

## Chapter 1. Heat Equation

### Section 1.2

1.2.9 (d) Circular cross section means that  $P = 2\pi r$ ,  $A = \pi r^2$ , and thus  $P/A = 2/r$ , where  $r$  is the radius. Also  $\gamma = 0$ .

1.2.9 (e)  $u(x, t) = u(t)$  implies that

$$c\rho \frac{du}{dt} = -\frac{2h}{r}u.$$

The solution of this first-order linear differential equation with constant coefficients, which satisfies the initial condition  $u(0) = u_0$ , is

$$u(t) = u_0 \exp\left[-\frac{2h}{c\rho r}t\right].$$

### Section 1.3

1.3.2  $\partial u/\partial x$  is continuous if  $K_0(x_0-) = K_0(x_0+)$ , that is, if the conductivity is continuous.

### Section 1.4

1.4.1 (a) Equilibrium satisfies (1.4.14),  $d^2u/dx^2 = 0$ , whose general solution is (1.4.17),  $u = c_1 + c_2x$ . The boundary condition  $u(0) = 0$  implies  $c_1 = 0$  and  $u(L) = T$  implies  $c_2 = T/L$  so that  $u = Tx/L$ .

1.4.1 (d) Equilibrium satisfies (1.4.14),  $d^2u/dx^2 = 0$ , whose general solution (1.4.17),  $u = c_1 + c_2x$ . From the boundary conditions,  $u(0) = T$  yields  $T = c_1$  and  $du/dx(L) = \alpha$  yields  $\alpha = c_2$ . Thus  $u = T + \alpha x$ .

1.4.1 (f) In equilibrium, (1.2.9) becomes  $d^2u/dx^2 = -Q/K_0 = -x^2$ , whose general solution (by integrating twice) is  $u = -x^4/12 + c_1 + c_2x$ . The boundary condition  $u(0) = T$  yields  $c_1 = T$ , while  $du/dx(L) = 0$  yields  $c_2 = L^3/3$ . Thus  $u = -x^4/12 + L^3x/3 + T$ .

1.4.1 (h) Equilibrium satisfies  $d^2u/dx^2 = 0$ . One integration yields  $du/dx = c_2$ , the second integration yields the general solution  $u = c_1 + c_2x$ .

$$\begin{aligned} x = 0 : \quad c_2 - (c_1 - T) &= 0 \\ x = L : \quad c_2 &= \alpha \text{ and thus } c_1 = T + \alpha. \end{aligned}$$

Therefore,  $u = (T + \alpha) + \alpha x = T + \alpha(x + 1)$ .

1.4.7 (a) For equilibrium:

$$\frac{d^2u}{dx^2} = -1 \text{ implies } u = -\frac{x^2}{2} + c_1x + c_2 \text{ and } \frac{du}{dx} = -x + c_1.$$

From the boundary conditions  $\frac{du}{dx}(0) = 1$  and  $\frac{du}{dx}(L) = \beta$ ,  $c_1 = 1$  and  $-L + c_1 = \beta$  which is consistent only if  $\beta + L = 1$ . If  $\beta = 1 - L$ , there is an equilibrium solution ( $u = -\frac{x^2}{2} + x + c_2$ ). If  $\beta \neq 1 - L$ , there isn't an equilibrium solution. The difficulty is caused by the heat flow being specified at both ends and a source specified inside. An equilibrium will exist only if these three are in balance. This balance can be mathematically verified from conservation of energy:

$$\frac{d}{dt} \int_0^L c\rho u \, dx = -\frac{du}{dx}(0) + \frac{du}{dx}(L) + \int_0^L Q_0 \, dx = -1 + \beta + L.$$

If  $\beta + L = 1$ , then the total thermal energy is constant and the initial energy = the final energy:

$$\int_0^L f(x) \, dx = \int_0^L \left(-\frac{x^2}{2} + x + c_2\right) \, dx, \text{ which determines } c_2.$$

If  $\beta + L \neq 1$ , then the total thermal energy is always changing in time and an equilibrium is never reached.

## Section 1.5

- 1.5.9 (a) In equilibrium, (1.5.14) using (1.5.19) becomes  $\frac{d}{dr} \left( r \frac{du}{dr} \right) = 0$ . Integrating once yields  $r du/dr = c_1$  and integrating a second time (after dividing by  $r$ ) yields  $u = c_1 \ln r + c_2$ . An alternate general solution is  $u = c_1 \ln(r/r_1) + c_3$ . The boundary condition  $u(r_1) = T_1$  yields  $c_3 = T_1$ , while  $u(r_2) = T_2$  yields  $c_1 = (T_2 - T_1)/\ln(r_2/r_1)$ . Thus,  $u = \frac{1}{\ln(r_2/r_1)} [(T_2 - T_1) \ln r/r_1 + T_1 \ln(r_2/r_1)]$ .
- 1.5.11 For equilibrium, the radial flow at  $r = a$ ,  $2\pi a\beta$ , must equal the radial flow at  $r = b$ ,  $2\pi b$ . Thus  $\beta = b/a$ .
- 1.5.13 From exercise 1.5.12, in equilibrium  $\frac{d}{dr} \left( r^2 \frac{du}{dr} \right) = 0$ . Integrating once yields  $r^2 du/dr = c_1$  and integrating a second time (after dividing by  $r^2$ ) yields  $u = -c_1/r + c_2$ . The boundary conditions  $u(4) = 80$  and  $u(1) = 0$  yields  $80 = -c_1/4 + c_2$  and  $0 = -c_1 + c_2$ . Thus  $c_1 = c_2 = 320/3$  or  $u = \frac{320}{3} \left( 1 - \frac{1}{r} \right)$ .

## Chapter 2. Method of Separation of Variables

### Section 2.3

2.3.1 (a)  $u(r, t) = \phi(r)h(t)$  yields  $\phi \frac{dh}{dt} = \frac{kh}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right)$ . Dividing by  $k\phi h$  yields  $\frac{1}{kh} \frac{dh}{dt} = \frac{1}{r\phi} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) = -\lambda$  or  $\frac{dh}{dt} = -\lambda kh$  and  $\frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) = -\lambda \phi$ .

2.3.1 (c)  $u(x, y) = \phi(x)h(y)$  yields  $h \frac{d^2\phi}{dx^2} + \phi \frac{d^2h}{dy^2} = 0$ . Dividing by  $\phi h$  yields  $\frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\frac{1}{h} \frac{d^2h}{dy^2} = -\lambda$  or  $\frac{d^2\phi}{dx^2} = -\lambda \phi$  and  $\frac{d^2h}{dy^2} = \lambda h$ .

2.3.1 (e)  $u(x, t) = \phi(x)h(t)$  yields  $\phi(x) \frac{dh}{dt} = kh(t) \frac{d^4\phi}{dx^4}$ . Dividing by  $k\phi h$ , yields  $\frac{1}{kh} \frac{dh}{dt} = \frac{1}{\phi} \frac{d^4\phi}{dx^4} = \lambda$ .

2.3.1 (f)  $u(x, t) = \phi(x)h(t)$  yields  $\phi(x) \frac{d^2h}{dt^2} = c^2 h(t) \frac{d^2\phi}{dx^2}$ . Dividing by  $c^2\phi h$ , yields  $\frac{1}{c^2 h} \frac{d^2h}{dt^2} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\lambda$ .

2.3.2 (b)  $\lambda = (n\pi/L)^2$  with  $L = 1$  so that  $\lambda = n^2\pi^2$ ,  $n = 1, 2, \dots$

2.3.2 (d)

(i) If  $\lambda > 0$ ,  $\phi = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$ .  $\phi(0) = 0$  implies  $c_1 = 0$ , while  $\frac{d\phi}{dx}(L) = 0$  implies  $c_2 \sqrt{\lambda} \cos \sqrt{\lambda}L = 0$ . Thus  $\sqrt{\lambda}L = -\pi/2 + n\pi$  ( $n = 1, 2, \dots$ ).

(ii) If  $\lambda = 0$ ,  $\phi = c_1 + c_2x$ .  $\phi(0) = 0$  implies  $c_1 = 0$  and  $d\phi/dx(L) = 0$  implies  $c_2 = 0$ . Therefore  $\lambda = 0$  is not an eigenvalue.

(iii) If  $\lambda < 0$ , let  $\lambda = -s$  and  $\phi = c_1 \cosh \sqrt{s}x + c_2 \sinh \sqrt{s}x$ .  $\phi(0) = 0$  implies  $c_1 = 0$  and  $d\phi/dx(L) = 0$  implies  $c_2 \sqrt{s} \cosh \sqrt{s}L = 0$ . Thus  $c_2 = 0$  and hence there are no eigenvalues with  $\lambda < 0$ .

2.3.2 (f) The simplest method is to let  $x' = x - a$ . Then  $d^2\phi/dx'^2 + \lambda\phi = 0$  with  $\phi(0) = 0$  and  $\phi(b-a) = 0$ . Thus (from p. 46)  $L = b - a$  and  $\lambda = [n\pi/(b-a)]^2$ ,  $n = 1, 2, \dots$

2.3.3 From (2.3.30),  $u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}$ . The initial condition yields  $2 \cos \frac{3\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$ . From (2.3.35),  $B_n = \frac{2}{L} \int_0^L 2 \cos \frac{3\pi x}{L} \sin \frac{n\pi x}{L} dx$ .

2.3.4 (a) Total heat energy =  $\int_0^L c\rho u A dx = c\rho A \sum_{n=1}^{\infty} B_n e^{-k(\frac{n\pi}{L})^2 t} \frac{1 - \cos \frac{n\pi}{L}}$ , using (2.3.30) where  $B_n$  satisfies (2.3.35).

2.3.4 (b)

heat flux to right =  $-K_0 \partial u / \partial x$

total heat flow to right =  $-K_0 A \partial u / \partial x$

heat flow out at  $x = 0 = K_0 A \frac{\partial u}{\partial x} \Big|_{x=0}$

heat flow out ( $x = L$ ) =  $-K_0 A \frac{\partial u}{\partial x} \Big|_{x=L}$

2.3.4 (c) From conservation of thermal energy,  $\frac{d}{dt} \int_0^L u dx = k \frac{\partial u}{\partial x} \Big|_0^L = k \frac{\partial u}{\partial x}(L) - k \frac{\partial u}{\partial x}(0)$ . Integrating from  $t = 0$  yields

$$\underbrace{\int_0^L u(x, t) dx}_{\text{heat energy at } t} - \underbrace{\int_0^L u(x, 0) dx}_{\text{initial heat energy}} = k \underbrace{\int_0^t \left[ \frac{\partial u}{\partial x}(L) - \frac{\partial u}{\partial x}(0) \right] dx}_{\text{integral of flow in at } x=L} - \underbrace{\int_0^t \left[ \frac{\partial u}{\partial x}(L) - \frac{\partial u}{\partial x}(0) \right] dx}_{\text{integral of flow out at } x=L}.$$

2.3.8 (a) The general solution of  $k \frac{d^2 u}{dx^2} = \alpha u$  ( $\alpha > 0$ ) is  $u(x) = a \cosh \sqrt{\frac{\alpha}{k}}x + b \sinh \sqrt{\frac{\alpha}{k}}x$ . The boundary condition  $u(0) = 0$  yields  $a = 0$ , while  $u(L) = 0$  yields  $b = 0$ . Thus  $u = 0$ .

2.3.8 (b) Separation of variables,  $u = \phi(x)h(t)$  or  $\phi \frac{dh}{dt} + \alpha\phi h = kh \frac{d^2\phi}{dx^2}$ , yields two ordinary differential equations (divide by  $k\phi h$ ):  $\frac{1}{kh} \frac{dh}{dt} + \frac{\alpha}{k} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\lambda$ . Applying the boundary conditions, yields the eigenvalues  $\lambda = (n\pi/L)^2$  and corresponding eigenfunctions  $\phi = \sin \frac{n\pi x}{L}$ . The time-dependent part are exponentials,  $h = e^{-\lambda kt} e^{-\alpha t}$ . Thus by superposition,  $u(x, t) = e^{-\alpha t} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}$ , where the initial conditions  $u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$  yields  $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$ . As  $t \rightarrow \infty$ ,  $u \rightarrow 0$ , the only equilibrium solution.

2.3.9 (a) If  $\alpha < 0$ , the general equilibrium solution is  $u(x) = a \cos \sqrt{\frac{-\alpha}{k}} x + b \sin \sqrt{\frac{-\alpha}{k}} x$ . The boundary condition  $u(0) = 0$  yields  $a = 0$ , while  $u(L) = 0$  yields  $b \sin \sqrt{\frac{-\alpha}{k}} L = 0$ . Thus if  $\sqrt{\frac{-\alpha}{k}} L \neq n\pi$ ,  $u = 0$  is the only equilibrium solution. However, if  $\sqrt{\frac{-\alpha}{k}} L = n\pi$ , then  $u = A \sin \frac{n\pi x}{L}$  is an equilibrium solution.

2.3.9 (b) Solution obtained in 2.3.8 is correct. If  $-\frac{\alpha}{k} = \left(\frac{\pi}{L}\right)^2$ ,  $u(x, t) \rightarrow b_1 \sin \frac{\pi x}{L}$ , the equilibrium solution. If  $-\frac{\alpha}{k} < \left(\frac{\pi}{L}\right)^2$ , then  $u \rightarrow 0$  as  $t \rightarrow \infty$ . However, if  $-\frac{\alpha}{k} > \left(\frac{\pi}{L}\right)^2$ ,  $u \rightarrow \infty$  (if  $b_1 \neq 0$ ). Note that  $b_1 > 0$  if  $f(x) \geq 0$ . Other more unusual events can occur if  $b_1 = 0$ . [Essentially, the other possible equilibrium solutions are unstable.]

## Section 2.4

2.4.1 The solution is given by (2.4.19), where the coefficients satisfy (2.4.21) and hence (2.4.23-24).

$$(a) A_0 = \frac{1}{L} \int_{L/2}^L 1 dx = \frac{1}{2}, A_n = \frac{2}{L} \int_{L/2}^L \cos \frac{n\pi x}{L} dx = \frac{2}{L} \cdot \frac{L}{n\pi} \sin \frac{n\pi x}{L} \Big|_{L/2}^L = -\frac{2}{n\pi} \sin \frac{n\pi}{2}$$

(b) by inspection  $A_0 = 6, A_3 = 4$ , others = 0.

$$(c) A_0 = \frac{-2}{L} \int_0^L \sin \frac{\pi x}{L} dx = \frac{2}{\pi} \cos \frac{\pi x}{L} \Big|_0^L = \frac{2}{\pi} (1 - \cos \pi) = 4/\pi, A_n = \frac{-4}{L} \int_0^L \sin \frac{\pi x}{L} \cos \frac{n\pi x}{L} dx$$

(d) by inspection  $A_8 = -3$ , others = 0.

2.4.3 Let  $x' = x - \pi$ . Then the boundary value problem becomes  $d^2\phi/dx'^2 = -\lambda\phi$  subject to  $\phi(-\pi) = \phi(\pi)$  and  $d\phi/dx'(-\pi) = d\phi/dx'(\pi)$ . Thus, the eigenvalues are  $\lambda = (n\pi/L)^2 = n^2\pi^2$ , since  $L = \pi, n = 0, 1, 2, \dots$  with the corresponding eigenfunctions being both  $\sin n\pi x'/L = \sin n(x-\pi) = (-1)^n \sin nx \Rightarrow \sin nx$  and  $\cos n\pi x'/L = \cos n(x-\pi) = (-1)^n \cos nx \Rightarrow \cos nx$ .

## Section 2.5

2.5.1 (a) Separation of variables,  $u(x, y) = h(x)\phi(y)$ , implies that  $\frac{1}{h} \frac{d^2h}{dx^2} = -\frac{1}{\phi} \frac{d^2\phi}{dy^2} = -\lambda$ . Thus  $d^2h/dx^2 = -\lambda h$  subject to  $h'(0) = 0$  and  $h'(L) = 0$ . Thus as before,  $\lambda = (n\pi/L)^2, n = 0, 1, 2, \dots$  with  $h(x) = \cos n\pi x/L$ . Furthermore,  $\frac{d^2\phi}{dy^2} = \lambda\phi = \left(\frac{n\pi}{L}\right)^2 \phi$  so that

$$n = 0 : \phi = c_1 + c_2 y, \text{ where } \phi(0) = 0 \text{ yields } c_1 = 0$$

$$n \neq 0 : \phi = c_1 \cosh \frac{n\pi y}{L} + c_2 \sinh \frac{n\pi y}{L}, \text{ where } \phi(0) = 0 \text{ yields } c_1 = 0.$$

The result of superposition is

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}.$$

The nonhomogeneous boundary condition yields

$$f(x) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi H}{L} \cos \frac{n\pi x}{L},$$

so that

$$A_0 H = \frac{1}{L} \int_0^L f(x) dx \text{ and } A_n \sinh \frac{n\pi H}{L} = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx.$$

2.5.1 (c) Separation of variables,  $u = h(x)\phi(y)$ , yields  $\frac{1}{h} \frac{d^2 h}{dx^2} = -\frac{1}{\phi} \frac{d^2 \phi}{dy^2} = \lambda$ . The boundary conditions  $\phi(0) = 0$  and  $\phi(H) = 0$  yield an eigenvalue problem in  $y$ , whose solution is  $\lambda = (n\pi/H)^2$  with  $\phi = \sin n\pi y/H, n = 1, 2, 3, \dots$ . The solution of the  $x$ -dependent equation is  $h(x) = \cosh n\pi x/H$  using  $dh/dx(0) = 0$ . By superposition:

$$u(x, y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi x}{H} \sin \frac{n\pi y}{H}.$$

The nonhomogeneous boundary condition at  $x = L$  yields  $g(y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi L}{H} \sin \frac{n\pi y}{H}$ , so that  $A_n$  is determined by  $A_n \cosh \frac{n\pi L}{H} = \frac{2}{H} \int_0^H g(y) \sin \frac{n\pi y}{H} dy$ .

2.5.1 (e) Separation of variables,  $u = \phi(x)h(y)$ , yields the eigenvalues  $\lambda = (n\pi/L)^2$  and corresponding eigenfunctions  $\phi = \sin n\pi x/L, n = 1, 2, 3, \dots$ . The  $y$ -dependent differential equation,  $\frac{d^2 h}{dy^2} = \left(\frac{n\pi}{L}\right)^2 h$ , satisfies  $h(0) - \frac{dh}{dy}(0) = 0$ . The general solution  $h = c_1 \cosh \frac{n\pi y}{L} + c_2 \sinh \frac{n\pi y}{L}$  obeys  $h(0) = c_1$ , while  $\frac{dh}{dy} = \frac{n\pi}{L} (c_1 \sinh \frac{n\pi y}{L} + c_2 \cosh \frac{n\pi y}{L})$  obeys  $\frac{dh}{dy}(0) = c_2 \frac{n\pi}{L}$ . Thus,  $c_1 = c_2 \frac{n\pi}{L}$  and hence  $h_n(y) = \cosh \frac{n\pi y}{L} + \frac{L}{n\pi} \sinh \frac{n\pi y}{L}$ . Superposition yields

$$u(x, y) = \sum_{n=1}^{\infty} A_n h_n(y) \sin n\pi x/L,$$

where  $A_n$  is determined from the boundary condition,  $f(x) = \sum_{n=1}^{\infty} A_n h_n(H) \sin n\pi x/L$ , and hence

$$A_n h_n(H) = \frac{2}{L} \int_0^L f(x) \sin n\pi x/L dx.$$

2.5.2 (a) From physical reasoning (or exercise 1.5.8), the total heat flow across the boundary must equal zero in equilibrium (without sources, i.e. Laplace's equation). Thus  $\int_0^L f(x) dx = 0$  for a solution.

2.5.3 In order for  $u$  to be bounded as  $r \rightarrow \infty, c_1 = 0$  in (2.5.43) and  $\bar{c}_2 = 0$  in (2.5.44). Thus,

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^{-n} \cos n\theta + \sum_{n=1}^{\infty} B_n r^{-n} \sin n\theta.$$

(a) The boundary condition yields  $A_0 = \ln 2, A_3 a^{-3} = 4$ , other  $A_n = 0, B_n = 0$ .

(b) The boundary conditions yield (2.5.46) with  $a^{-n}$  replacing  $a^n$ . Thus, the coefficients are determined by (2.5.47) with  $a^n$  replaced by  $a^{-n}$ .

2.5.4 By substituting (2.5.47) into (2.5.45) and interchanging the orders of summation and integration

$$u(r, \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\bar{\theta}) \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (\cos n\theta \cos n\bar{\theta} + \sin n\theta \sin n\bar{\theta}) \right] d\bar{\theta}.$$

Noting the trigonometric addition formula and  $\cos z = \text{Re}[e^{iz}]$ , we obtain

$$u(r, \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\bar{\theta}) \left[ -\frac{1}{2} + \text{Re} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta-\bar{\theta})} \right] d\bar{\theta}.$$

Summing the geometric series enables the bracketed term to be replaced by

$$-\frac{1}{2} + \text{Re} \frac{1}{1 - \frac{r}{a} e^{i(\theta-\bar{\theta})}} = -\frac{1}{2} + \frac{1 - \frac{r}{a} \cos(\theta-\bar{\theta})}{1 + \frac{r^2}{a^2} - \frac{2r}{a} \cos(\theta-\bar{\theta})} = \frac{\frac{1}{2} - \frac{1}{2} \frac{r^2}{a^2}}{1 + \frac{r^2}{a^2} - \frac{2r}{a} \cos(\theta-\bar{\theta})}.$$

2.5.5 (a) The eigenvalue problem is  $d^2\phi/d\theta^2 = -\lambda\phi$  subject to  $d\phi/d\theta(0) = 0$  and  $\phi(\pi/2) = 0$ . It can be shown that  $\lambda > 0$  so that  $\phi = \cos\sqrt{\lambda}\theta$  where  $\phi(\pi/2) = 0$  implies that  $\cos\sqrt{\lambda}\pi/2 = 0$  or  $\sqrt{\lambda}\pi/2 = -\pi/2 + n\pi, n = 1, 2, 3, \dots$ . The eigenvalues are  $\lambda = (2n - 1)^2$ . The radially dependent term satisfies (2.5.40), and hence the boundedness condition at  $r = 0$  yields  $G(r) = r^{2n-1}$ . Superposition yields

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^{2n-1} \cos(2n - 1)\theta.$$

The nonhomogeneous boundary condition becomes

$$f(\theta) = \sum_{n=1}^{\infty} A_n \cos(2n - 1)\theta \quad \text{or} \quad A_n = \frac{4}{\pi} \int_0^{\pi/2} f(\theta) \cos(2n - 1)\theta d\theta.$$

2.5.5 (c) The boundary conditions of (2.5.37) must be replaced by  $\phi(0) = 0$  and  $\phi(\pi/2) = 0$ . Thus,  $L = \pi/2$ , so that  $\lambda = (n\pi/L)^2 = (2n)^2$  and  $\phi = \sin \frac{n\pi\theta}{L} = \sin 2n\theta$ . The radial part that remains bounded at  $r = 0$  is  $G = r^{\sqrt{\lambda}} = r^{2n}$ . By superposition,

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin 2n\theta.$$

To apply the nonhomogeneous boundary condition, we differentiate with respect to  $r$ :

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} A_n (2n) r^{2n-1} \sin 2n\theta.$$

The bc at  $r = 1$ ,  $f(\theta) = \sum_{n=1}^{\infty} 2nA_n \sin 2n\theta$ , determines  $A_n, 2nA_n = \frac{4}{\pi} \int_0^{\pi/2} f(\theta) \sin 2n\theta d\theta$ .

2.5.6 (a) The boundary conditions of (2.5.37) must be replaced by  $\phi(0) = 0$  and  $\phi(\pi) = 0$ . Thus  $L = \pi$ , so that the eigenvalues are  $\lambda = (n\pi/L)^2 = n^2$  and corresponding eigenfunctions  $\phi = \sin n\pi\theta/L = \sin n\theta, n = 1, 2, 3, \dots$ . The radial part which is bounded at  $r = 0$  is  $G = r^{\sqrt{\lambda}} = r^n$ . Thus by superposition

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

The bc at  $r = a$ ,  $g(\theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta$ , determines  $A_n, A_n a^n = \frac{2}{\pi} \int_0^{\pi} g(\theta) \sin n\theta d\theta$ .

2.5.7 (b) The boundary conditions of (2.5.37) must be replaced by  $\phi'(0) = 0$  and  $\phi'(\pi/3) = 0$ . This will yield a cosine series with  $L = \pi/3, \lambda = (n\pi/L)^2 = (3n)^2$  and  $\phi = \cos n\pi\theta/L = \cos 3n\theta, n = 0, 1, 2, \dots$ . The radial part which is bounded at  $r = 0$  is  $G = r^{\sqrt{\lambda}} = r^{3n}$ . Thus by superposition

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^{3n} \cos 3n\theta.$$

The boundary condition at  $r = a$ ,  $g(\theta) = \sum_{n=0}^{\infty} A_n a^{3n} \cos 3n\theta$ , determines  $A_n$ :  $A_0 = \frac{3}{\pi} \int_0^{\pi/3} g(\theta) d\theta$  and  $(n \neq 0) A_n a^{3n} = \frac{6}{\pi} \int_0^{\pi/3} g(\theta) \cos 3n\theta d\theta$ .

2.5.8 (a) There is a full Fourier series in  $\theta$ . It is easier (but equivalent) to choose radial solutions that satisfy the corresponding homogeneous boundary condition. Instead of  $r^n$  and  $r^{-n}$  ( $1$  and  $\ln r$  for  $n = 0$ ), we choose  $\phi_1(r)$  such that  $\phi_1(a) = 0$  and  $\phi_2(r)$  such that  $\phi_2(b) = 0$ :

$$\phi_1(r) = \begin{cases} \ln(r/a) & n = 0 \\ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n & n \neq 0 \end{cases} \quad \phi_2(r) = \begin{cases} \ln(r/b) & n = 0 \\ \left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n & n \neq 0 \end{cases}.$$

Then by superposition

$$u(r, \theta) = \sum_{n=0}^{\infty} \cos n\theta [A_n \phi_1(r) + B_n \phi_2(r)] + \sum_{n=1}^{\infty} \sin n\theta [C_n \phi_1(r) + D_n \phi_2(r)].$$

The boundary conditions at  $r = a$  and  $r = b$ ,

$$f(\theta) = \sum_{n=0}^{\infty} \cos n\theta [A_n \phi_1(a) + B_n \phi_2(a)] + \sum_{n=1}^{\infty} \sin n\theta [C_n \phi_1(a) + D_n \phi_2(a)]$$

$$g(\theta) = \sum_{n=0}^{\infty} \cos n\theta [A_n \phi_1(b) + B_n \phi_2(b)] + \sum_{n=1}^{\infty} \sin n\theta [C_n \phi_1(b) + D_n \phi_2(b)]$$

easily determine  $A_n, B_n, C_n, D_n$  since  $\phi_1(a) = 0$  and  $\phi_2(b) = 0 : D_n \phi_2(a) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$ , etc.

2.5.9 (a) The boundary conditions of (2.5.37) must be replaced by  $\phi(0) = 0$  and  $\phi(\pi/2) = 0$ . This is a sine series with  $L = \pi/2$  so that  $\lambda = (n\pi/L)^2 = (2n)^2$  and the eigenfunctions are  $\phi = \sin n\pi\theta/L = \sin 2n\theta, n = 1, 2, 3, \dots$ . The radial part which is zero at  $r = a$  is  $G = (r/a)^{2n} - (a/r)^{2n}$ . Thus by superposition,

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n \left[ \left( \frac{r}{a} \right)^{2n} - \left( \frac{a}{r} \right)^{2n} \right] \sin 2n\theta.$$

The nonhomogeneous boundary condition,  $f(\theta) = \sum_{n=1}^{\infty} A_n \left[ \left( \frac{b}{a} \right)^{2n} - \left( \frac{a}{b} \right)^{2n} \right] \sin 2n\theta$ , determines  $A_n : A_n \left[ \left( \frac{b}{a} \right)^{2n} - \left( \frac{a}{b} \right)^{2n} \right] = \frac{4}{\pi} \int_0^{\pi/2} f(\theta) \sin 2n\theta d\theta$ .

2.5.9 (b) The two homogeneous boundary conditions are in  $r$ , and hence  $\phi(r)$  must be an eigenvalue problem. By separation of variables,  $u = \phi(r)G(\theta), d^2G/d\theta^2 = \lambda G$  and  $r^2 \frac{d^2\phi}{dr^2} + r \frac{d\phi}{dr} + \lambda\phi = 0$ . The radial equation is equidimensional (see p.78) and solutions are in the form  $\phi = r^p$ . Thus  $p^2 = -\lambda$  (with  $\lambda > 0$ ) so that  $p = \pm i\sqrt{\lambda}$ .  $r^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda} \ln r}$ . Thus real solutions are  $\cos(\sqrt{\lambda} \ln r)$  and  $\sin(\sqrt{\lambda} \ln r)$ . It is more convenient to use independent solutions which simplify at  $r = a, \cos[\sqrt{\lambda} \ln(r/a)]$  and  $\sin[\sqrt{\lambda} \ln(r/a)]$ . Thus the general solution is

$$\phi = c_1 \cos[\sqrt{\lambda} \ln(r/a)] + c_2 \sin[\sqrt{\lambda} \ln(r/a)].$$

The homogeneous condition  $\phi(a) = 0$  yields  $0 = c_1$ , while  $\phi(b) = 0$  implies  $\sin[\sqrt{\lambda} \ln(r/a)] = 0$ . Thus  $\sqrt{\lambda} \ln(b/a) = n\pi, n = 1, 2, 3, \dots$  and the corresponding eigenfunctions are  $\phi = \sin \left[ n\pi \frac{\ln(r/a)}{\ln(b/a)} \right]$ . The solution of the  $\theta$ -equation satisfying  $G(0) = 0$  is  $G = \sinh \sqrt{\lambda}\theta = \sinh \frac{n\pi\theta}{\ln(b/a)}$ . Thus by superposition

$$u = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi\theta}{\ln(b/a)} \sin \left[ n\pi \frac{\ln(r/a)}{\ln(b/a)} \right].$$

The nonhomogeneous boundary condition,

$$f(r) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi^2}{2 \ln(b/a)} \sin \left[ n\pi \frac{\ln(r/a)}{\ln(b/a)} \right],$$

will determine  $A_n$ . One method (for another, see exercise 5.3.9) is to let  $z = \ln(r/a)/\ln(b/a)$ . Then  $a < r < b$ , lets  $0 < z < 1$ . This is a sine series in  $z$  (with  $L = 1$ ) and hence

$$A_n \sinh \frac{n\pi^2}{2 \ln(b/a)} = 2 \int_0^1 f(r) \sin \left[ n\pi \frac{\ln(r/a)}{\ln(b/a)} \right] dz.$$

But  $dz = dr/r \ln(b/a)$ . Thus

$$A_n \sinh \frac{n\pi^2}{2 \ln(b/a)} = \frac{2}{\ln(b/a)} \int_0^1 f(r) \sin \left[ n\pi \frac{\ln(r/a)}{\ln(b/a)} \right] dr/r.$$