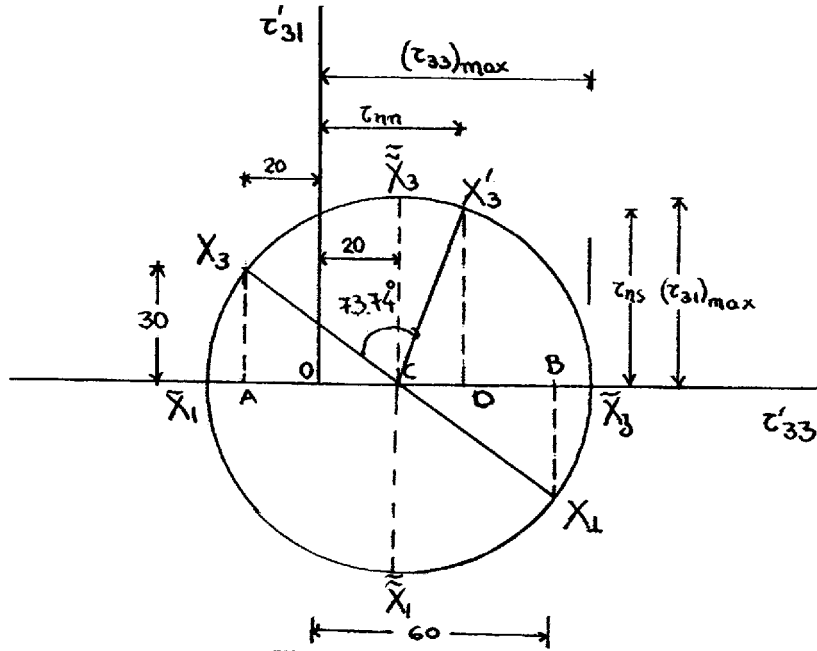


Figure b Mohr's circle

The point of Mohr's circle whose coordinates are the components of stress acting on the plane normal to the unit vector \mathbf{i}_n , referring to Fig. c, is located $2 \times 36.87^\circ = 73.74^\circ$ clockwise from point X_3 .

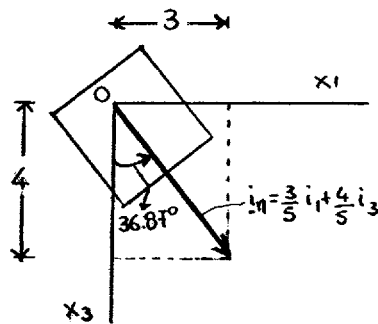


Figure c

Thus, referring to Fig. b, we have that

$$\tau_{nm} = \overline{OC} + \overline{CD} = 20 + R \cos(69.39^\circ) = 37.6 \text{ MPa}$$

$$\tau_{ns} = R \sin(69.39^\circ) = 46.80 \text{ MPa}$$

(b) Referring to Fig. b we see that the maximum normal component of stress is the abscissa of point \tilde{X}_3 . Thus,

$$(\tau'_{33})_{\max} = \overline{OC} + R = 20 + 50 = 70 \text{ MPa}$$

Moreover, $\tan 2\tilde{\varphi}_{33} = -30 / 40 = -0.75$ and

$$2\tilde{\varphi}_{33} = 143.14^\circ \quad \text{or} \quad \tilde{\varphi}_{33} = 71.57^\circ \text{ counter-clockwise}$$

In Fig. d we show the plane on which the maximum normal component of stress acts.

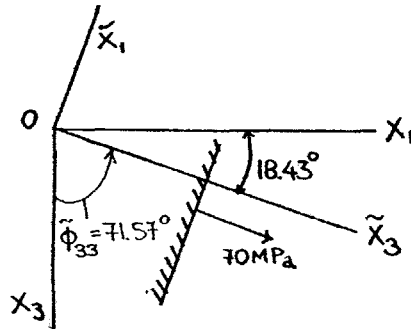


Figure d Principal directions & maximum component of stress

(c) The maximum shearing component of stress is the ordinate of point $\tilde{\tilde{X}}_3$. Thus,

$$(\tau'_{31})_{\max} = R = 50 \text{ MPa}$$

Moreover $\angle X_3\tilde{\tilde{X}}_3 = 143.14^\circ - 90^\circ = 53.14^\circ$ and $\tilde{\tilde{\varphi}}_{33} = 26.57^\circ$ counter-clockwise

In Fig. e we show the plane on which the maximum shearing component of stress acts.

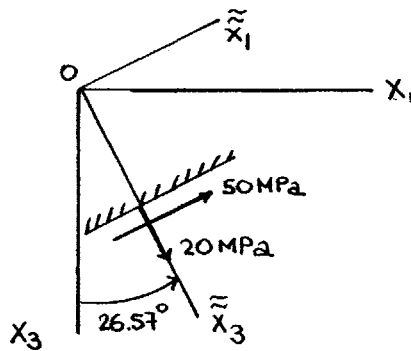


Figure e Plane of maximum shear stress

(d) Referring to relation (1.61) the normal component of stress acting on the plane normal to unit vector $\mathbf{i}_n = 1/3 (\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3)$ is

$$\tau_{nn} = \{n\}^T [z] \{n\} = \frac{1}{\sqrt{3}} [1 \ -1 \ 1] \begin{bmatrix} 60 & 0 & 30 \\ 0 & 0 & 0 \\ 30 & 0 & -20 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}} = \frac{100}{3} \text{ MPa}$$

Problem 1.13

The state of stress of the particle of Fig. 1P13 is

$$[\tau] = \begin{bmatrix} -120 & 0 & -30 \\ 0 & 0 & 0 \\ -30 & 0 & -40 \end{bmatrix}$$

(a) The unit vector \mathbf{i}_s normal to \mathbf{i}_n , which lies in the x_1, x_3 plane, is

$$\mathbf{i}_s = \frac{4}{5}\mathbf{i}_1 - \frac{3}{5}\mathbf{i}_3$$

Thus referring to relation (1.61) we have

$$\tau_{nn} = \{\eta\}^T [\tau] \{\eta\} = \frac{1}{5} \begin{bmatrix} 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} -120 & 0 & -30 \\ 0 & 0 & 0 \\ -30 & 0 & -40 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \frac{1}{5} = -97.6 \text{ MPa}$$

Referring to relation (1.64a) we have

$$\tau_{ns} = \{\eta\}^T [\tau] \{s\} = \frac{1}{5} \begin{bmatrix} 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} -120 & 0 & -30 \\ 0 & 0 & 0 \\ -30 & 0 & -40 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} \frac{1}{5} = -46.8 \text{ MPa}$$

The negative sign indicates that τ_{ns} is in the opposite direction of the unit vector \mathbf{i}_s . That is as we show in Fig. a.

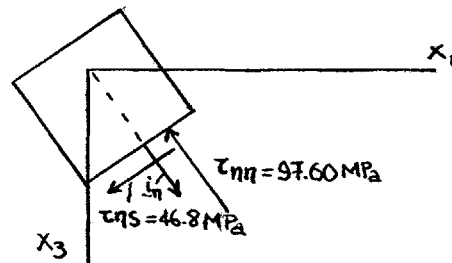


Figure a Results

Another way of solving this problem is using Mohr's circle. We plot the circle using two points X_3 $(-40, -30)$ and X_1 $(-120, 30)$. These points are located on the same diameter of the circle. Referring to Fig. b we see that the center, C of the circle is located on the τ'_{33} axis at a distance $1/2 (\tau_{33} + \tau_{11}) = -80$ from the origin O . From geometric considerations we have also that the radius of the circle is

$$R = (40^2 + 30^2)^{1/2} = 50$$

The point of Mohr's circle whose coordinates are the components of stress acting on the plane normal to the unit vector \mathbf{i}_n , which referring to Fig. c of the solution of problem 1.12, is located $2 \times 36.87^\circ = 73.74^\circ$ clockwise from point X_3 . So from Fig. b we have that

$$\tau_{nn} = -(\overline{OC} + \overline{CD}) = -[80 + R \cos(69.39^\circ)] = -97.60 \text{ MPa}$$

$$\tau_{ns} = -R \sin(69.39^\circ) = -46.80 \text{ MPa}$$

In Fig. d we show the plane on which the maximum shearing component of stress acts

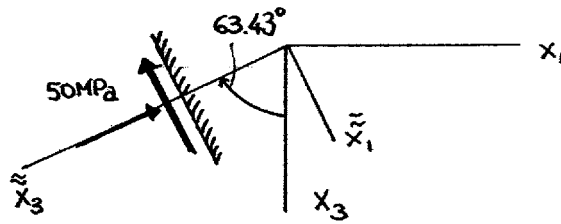


Figure d Plane of maximum shear stress

(d) Referring to relation (1.61) the normal component of stress acting on the plane normal to unit vector $\mathbf{i}_n = 1/3 (\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3)$ is

$$\tau_{nn} = \{\mathbf{n}\}^T [\boldsymbol{\tau}] \{\mathbf{n}\} = \frac{1}{\sqrt{3}} [1 \ -1 \ 1] \begin{bmatrix} -120 & 0 & -30 \\ 0 & 0 & 0 \\ -30 & 0 & -40 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}} = -\frac{220}{3}$$

Problem 1.14

(a) Referring to relations (1.78) to (1.80) the invariants of the tensor $[A]$ are

$$\Pi_1 = 3 \quad \Pi_2 = -21 \quad \Pi_3 = 4$$

Thus, relation (1.105) becomes

$$A_K^3 - 3A_K^2 - 21A_K - 4 = 0$$

The roots of this equation are

$$A_1 = 6.3863 \quad A_2 = -0.1963 \quad A_3 = -3.19$$

Substituting $A_1 = 6.3863$ into the first two of relations (1.103) we obtain

$$\begin{aligned} (3 - 6.3863) \lambda_{11}^{(1)} + 4 \lambda_{12}^{(1)} + \lambda_{13}^{(1)} &= 0 \\ 4 \lambda_{11}^{(1)} + (0 - 6.3863) \lambda_{12}^{(1)} + 2 \lambda_{13}^{(1)} &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} \lambda_{12}^{(1)} &= 0.7488 \lambda_{11}^{(1)} \\ \lambda_{13}^{(1)} &= 0.3911 \lambda_{11}^{(1)} \end{aligned} \quad (a)$$

Referring to equation (1.96), from the orthogonality condition $\mathbf{i}_1^{(1)} \cdot \mathbf{i}_1^{(1)} = 1$ we have

$$(\lambda_{11}^{(1)})^2 + (\lambda_{12}^{(1)})^2 + (\lambda_{13}^{(1)})^2 = 1 \quad (b)$$

Solving relations (a) and (b) we get

$$\lambda_{11}^{(1)} = \pm 0.7639 \quad \lambda_{12}^{(1)} = \pm 0.5720 \quad \lambda_{13}^{(1)} = \pm 0.2988$$

The set of direction cosines with the upper sign specifies the same direction as the set of direction cosines with the lower sign but opposite sense.

Substituting $A_2 = -0.1963$ into the first of relations (1.103) we get

$$(3 + 0.1963) \lambda_{21}^{(2)} + 4 \lambda_{22}^{(2)} + \lambda_{23}^{(2)} = 0$$

$$4 \lambda_{21}^{(2)} + 0.1963 \lambda_{22}^{(2)} + 2 \lambda_{23}^{(2)} = 0$$

Thus,

$$\lambda_{22}^{(2)} = -0.3066 \lambda_{21}^{(2)} \quad (c)$$

$$\lambda_{23}^{(2)} = -1.9699 \lambda_{21}^{(2)}$$

and

$$(\lambda_{21}^{(2)})^2 + (\lambda_{22}^{(2)})^2 + (\lambda_{23}^{(2)})^2 = 1 \quad (d)$$

Solving relations (c) and (d) we have

$$\lambda_{21}^{(2)} = \pm 0.4484 \quad \lambda_{22}^{(2)} = \mp 0.1375 \quad \lambda_{23}^{(2)} = \mp 0.8833$$

Substituting $A_3 = -3.19$ into the first two of relations (1.103) we have

$$(3 + 3.19) \lambda_{31}^{(3)} + 4 \lambda_{32}^{(3)} + \lambda_{33}^{(3)} = 0$$

$$4 \lambda_{31}^{(3)} + 3.19 \lambda_{32}^{(3)} + 2 \lambda_{33}^{(3)} = 0$$

Thus,

$$\lambda_{32}^{(3)} = -1.7422 \lambda_{31}^{(3)} \quad (e)$$

$$\lambda_{33}^{(3)} = 0.7788 \lambda_{31}^{(3)}$$

and

$$(\lambda_{31}^{(3)})^2 + (\lambda_{32}^{(3)})^2 + (\lambda_{33}^{(3)})^2 = 1 \quad (f)$$

Solving relations (e) and (f) we get

$$\lambda_{31}^{(3)} = \pm 0.4641 \quad \lambda_{32}^{(3)} = \mp 0.8086 \quad \lambda_{33}^{(3)} = \pm 0.3614$$

Thus, the principal directions of the tensor are

$$\tilde{i}_1^{(1)} = 0.7639 \hat{i}_1 + 0.5720 \hat{i}_2 + 0.2988 \hat{i}_3$$

$$\tilde{i}_2^{(2)} = 0.4484 \hat{i}_1 - 0.1375 \hat{i}_2 - 0.8833 \hat{i}_3 \quad (g)$$

$$\tilde{i}_3^{(3)} = 0.4641 \hat{i}_1 - 0.8086 \hat{i}_2 + 0.3614 \hat{i}_3$$

As a check we apply the orthogonality conditions $\tilde{i}_1^{(1)} \cdot \tilde{i}_2^{(2)} = 0$ and $\tilde{i}_1^{(1)} \cdot \tilde{i}_3^{(3)} = 0$.

Referring to relations (1.25a) we have

$$\lambda_{11}^{(1)} \lambda_{12}^{(2)} + \lambda_{21}^{(1)} \lambda_{22}^{(2)} + \lambda_{31}^{(1)} \lambda_{32}^{(2)} = 0$$

$$\lambda_{11}^{(1)} \lambda_{13}^{(3)} + \lambda_{21}^{(1)} \lambda_{23}^{(3)} + \lambda_{31}^{(1)} \lambda_{33}^{(3)} = 0$$

or

$$0.7639 \cdot 0.4484 - 0.5720 \cdot 0.1375 - 0.2988 \cdot 0.8833 \approx 0$$

$$0.7639 \cdot 0.4641 - 0.5720 \cdot 0.8086 + 0.2988 \cdot 0.3614 \approx 0$$

(b) Referring to relations (1.143) the maximum values of the non-diagonal components of the tensor are

$$(A'_{12})_{\max} = \frac{A_1 - A_2}{2} = 3.2913$$

$$(A'_{13})_{\max} = \frac{A_1 - A_3}{2} = 4.7882$$

$$(A'_{23})_{\max} = \frac{A_2 - A_3}{2} = 1.4969$$

Problem 1.15

(a) Referring to relations (1.78) to (1.80) the invariants of the tensor $[A]$ are

$$\mathbb{I}_1 = 7 \quad \mathbb{I}_2 = 0 \quad \mathbb{I}_3 = -25$$

Thus, relation (1.105) becomes

$$A_k^3 - 7A_k^2 + 25 = 0$$

The roots of this equation are

$$A_1 = 6.3872 \quad A_2 = 2.3084 \quad A_3 = -1.6956$$

Substituting $A_1 = 6.3872$ into the first two of relations (1.103) we obtain

$$(1 - 6.3872) \lambda_{11}^{(1)} - 2 \lambda_{12}^{(1)} + \lambda_{13}^{(1)} = 0$$

$$-2 \lambda_{11}^{(1)} + (4 - 6.3872) \lambda_{12}^{(1)} + 3 \lambda_{13}^{(1)} = 0$$

Thus,

$$\lambda_{12}^{(1)} = -3.9198 \lambda_{11}^{(1)} \quad (a)$$

$$\lambda_{13}^{(1)} = -2.4526 \lambda_{11}^{(1)}$$

Referring to equation (1.96), from the orthogonality condition $\mathbf{i}_1^{(1)} \cdot \mathbf{i}_1^{(1)} = 1$ we have

$$(\lambda_{11}^{(1)})^2 + (\lambda_{12}^{(1)})^2 + (\lambda_{13}^{(1)})^2 = 1 \quad (b)$$

Solving relations (a) and (b) we get

$$\lambda_{11}^{(1)} = \pm 0.2114 \quad \lambda_{12}^{(1)} = \mp 0.8285 \quad \lambda_{13}^{(1)} = \mp 0.5185$$

The set of direction cosines with the upper sign specifies the same direction as the set of direction cosines with the lower sign but opposite sense.

Substituting $A_2 = 2.3084$ into the first of relations (1.103) we get

$$(1 - 2.3084) \lambda_{21}^{(2)} - 2 \lambda_{22}^{(2)} + \lambda_{23}^{(2)} = 0$$

$$-2 \lambda_{21}^{(2)} + (4 - 2.3084) \lambda_{22}^{(2)} + 3 \lambda_{23}^{(2)} = 0$$

Thus,

$$\lambda_{22}^{(2)} = -0.2503 \lambda_{21}^{(2)} \quad (c)$$

$$\lambda_{23}^{(2)} = 0.8078 \lambda_{21}^{(2)}$$

and

$$(\lambda_{21}^{(2)})^2 + (\lambda_{22}^{(2)})^2 + (\lambda_{23}^{(2)})^2 = 1 \quad (d)$$

Solving relations (c) and (d) we have

$$\lambda_{21}^{(2)} = \pm 0.7636 \quad \lambda_{22}^{(2)} = \mp 0.1911 \quad \lambda_{23}^{(2)} = \pm 0.6168$$

Substituting $A_3 = -1.6956$ into the first two of relations (1.103) we have

$$\begin{aligned} (1 + 1.6956)\lambda_{31}^{(3)} - 2\lambda_{32}^{(3)} + \lambda_{33}^{(3)} &= 0 \\ -2\lambda_{31}^{(3)} + (4 + 1.6956)\lambda_{32}^{(3)} + 3\lambda_{33}^{(3)} &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} \lambda_{32}^{(3)} &= 0.8624 \lambda_{31}^{(3)} \\ \lambda_{33}^{(3)} &= -0.9708 \lambda_{31}^{(3)} \end{aligned} \quad (e)$$

and

$$(\lambda_{31}^{(3)})^2 + (\lambda_{32}^{(3)})^2 + (\lambda_{33}^{(3)})^2 = 1 \quad (f)$$

Solving relations (e) and (f) we get

$$\lambda_{31}^{(3)} = \pm 0.6101 \quad \lambda_{32}^{(3)} = \pm 0.5262 \quad \lambda_{33}^{(3)} = \mp 0.5923$$

Thus, the principal directions of the tensor are

$$\begin{aligned} \tilde{i}_1^{(1)} &= 0.2114 \underline{i}_1 - 0.8285 \underline{i}_2 - 0.5185 \underline{i}_3 \\ \tilde{i}_2^{(2)} &= 0.7636 \underline{i}_1 - 0.1911 \underline{i}_2 + 0.6168 \underline{i}_3 \\ \tilde{i}_3^{(3)} &= 0.6101 \underline{i}_1 + 0.5262 \underline{i}_2 - 0.5923 \underline{i}_3 \end{aligned} \quad (g)$$

(b) Referring to relations (1.143) the maximum values of the non-diagonal components of the tensor are

$$\begin{aligned} (A'_{12})_{\max} &= \frac{A_1 - A_2}{2} = 2.0394 \\ (A'_{13})_{\max} &= \frac{A_1 - A_3}{2} = 4.0414 \\ (A'_{23})_{\max} &= \frac{A_2 - A_3}{2} = 2.002 \end{aligned}$$

Problem 1.16

(a) Referring to relations (1.78) to (1.80) the invariants of the tensor are

$$\Pi_1 = 0 \quad \Pi_2 = -25 \quad \Pi_3 = 0$$

Thus, relation (1.105) becomes

$$A_K^3 - 25 A_K = 0$$

The roots of this equation are

$$A_1 = 0 \quad A_2 = 5 \quad A_3 = -5$$

Substituting $A_1 = 0$ into the first two of relations (1.103) we obtain

$$\begin{aligned} 4\lambda_{12}^{(1)} + 3\lambda_{13}^{(1)} &= 0 \\ 4\lambda_{11}^{(1)} &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} \lambda_{12}^{(1)} &= -0.75\lambda_{13}^{(1)} \\ \lambda_{11}^{(1)} &= 0 \end{aligned} \tag{a}$$

Referring to equation (1.96), from the orthogonality condition $\mathbf{i}_1^{(1)} \cdot \mathbf{i}_1^{(1)} = 1$ we have

$$(\lambda_{11}^{(1)})^2 + (\lambda_{12}^{(1)})^2 + (\lambda_{13}^{(1)})^2 = 1 \tag{b}$$

Solving relations (a) and (b) we get

$$\lambda_{11}^{(1)} = 0 \quad \lambda_{12}^{(1)} = \mp \frac{3}{5} \quad \lambda_{13}^{(1)} = \pm \frac{4}{5}$$

The set of direction cosines with the upper sign specifies the same direction as the set of direction cosines with the lower sign but opposite sense.

Substituting $A_2 = 5$ into the first of relations (1.103) we get

$$\begin{aligned} -5\lambda_{21}^{(2)} + 4\lambda_{22}^{(2)} + 3\lambda_{23}^{(2)} &= 0 \\ 4\lambda_{21}^{(2)} - 5\lambda_{22}^{(2)} &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} \lambda_{21}^{(2)} &= \frac{5}{4}\lambda_{22}^{(2)} \\ \lambda_{23}^{(2)} &= \frac{3}{4}\lambda_{22}^{(2)} \end{aligned} \tag{c}$$

and

$$(\lambda_{21}^{(2)})^2 + (\lambda_{22}^{(2)})^2 + (\lambda_{23}^{(2)})^2 = 1 \tag{d}$$

Solving relations (c) and (d) we have

$$\lambda_{21}^{(2)} = \pm \frac{1}{\sqrt{2}} \quad \lambda_{22}^{(2)} = \pm \frac{4}{5\sqrt{2}} \quad \lambda_{23}^{(2)} = \pm \frac{3}{5\sqrt{2}}$$

Substituting $A_3 = -5$ into the first two of relations (1.103) we have

$$\begin{aligned} 5\lambda_{31}^{(3)} + 4\lambda_{32}^{(3)} + 3\lambda_{33}^{(3)} &= 0 \\ 4\lambda_{31}^{(3)} + 5\lambda_{32}^{(3)} &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} \lambda_{31}^{(3)} &= -\frac{5}{4}\lambda_{32}^{(3)} \\ \lambda_{33}^{(3)} &= \frac{3}{4}\lambda_{32}^{(3)} \end{aligned} \tag{e}$$

and

$$(\lambda_{31}^{(3)})^2 + (\lambda_{32}^{(3)})^2 + (\lambda_{33}^{(3)})^2 = 1 \tag{f}$$

Solving relations (e) and (f) we get

$$\lambda_{31}^{(3)} = -\frac{1}{\sqrt{2}} \quad \lambda_{32}^{(3)} = +\frac{4}{5\sqrt{2}} \quad \lambda_{33}^{(3)} = \pm \frac{3}{5\sqrt{2}}$$

Thus, the principal directions of the tensor are

$$\begin{aligned} \tilde{i}_1^{(1)} &= -\frac{3}{5}i_2 + \frac{4}{5}i_3 \\ \tilde{i}_2^{(2)} &= \frac{1}{\sqrt{2}}i_1 + \frac{4}{5\sqrt{2}}i_2 + \frac{3}{5\sqrt{2}}i_3 \\ \tilde{i}_3^{(3)} &= -\frac{1}{\sqrt{2}}i_1 + \frac{4}{5\sqrt{2}}i_2 + \frac{3}{5\sqrt{2}}i_3 \end{aligned} \quad (g)$$

As a check we apply the orthogonality conditions $\tilde{i}_1^{(1)} \cdot \tilde{i}_2^{(2)} = 0$ and $\tilde{i}_1^{(1)} \cdot \tilde{i}_3^{(3)} = 0$.

Referring to relations (1.25a) we have

$$\begin{aligned} \lambda_{11}^{(1)}\lambda_{12}^{(2)} + \lambda_{21}^{(1)}\lambda_{22}^{(2)} + \lambda_{31}^{(1)}\lambda_{32}^{(2)} &= 0 \\ \lambda_{11}^{(1)}\lambda_{13}^{(3)} + \lambda_{21}^{(1)}\lambda_{23}^{(3)} + \lambda_{31}^{(1)}\lambda_{33}^{(3)} &= 0 \end{aligned}$$

or

$$\begin{aligned} 0 \cdot \frac{1}{\sqrt{2}} - \frac{3}{5} \cdot \frac{4}{5\sqrt{2}} + \frac{4}{5} \cdot \frac{3}{5\sqrt{2}} &= 0 \\ 0 \cdot \left(-\frac{1}{\sqrt{2}}\right) - \frac{3}{5} \cdot \frac{4}{5\sqrt{2}} + \frac{4}{5} \cdot \frac{3}{5\sqrt{2}} &= 0 \end{aligned}$$

(b) Referring to Fig. 1P16 the transformation matrix is

$$[\Lambda_S] = \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (h)$$

Substituting (h) into relation (1.73a) we have

$$\begin{aligned} [A'] &= \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3.464 & 2 & 2.598 \\ 2 & -3.464 & -1.5 \\ 2.598 & -1.5 & 0 \end{bmatrix} \end{aligned}$$