1 General Introduction

Mathematics is the language with which God has written the universe.

— Galileo Galilei

1.1 Newton's second law can be expressed as

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{F}$ is the net force acting on the body, m mass of the body, and $\mathbf{a}$ is the acceleration of the body in the direction of the net force. Determine the mathematical model, that is, the governing equation of a free-falling body. Consider only the forces due to gravity and the air resistance. Assume that the air resistance is linearly proportional to the velocity of the falling body.

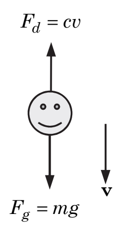

Fig. P1.1

Solution: From the free-body-diagram it follows that

$$m\frac{dv}{dt} = F_g - F_d, \quad F_g = mg, \quad F_d = cv$$

where v is the downward velocity (m/s) of the body, F_g is the downward force (N or kg m/s^2) due to gravity, F_d is the upward drag force, m is the mass (kg) of the body, g the acceleration (m/s^2) due to gravity, and c is the proportionality constant (drag coefficient, kg/s). The equation of motion is

$$\frac{dv}{dt} + \alpha v = g, \quad \alpha = \frac{c}{m}$$

1.2 A cylindrical storage tank of diameter D contains a liquid at depth (or head) $h(x,t)$. Liquid is supplied to the tank at a rate of q_i (m^3/day) and drained at a rate of q_0 (m^3/day). Use the principle of conservation of mass to arrive at the governing equation of the flow problem.

Solution: The conservation of mass requires

time rate of change in mass = mass inflow - mass outflow

The above equation for the problem at hand becomes

$$\frac{d}{dt}(\rho A h) = \rho q_i - \rho q_0 \quad \text{or} \quad \frac{d(Ah)}{dt} = q_i - q_0$$

where A is the area of cross section of the tank ($A = \pi D^2/4$) and ρ is the mass density of the liquid.

1.3 Consider the simple pendulum of **Example 1.3.1**. Write a computer program to numerically solve the *linear* equation (1.2.4) using Euler's (or forward difference) finite difference scheme. Tabulate the numerical results for two different time steps $\Delta t = 0.05$ and $\Delta t = 0.025$ along with the exact linear solution.

Solution: One needs to program the finite difference equations in Eq. (1.3.6). Table 1.3 contains representative numerical results.

Table 1.3 Comparison of various approximate solutions of the equation $(d^2\theta/dt^2) + \lambda^2\theta = 0$ with its exact linear solution.

t	Exact	Approx. solution, θ		Exact	Approx. solution, v	
	θ	$\Delta t = 0.05$	$\Delta t = 0.025$	v	$\Delta t = 0.05$	$\Delta t = 0.025$
0.00	0.78540	0.78540	0.78540	−0.00000	−0.00000	−0.00000
0.05	0.76965	0.78540	0.77749	−0.62801	−0.63225	−0.63225
0.10	0.72302	0.75379	0.73806	−1.23083	−1.26449	−1.25177
0.15	0.64739	0.69056	0.66804	−1.78428	−1.87129	−1.83331
0.20	0.54578	0.59700	0.56966	−2.26615	−2.42719	−2.35264
0.25	0.42229	0.47564	0.44629	−2.65711	−2.90777	−2.78754
0.30	0.28185	0.33025	0.30243	−2.94148	−3.29066	−3.11875
0.35	0.13011	0.16572	0.14344	−3.10785	−3.55651	−3.33082
0.40	−0.02685	−0.01211	−0.02454	−3.14955	−3.68991	−3.41278
0.45	−0.18274	−0.19661	−0.19493	−3.06491	−3.68016	−3.35868
0.50	−0.33129	−0.38061	−0.36091	−2.85732	−3.52190	−3.16797
0.60	−0.58310	−0.71748	−0.65276	−2.11119	−2.76735	−2.40181
0.80	−0.78356	−1.07657	−0.92032	0.21536	0.11379	0.21327
1.00	−0.50591	−0.79648	−0.62784	2.41051	3.41351	2.91148

1.4 Consider the simple pendulum of **Example 1.3.1**. Write a computer program to numerically solve the *nonlinear* equation (1.2.3) using Euler's (or forward difference) finite difference scheme. Tabulate the numerical results for two different time steps $\Delta t = 0.05$ and $\Delta t = 0.025$ along with the exact linear solution.

Solution: In order to use the forward finite difference scheme in Eq. (1.3.2), we rewrite the nonlinear equation in (1.2.3) as a pair of first-order equations

$$\frac{d\theta}{dt} = v, \quad \frac{dv}{dt} = -\lambda^2 \sin\theta$$

Applying the scheme of Eq. (1.3.2) to the two equations at hand, we obtain

$$\theta_{i+1} = \theta_i + \Delta t\ v_i; \quad v_{i+1} = v_i - \Delta t\ \lambda^2 \sin\theta_i$$

The above equations can be programmed to solve for (θ_i, v_i). Table 1.4 contains representative numerical results.

1.5 An improvement of Euler's method is provided by Heun's method, which uses the average of the derivatives at the two ends of the interval to estimate the slope. Applied to the equation

$$\frac{du}{dt} = f(t,u)$$

Heun's scheme has the form

$$u_{i+1} = u_i + \frac{\Delta t}{2}\left[f(t_i,u_i) + f(t_{i+1},u^0_{i+1})\right], \quad u^0_{i+1} = u_i + \Delta t \ f(t_i,u_i)$$

Table 1.4 Comparison of various approximate solutions of the equation $(d^2\theta/dt^2) + \lambda^2 \sin\theta = 0$ with exact linear solution.

	Exact	Approx. solution, θ		Exact	Approx. solution, v	
t	θ	Euler's	Heun's	v	Euler's	Heun's
0.00	0.78540	0.78540	0.78540	−0.00000	−0.00000	−0.00000
0.05	0.76965	0.78540	0.77828	−0.62801	−0.56922	−0.56922
0.10	0.72302	0.75694	0.74276	−1.23083	−1.13844	−1.13027
0.15	0.64739	0.70002	0.67944	−1.78428	−1.69123	−1.66622
0.20	0.54578	0.58980	0.56482	−2.26615	−2.20984	−2.15879
0.25	0.42229	0.50496	0.47627	−2.65711	−2.67459	−2.58816
0.30	0.28185	0.37123	0.34225	−2.94148	−3.06403	−2.93371
0.35	0.13011	0.21803	0.19218	−3.10785	−3.35605	−3.17573
0.40	−0.02685	0.05023	0.03148	−3.14955	−3.53018	−3.29791
0.45	−0.18274	−0.12628	−0.13374	−3.06491	−3.57060	−3.29007
0.50	−0.33129	−0.30481	−0.29690	−2.85732	−3.46921	−3.15014
0.60	−0.58310	−0.63965	−0.59131	−2.11119	−2.85712	−2.50787
0.80	−0.78356	−1.05068	−0.91171	0.21536	−0.50399	−0.28356
1.00	−0.50591	−0.94062	−0.74672	2.41051	2.29398	2.19765

The second equation is known as the *predictor* equation and the first equation is called the *corrector* equation. Apply Heun's method to Eq. (1.3.5) and obtain the numerical solution for $\Delta t = 0.05$.

Solution: Heun's method applied to the pair

$$\frac{d\theta}{dt} = v, \quad \frac{dv}{dt} = -\lambda^2 \sin\theta$$

yields the following discrete equations:

$$\begin{aligned}
\theta^0_{i+1} &= \theta_i + \Delta t \ v_i \\
v_{i+1} &= v_i - \lambda^2 \frac{\Delta t}{2}\left(\sin\theta_i + \sin\theta^0_{i+1}\right) \\
\theta_{i+1} &= \theta_i + \frac{\Delta t}{2}\left(v_i + v_{i+1}\right)
\end{aligned}$$

The numereical results obtained with the Heun's method and Euler's method are presented in Table 1.5.

Table 1.5 Numerical solutions of the nonlinear equation $d^2\theta/dt^2 + \lambda^2 \sin\theta = 0$ along with the exact solution of the linear equation $d^2\theta/dt^2 + \lambda^2\theta = 0$.

	Exact	Approx. solution, θ		Exact	Approx. solution, v	
t	θ	Euler's	Heun's	v	Euler's	Heun's
0.00	0.785398	0.785398	0.785398	−0.000000	−0.000000	−0.000000
0.05	0.769645	0.785398	0.771168	−0.628013	−0.569221	−0.569221
0.10	0.723017	0.756937	0.728680	−1.230833	−1.138442	−1.121957
0.20	0.545784	0.615453	0.564818	−2.266146	−2.209838	−1.121957
0.40	−0.026852	0.050228	0.015246	−3.149552	−3.530178	−3.073095
0.60	−0.583104	−0.639652	−0.544352	−2.111190	−2.857121	−2.194398
0.80	−0.783562	−1.050679	−0.787095	0.215362	−0.503993	−0.114453
1.00	−0.505912	−0.940622	−0.587339	2.410506	2.293983	2.023807

1.6 Show that the backward difference approximation of the boundary condition in Eq. (1.2.20) yields

$$\theta_{N+1} = \left(1 + \frac{\beta\Delta x}{k}\right)^{-1} \theta_N$$

and that it is the same as that in Eq. (1.3.9) when $\frac{\beta\Delta x}{k} < 1$.

Solution: Expanding in Taylor's series and keeping the first two terms, we obtain the required result:

$$\theta_{N+1} = \left(1 + \frac{\beta\Delta x}{k}\right)^{-1} \theta_N \approx \left(1 - \frac{\beta\Delta x}{k}\right)\theta_N$$

1.7 Write a computer program to solve the rod problem of **Example 1.3.2** using 8 intervals (i.e., $\Delta x = 0.00625$) and determine the solution at mesh points $x = 0.00625$, 0.0125, 0.01875, $\cdots$, 0.05 m.

Solution: For a subdivision of 8 subintervals, we have $\Delta x = 0.00625$, $D = 2 + (20 \times 0.00625)^2 = 2.015625$, and $D - 1 + (\beta\Delta x/k) = 1.028125$. Hence, the finite difference equations are

$$\begin{array}{rrrrrcl}
2.015625\,\theta_1 & -\theta_2 & & & & = & 300 \\
-\theta_1 & +2.015625\,\theta_2 & -\theta_3 & & & = & 0 \\
 & -\theta_2 & +2.015625\,\theta_3 & -\theta_4 & & = & 0 \\
 & & \vdots & & \vdots & & \\
 & & -\theta_6 & +2.015625\,\theta_7 & -\theta_8 & = & 0 \\
 & & & -\theta_7 & +1.028125\,\theta_8 & = & 0
\end{array}$$

The finite difference solutions is

$$\{\theta\} = \{271.46,\ 247.16,\ 226.72,\ 209.82,\ 196.21,\ 185.66,\ 178.00, 173.13\}^{\mathrm{T}}$$

The analytical solution at the same points is

$$\{\theta\} = \{272.25,\ 248.75,\ 229.15,\ 213.13,\ 200.44,\ 190.90,\ 184.33,\ 180.66\}^{\mathrm{T}}$$

1.8 Repeat **Problem 1.7** for 16 subdivisions and compare the finite difference solution with the analytical solution.

Solution: For a subdivision of 16 subintervals, we have $\Delta x = 0.003125$, $D = 2 + (20 \times 0.003125)^2 = 2.00390625$, and $D - 1 + (\beta\Delta x/k) = 1.01015625$. Hence, the finite difference equations are

$$\begin{array}{cccccccc}
2.00391\,\theta_1 & -\theta_2 & & & & & = & 300 \\
-\theta_1 & 2.00391\,\theta_2 & -\theta_3 & & & & = & 0 \\
 & -\theta_2 & 2.00391\,\theta_3 & -\theta_4 & & & = & 0 \\
 & & \vdots & & \vdots & & & \\
 & & -\theta_{13} & 2.00391\,\theta_{14} & -\theta_{15} & & = & 0 \\
 & & & -\theta_{15} & 1.01016\,\theta_{16} & & = & 0
\end{array}$$

The solution of these equations is

$$\{\theta\} = \{285.36,\ 271.84,\ 259.37,\ 247.92,\ 237.44,\ 227.89,\ 219.22, 211.42,\ \cdots, 176.78\}^{\mathrm{T}}$$

The analytical solution at the same points is

$$\{\theta\} = \{285.56,\ 272.25,\ 259.99,\ 248.75,\ 238.48,\ 229.15,\ 220.71, 213.13,\ \cdots, 180.66\}^{\mathrm{T}}$$

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2 Mathematical Preliminaries and Classical Variational Methods

The fact that an opinion has been widely held is no evidence whatever that it is not utterly absurd; indeed in view of the silliness of the majority of mankind, a widespread belief is more likely to be foolish than sensible. — Bertrand Russell

2.1 Let $\mathbf{R}$ denote a position vector $\mathbf{R} = \mathbf{x} = x_i\,\hat{\mathbf{e}}_i$ $(R^2 = x_i x_i)$ and $\mathbf{A}$ an arbitrary constant vector. Show that:

(a) $\nabla^2(R^p) = p\,(p+1)R^{p-2}$ (b) $\text{grad}\,(\mathbf{R}\cdot\mathbf{A}) = \mathbf{A}$

(c) $\text{div}\,(\mathbf{R}\times\mathbf{A}) = 0$ (d) $\text{div}\,(R\mathbf{A}) = \dfrac{1}{R}(\mathbf{R}\cdot\mathbf{A})$

Solution: First we establish following two identities:

$$
\begin{aligned}
\text{grad}(R) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\,(x_j x_j)^{\frac{1}{2}} = \hat{\mathbf{e}}_i \frac{1}{2}\,(x_j x_j)^{\frac{1}{2}-1}\,2x_i \\
&= \hat{\mathbf{e}}_i x_i\,(x_j x_j)^{-\frac{1}{2}} = \frac{\mathbf{R}}{R} \qquad (1)\\
\text{grad}(R^p) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\,(x_j x_j)^{\frac{p}{2}} = \hat{\mathbf{e}}_i \frac{p}{2}\,(x_j x_j)^{\frac{p}{2}-1}\,2x_i \\
&= p\hat{\mathbf{e}}_i x_i\,(x_j x_j)^{\frac{p-2}{2}} = pR^{p-2}\mathbf{R} \qquad (2)
\end{aligned}
$$

(**a**) Then we have

$$
\begin{aligned}
\nabla^2(R^p) &= \frac{\partial^2}{\partial x_i \partial x_i}\,(R^p) = \frac{\partial}{\partial x_i}\left(pR^{p-2}x_i\right) \\
&= p(p-2)R^{p-3}\frac{\partial R}{\partial x_i}x_i + pR^{p-2}\delta_{ii} = p(p-2)R^{p-3}\frac{x_i}{R}x_i + 3pR^{p-2} \\
&= [p(p-2)+3p]R^{p-2} = p(p+1)R^{p-2}
\end{aligned}
$$

(**b**) Since $\mathbf{A}$ is a constant vector, we have

$$
\begin{aligned}
\text{grad}(\mathbf{R}\cdot\mathbf{A}) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\,(x_j A_j) = \hat{\mathbf{e}}_i\left(\delta_{ij}A_j + x_j\frac{\partial A_j}{\partial x_i}\right) \\
&= \hat{\mathbf{e}}_i\,(A_i + 0) = \mathbf{A}
\end{aligned}
$$

(**c**) Carrying out the indicated operation, we obtain

$$
\begin{aligned}
\text{div}(\mathbf{R}\times\mathbf{A}) &= \hat{\mathbf{e}}_i \cdot \frac{\partial}{\partial x_i}\,(e_{jk\ell}x_j A_k \hat{\mathbf{e}}_\ell) = e_{jk\ell}\delta_{i\ell}\left(\frac{\partial x_j}{\partial x_i}A_k + x_j\frac{\partial A_k}{\partial x_i}\right) \\
&= (0+0) = 0
\end{aligned}
$$

(**d**) Carrying out the indicated operation, we obtain

$$\text{div}(R\mathbf{A}) = \hat{\mathbf{e}}_i \cdot \frac{\partial}{\partial x_i}(RA_j\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \left(\frac{\partial R}{\partial x_i} A_j\right) = \frac{x_i}{R} A_i = \frac{1}{R}\mathbf{R}\cdot\mathbf{A}$$

2.2 Let $\mathbf{A}$ and $\mathbf{B}$ be continuous vector functions of the position $\mathbf{x}$ with continuous first derivatives, and let F and G be continuous scalar functions of position $\mathbf{x}$ with continuous first and second derivatives. Show that:
(a) $\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{x}) = \mathbf{A} + \boldsymbol{\nabla}\mathbf{A}\cdot\mathbf{x}$
(b) $\boldsymbol{\nabla}\cdot(F\mathbf{A}) = \mathbf{A}\cdot\boldsymbol{\nabla}F + F\boldsymbol{\nabla}\cdot\mathbf{A}$

Solution:
(**a**) Since $\mathbf{A}$=$\mathbf{A}(\mathbf{x})$, we have

$$\begin{aligned}\text{grad}(\mathbf{A}\cdot\mathbf{x}) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(A_j x_j) = \hat{\mathbf{e}}_i \left(\frac{\partial A_j}{\partial x_i} x_j + A_j \frac{\partial x_j}{\partial x_i}\right) \\ &= \nabla\mathbf{A}\cdot\mathbf{x} + \mathbf{A}\end{aligned}$$

where $A_j \frac{\partial x_j}{\partial x_i} = A_j\delta_{ij} = A_i$ is used in arriving at the final expression.

(**b**) Since $F = F(\mathbf{x})$ and $\mathbf{A}$=$\mathbf{A}(\mathbf{x})$, we have

$$\begin{aligned}\nabla\cdot(F\mathbf{A}) &= \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\right)\cdot(\hat{\mathbf{e}}_j A_j\ F) = \delta_{ij}\left(\frac{\partial A_j}{\partial x_i} F + A_j \frac{\partial F}{\partial x_i}\right) \\ &= \frac{\partial A_i}{\partial x_i} F + A_i \frac{\partial F}{\partial x_i} = \nabla\cdot\mathbf{A}F + \mathbf{A}\cdot\nabla F\end{aligned}$$

2.3 Let $\mathbf{a}$ be continuous second-order tensor function of the position $\mathbf{x}$ with continuous first derivatives, and let u be a continuous scalar function of the position $\mathbf{x}$ with continuous first and second derivatives. Show that

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\mathbf{a}\cdot\boldsymbol{\nabla}u) = &\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(a_{rr}\frac{\partial u}{\partial r} + a_{r\theta}\frac{1}{r}\frac{\partial u}{\partial \theta} + a_{rz}\frac{\partial u}{\partial z}\right)\right] \\ &+ \frac{1}{r}\frac{\partial}{\partial \theta}\left(a_{\theta r}\frac{\partial u}{\partial r} + a_{\theta\theta}\frac{1}{r}\frac{\partial u}{\partial \theta} + a_{\theta z}\frac{\partial u}{\partial z}\right) \\ &+ \frac{\partial}{\partial z}\left(a_{zr}\frac{\partial u}{\partial r} + a_{z\theta}\frac{1}{r}\frac{\partial u}{\partial \theta} + a_{zz}\frac{\partial u}{\partial z}\right)\end{aligned}$$

Solution: First note that

$$\boldsymbol{\nabla}u = \hat{\mathbf{e}}_r \frac{\partial u}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial u}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial u}{\partial z}$$

$$\begin{aligned}\mathbf{a} = &\, a_{rr}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + a_{r\theta}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + a_{rz}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z + a_{\theta r}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r + a_{\theta\theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta + a_{\theta z}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z \\ &+ a_{zr}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r + a_{z\theta}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta + a_{zz}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z \\ \mathbf{a}\cdot\boldsymbol{\nabla}u = &\left(a_{rr}\hat{\mathbf{e}}_r + a_{\theta r}\hat{\mathbf{e}}_\theta + a_{zr}\hat{\mathbf{e}}_z\right)\frac{\partial u}{\partial r} + \left(a_{r\theta}\hat{\mathbf{e}}_r + a_{\theta\theta}\hat{\mathbf{e}}_\theta + a_{z\theta}\hat{\mathbf{e}}_z\right)\frac{1}{r}\frac{\partial u}{\partial \theta} \\ &+ \left(a_{rz}\hat{\mathbf{e}}_r + a_{\theta z}\hat{\mathbf{e}}_\theta + a_{zz}\hat{\mathbf{e}}_z\right)\frac{\partial u}{\partial z}\end{aligned}$$

Then, using Eq. (2.2.12), we have

$$\nabla \cdot (\mathbf{a} \cdot \nabla u) = \frac{\partial}{\partial r}\left(a_{rr}\frac{\partial u}{\partial r} + a_{r\theta}\frac{1}{r}\frac{\partial u}{\partial \theta} + a_{rz}\frac{\partial u}{\partial z}\right) + a_{rr}\frac{1}{r}\frac{\partial u}{\partial r} + a_{r\theta}\frac{1}{r^2}\frac{\partial u}{\partial \theta} + a_{rz}\frac{1}{r}\frac{\partial u}{\partial z}$$
$$+ \frac{1}{r}\frac{\partial}{\partial \theta}\left(a_{\theta r}\frac{\partial u}{\partial r} + a_{\theta\theta}\frac{1}{r}\frac{\partial u}{\partial \theta} + a_{\theta z}\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}\left(a_{zr}\frac{\partial u}{\partial r} + a_{z\theta}\frac{1}{r}\frac{\partial u}{\partial \theta} + a_{zz}\frac{\partial u}{\partial z}\right)$$

Since

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r a_{rr}\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(a_{rr}\frac{\partial u}{\partial r}\right) + a_{rr}\frac{1}{r}\frac{\partial u}{\partial r}$$

and so on, we arrive at the desired result.

In **Problems 2.4–2.9**, construct the weak forms and, whenever possible, the associated quadratic functionals (I).

2.4 A linear differential equation:

$$-\frac{d}{dx}\left[(1+2x^2)\frac{du}{dx}\right] + u = x^2$$

$$u(0) = 1\ , \quad \left(\frac{du}{dx}\right)_{x=1} = 2$$

Solution: First, compare the given equations with the model equation in (2.4.1) and its boundary conditions (2.4.2). We find that $a(x) = 1 + 2x^2$, $c(x) = 1$, $f(x) = x^2$, $L = 1$, $u_0 = 1$, $\beta = 0$, and $Q_L = 3 \times 2$. Hence, the weak form is obtained directly from (2.4.18) as

$$0 = \int_0^L \left[(1+2x^2)\frac{dw_i}{dx}\frac{du_n}{dx} + w_i u_n - w_i\, x^2\right]dx - 6w_i(1) \qquad (2)$$

The bilinear and linear forms are given by Eq. (2.4.19) as

$$\begin{aligned} B(w_i, u_n) &= \int_0^1 \left[(1+2x^2)\frac{dw_i}{dx}\frac{du_n}{dx} + w_i u_n\right] dx \text{ (symmetric)} \\ \ell(w_i) &= \int_0^1 x^2\, w_i\ dx + 6w_i(1) \end{aligned} \qquad (3)$$

Since $B(\cdot,\cdot)$ is symmetric in its arguments and $\ell(w_i)$ is linear in w_i, a quadratic functional can be constructed and it is given by [see Eq. (2.4.25)]

$$\begin{aligned} I(u_n) &= \frac{1}{2}B(u_n, u_n) - \ell(u_n) \\ &= \frac{1}{2}\int_0^1 \left[(1+2x^2)\left(\frac{du_n}{dx}\right)^2 + u_n^2\right] dx - \int_0^1 x^2\, u_n\ dx - 6u_n(1) \end{aligned}$$

2.5 A nonlinear equation:

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right)+f=0 \quad \text{for} \quad 0<x<1$$

$$\left(u\frac{du}{dx}\right)\Big|_{x=0}=0 \quad u(1)=\sqrt{2}$$

Solution: Compared to the model equation (2.4.1), we have $a(x) = u_n(x)$, $f(x) \to -f(x)$, and $L = 1$. Following the three-step procedure, we write the weak form

$$0=\int_0^1 w_i\left[-\frac{d}{dx}\left(u_n\frac{du_n}{dx}\right)+f\right]\,dx \tag{1}$$

$$=\int_0^1\left(u_n\frac{dw_i}{dx}\frac{du_n}{dx}+w_i f\right)\,dx-\left[w_i\left(u_n\frac{du_n}{dx}\right)\right]_0^1 \tag{2}$$

Using the boundary conditions, $w_i(1) = 0$ (because u_n is specified at $x = 1$) and $(du_n/dx) = 0$ at $x = 0$, we obtain

$$0=\int_0^1\left(u_n\frac{dw_i}{dx}\frac{du_n}{dx}+w_i f\right)dx \tag{3}$$

For this problem, the weak form does not contain an expression that is linear in both u_n and w_i; the expression is linear in w_i but not linear in u_n. Therefore, a quadratic functional does not exist for this case. The expressions for $B(\cdot,\cdot)$ and $\ell(\cdot)$ are given by

$$B(w_i,u_n)=\int_0^1 u_n\frac{dw_i}{dx}\frac{du_n}{dx}dx \quad \text{(not linear in } u_n \text{ and not symmetric in } u_n \text{ and } w_i\text{)}$$

$$\ell(w_i)=-\int_0^1 w_i f\,dx \tag{4}$$

2.6 The Euler-Bernoulli-von Kármán nonlinear theory of beams:

$$-\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}=f \quad \text{for} \quad 0<x<L$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)-\frac{d}{dx}\left\{EA\frac{dw}{dx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}=q$$

$$u=w=0 \quad \text{at} \quad x=0,\ L; \quad \left(\frac{dw}{dx}\right)\Big|_{x=0}=0; \quad \left(EI\frac{d^2w}{dx^2}\right)\Big|_{x=L}=M_0$$

where EA, EI, f, and q are functions of x and M_0 is a constant. Here u denotes the axial displacement and w the transverse deflection of the beam.

Solution: The first step of the formulation is to multiply each equation with a weight function, say v_{1i} for the first equation and v_{2i} for the second equation, and integrate over the interval $(0, L)$. In the second step, carry out the integration-by-parts once in the first equation, twice in the first term of the second equation, and once in the second part of the second equation. Then use the fact that $v_{1i}(0) = v_{1i}(L) = 0$ (because u is specified there), $v_{2i}(0) = v_{2i}(L) = 0$ (because w is specified), and $(dv_{2i}/dx)(0) = 0$ (because dw/dx is specified at $x = 0$). In addition, we have $EI(d^2w/dx^2) = M_0$ at $x = L$. The final weak forms are given by (assuming m-parameter approximation of u and n-parameter approximation of w)

$$0 = \int_0^L \left\{ EA\frac{dv_{1i}}{dx}\left[\frac{du_m}{dx} + \frac{1}{2}\left(\frac{dw_n}{dx}\right)^2\right] - v_{1i}f \right\} dx \quad (i = 1, 2, \cdots, m) \tag{1a}$$

$$\begin{aligned} 0 = &\int_0^L \left\{ EI\frac{d^2v_{2i}}{dx^2}\frac{d^2w_n}{dx^2} + EA\frac{dv_{2i}}{dx}\frac{dw_n}{dx}\left[\frac{du_m}{dx} + \frac{1}{2}\left(\frac{dw_n}{dx}\right)^2\right] - v_{2i}q \right\} dx \\ &- \left(\frac{dv_{2i}}{dx}\right)\Bigg|_L M_0 \quad (i = 1, 2, \cdots, n) \end{aligned} \tag{1b}$$

Note that for this case the weak form is not linear in u_m or w_n. However, a functional can be constructed for this using the potential operator theory (see Oden and Reddy [10] and Reddy [3]). The functional is given by

$$\begin{aligned} \Pi(u, w) = &\int_0^L \left\{ \frac{EA}{2}\left[\left(\frac{du}{dx}\right)^2 + \frac{du}{dx}\left(\frac{dw}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^4\right] + \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 \right. \\ &\left. + uf + wq \right\} dx - \frac{dw}{dx}\Bigg|_L M_0 \end{aligned}$$

2.7 A general second-order equation:

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + f = 0 \quad \text{in} \quad \Omega$$

$$u = u_0 \quad \text{on} \quad \Gamma_1, \qquad \left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right)n_x + \left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)n_y = t_0 \quad \text{on} \quad \Gamma_2$$

where $a_{ij} = a_{ji}$ $(i, j = 1, 2)$ and f are given functions of position (x, y) in a two-dimensional domain Ω and u_0 and t_0 are known functions on portions Γ_1 and Γ_2 of the boundary Γ: $\Gamma_1 + \Gamma_2 = \Gamma$.

Solution: Multiplying with the weight function w_i and integrating by parts, we obtain the weak form

$$\begin{aligned} 0 = &\int_\Omega \left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_n}{\partial x} + a_{12}\frac{\partial u_n}{\partial y}\right) + \frac{\partial w_i}{\partial y}\left(a_{21}\frac{\partial u_n}{\partial x} + a_{22}\frac{\partial u_n}{\partial y}\right) + w_i f\right] dxdy \\ &- \oint_\Gamma w_i \left[\left(a_{11}\frac{\partial u_n}{\partial x} + a_{12}\frac{\partial u_n}{\partial y}\right)n_x + \left(a_{21}\frac{\partial u_n}{\partial x} + a_{22}\frac{\partial u_n}{\partial y}\right)n_y\right] ds \\ = &\int_\Omega \left[\frac{\partial w_i}{\partial x}\left(a_{11}\frac{\partial u_n}{\partial x} + a_{12}\frac{\partial u_n}{\partial y}\right) + \frac{\partial w_i}{\partial y}\left(a_{21}\frac{\partial u_n}{\partial x} + a_{22}\frac{\partial u_n}{\partial y}\right) + w_i f\right] dxdy \\ &- \int_{\Gamma_2} v t_0 \, ds \end{aligned}$$

where $w_i = 0$ on Γ_1. The bilinear form (which is symmetric only if $a_{12} = a_{21}$) and linear form are:

$$B(w_i, u_n) = \int_\Omega \left(a_{11} \frac{\partial w_i}{\partial x} \frac{\partial u_n}{\partial x} + a_{12} \frac{\partial w_i}{\partial x} \frac{\partial u_n}{\partial y} + a_{21} \frac{\partial w_i}{\partial y} \frac{\partial u_n}{\partial x} + a_{22} \frac{\partial w_i}{\partial y} \frac{\partial u_n}{\partial y} \right) dxdy$$

$$\ell(w_i) = -\int_\Omega w_i f \; dxdy + \int_{\Gamma_2} w_i \; t_0 \; ds$$

The quadratic functional, when $a_{12} = a_{21}$, is given by

$$I(u) = \frac{1}{2} \int_\Omega \left[a_{11} \left(\frac{\partial u}{\partial x} \right)^2 + 2a_{12} \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + a_{22} \left(\frac{\partial u}{\partial y} \right)^2 \right] dxdy + \int_\Omega uf \; dxdy - \int_{\Gamma_2} u \; t_0 \; ds$$

2.8 Two-dimensional heat transfer in an axisymmetric geometry:

$$-\left[\frac{1}{r} \frac{\partial}{\partial r} \left(rk \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) \right] = g \quad \text{in} \quad \Omega \tag{1}$$

$$T = \hat{T} \quad \text{on} \quad \Gamma_T \quad \text{and} \quad q_n \equiv rk \left(\frac{\partial T_n}{\partial r} n_r + \frac{\partial T_n}{\partial z} n_z \right) = \hat{q}_n \quad \text{on} \quad \Gamma_q \tag{2}$$

Solution: The weak form of this equation can be developed following the three-step procedure with the area element being $dA = rdrdz$ and $w_i = 0$ on Γ_T:

$$\begin{aligned} 0 &= -\int_\Omega w_i \left\{ \left[\frac{1}{r} \frac{\partial}{\partial r} \left(rk \frac{\partial T_n}{\partial r} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T_n}{\partial z} \right) \right] + g \right\} rdrdz \\ &= \int_\Omega \left(k \frac{\partial w_i}{\partial r} \frac{\partial T_n}{\partial r} + \frac{\partial w_i}{\partial z} \frac{\partial T_n}{\partial z} - w_i g \right) rdrdz \\ &\quad - \oint_\Gamma rkw_i \left(\frac{\partial T_n}{\partial r} n_r + \frac{\partial T_n}{\partial z} n_z \right) ds \\ &= \int_\Omega \left(k \frac{\partial w_i}{\partial r} \frac{\partial T_n}{\partial r} + \frac{\partial w_i}{\partial z} \frac{\partial T_n}{\partial z} - w_i g \right) rdrdz - \int_{\Gamma_q} w_i \hat{q}_n \, ds \end{aligned} \tag{3}$$

where (n_r, n_z) are the direction cosines of unit normal to the boundary. The bilinear and linear forms are

$$\begin{aligned} B(w_i, T_n) &= \int_\Omega k \left(\frac{\partial w_i}{\partial r} \frac{\partial T_n}{\partial r} + \frac{\partial w_i}{\partial z} \frac{\partial T_n}{\partial z} \right) rdrdz \\ \ell(w_i) &= \int_\Omega w_i g \, rdrdz + \int_{\Gamma_q} w_i \hat{q}_n \, ds \end{aligned} \tag{4}$$

The quadratic functional is given by

$$I(T) = \int_\Omega \frac{k}{2} \left[\left(\frac{\partial T}{\partial r} \right)^2 + \left(\frac{\partial T}{\partial z} \right)^2 \right] rdrdz - \int_\Omega Tg \, rdrdz - \int_{\Gamma_q} T\hat{q}_n \tag{5}$$

2.9 Governing equations for two-dimensional flow of viscous incompressible fluids:

$$
\begin{aligned}
-\mu\left[2\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\right]+\frac{\partial P}{\partial x}&=f_x\\
-\mu\left[2\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\right]+\frac{\partial P}{\partial y}&=f_y \quad \text{in } \Omega\\
\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}&=0
\end{aligned}
\tag{1}
$$

$$
v_x=\hat{v}_x, \quad v_y=\hat{v}_y \quad \text{on} \quad \Gamma_1 \tag{2}
$$

and

$$
\left.\begin{aligned}
\left[\left(2\mu\frac{\partial v_x}{\partial x}-P\right)n_x+\mu\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)n_y\right]&=\hat{t}_x\\
\left[\left(2\mu\frac{\partial v_y}{\partial y}-P\right)n_y+\mu\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)n_x\right]&=\hat{t}_y
\end{aligned}\right. \quad \text{on } \Gamma_2 \tag{3}
$$

Solution: For this set of three differential equations in two dimensions (see Chapter 11 and Reddy [7] for the physics behind the equations), we follow exactly the same procedure as before: use the three-step procedure for each equation. In the second step of the formulation, we must integrate by parts the terms involving P, u, and v, because these terms are required as a part of the natural boundary conditions given in Eq. (3). No integration by parts is used in the third equation, because the boundary terms resulting from such integration-by-parts do not constitute physical variables. We have

$$
\begin{aligned}
0&=\int_\Omega w_{1i}\left\{-\mu\left[2\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\right]+\frac{\partial P}{\partial x}-f_x\right\}dxdy\\
&=\int_\Omega\left[\frac{\partial w_{1i}}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}-P\right)+\mu\frac{\partial w_{1i}}{\partial y}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)-w_{1i}f_x\right]dxdy-\int_{\Gamma_2}w_{1i}\hat{t}_x\,ds
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
0&=\int_\Omega w_{2i}\left\{-\mu\left[2\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\right]+\frac{\partial P}{\partial y}-f_y\right\}dxdy\\
&=\int_\Omega\left[\frac{\partial w_{2i}}{\partial y}\left(2\mu\frac{\partial v_x}{\partial y}-P\right)+\mu\frac{\partial w_{2i}}{\partial x}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)-w_{2i}f_y\right]dxdy-\int_{\Gamma_2}w_{2i}\hat{t}_y\,ds
\end{aligned}
\tag{5}
$$

$$
0=\int_\Omega w_{3i}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dxdy \tag{6}
$$

where (w_{1i}, w_{2i}, w_{3i}) are weight functions such that $w_{1i}=w_{2i}=0$ on Γ_1. The bilinear form may be identified by adding all three equations in (4)–(6) and grouping the terms appropriately [$\mathbf{w}_i = (w_{1i}, w_{2i}, w_{3i})$ and $\mathbf{v} = (v_x, v_y)$]

$$B(\mathbf{w}_i, (\mathbf{v}, P)) = \int_\Omega \mu \left[2\left(\frac{\partial w_{1i}}{\partial x}\frac{\partial v_x}{\partial x} + \frac{\partial w_{2i}}{\partial y}\frac{\partial v_y}{\partial y} \right) + \left(\frac{\partial w_{1i}}{\partial y} + \frac{\partial w_{2i}}{\partial x} \right)\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \right] dxdy$$
$$- \int_\Omega \left[\left(\frac{\partial w_{1i}}{\partial x} + \frac{\partial w_{2i}}{\partial y} \right) P + \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) w_{3i} \right] dxdy \tag{7}$$

$$\ell(\mathbf{w}_i) = \int_\Omega (f_x w_{1i} + f_y w_{2i})\, dxdy + \int_{\Gamma_2} (\hat{t}_x w_{1i} + \hat{t}_y w_{2i})\, ds \tag{8}$$

Since $B(\cdot,\cdot)$ is symmetric and $\ell(\cdot)$ is linear in their arguments, we can construct the quadratic functional using Eq. (2.4.25) as follows [set $w_{1i} = v_x$, $w_{2i} = v_y$, and $w_{3i} = P$ in $B(\cdot,\cdot)$ and $\ell(\cdot)$ before substituting into Eq. (2.4.25)]:

$$I(\mathbf{v}, P) = \frac{1}{2}\int_\Omega \mu \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)^2 \right] dxdy$$
$$- \int_\Omega \left[\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) P + (f_x v_x + f_y v_y) \right] dxdy - \int_{\Gamma_2} (\hat{t}_x v_x + \hat{t}_y v_y)\, ds \tag{9}$$

2.10 Compute the coefficient matrix and the right-hand side of the n-parameter Ritz approximation of the equation

$$-\frac{d}{dx}\left[(1+x)\frac{du}{dx} \right] = 0 \quad \text{for} \quad 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 1$$

Use algebraic polynomials for the approximation functions. Specialize your result for $n = 2$ and compute the Ritz coefficients. *Answer:* $c_1 = \frac{55}{131}$ and $c_2 = -\frac{20}{131}$.

Solution: The weak form for this problem is given by

$$0 = \int_0^1 (1+x)\frac{dw}{dx}\frac{du}{dx}\, dx = B(w, u); \quad \ell(w) = 0$$

The variational problem is given by Eqs. (2.5.1) and (2.5.2), where [$\ell(\phi_i) = 0$ because there is no source term], and the Ritza equations are given by Eqs. (2.5.5) and (2.5.6) with

$$K_{ij} = B(\phi_i, \phi_j) = \int_0^1 (1+x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} dx \tag{1a}$$

$$F_i = -B(\phi_i, \phi_0) = -\int_0^1 (1+x)\frac{d\phi_i}{dx}\frac{d\phi_0}{dx} dx \tag{1b}$$

The approximation functions ϕ_0 and ϕ_i should be chosen such that

$$\phi_0(0) = 0,\ \phi_0(1) = 1\ ; \quad \phi_i(0) = \phi_i(1) = 0,\ (i = 1, 2, ..., n) \tag{2}$$

The following algebraic polynomials satisfy the above requirements:

$$\phi_0 = x\ , \quad \phi_i = x^i(1-x) \quad (i = 1, 2, \ldots, n) \tag{3}$$