

# CHAPTER 1

## Limits and Their Properties

### Section 1.1 A Preview of Calculus

1. Calculus is the mathematics of change. Precalculus is more static. Answers will vary. *Sample answer:*

Precalculus: Area of a rectangle

Calculus: Area under a curve

Precalculus: Work done by a constant force

Calculus: Work done by a variable force

Precalculus: Center of a rectangle

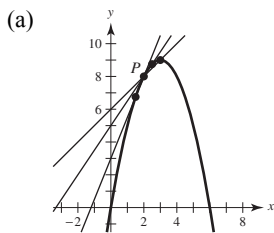
Calculus: Centroid of a region

2. A secant line through a point  $P$  is a line joining  $P$  and another point  $Q$  on the graph.

The slope of the tangent line  $P$  is the limit of the slopes of the secant lines joining  $P$  and  $Q$ , as  $Q$  approaches  $P$ .

3. Precalculus:  $(20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$
4. Calculus required: Velocity is not constant.  
 Distance  $\approx (20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$
5. Calculus required: Slope of the tangent line at  $x = 2$  is the rate of change, and equals about 0.16.
6. Precalculus: rate of change = slope = 0.08

8.  $f(x) = 6x - x^2$



(b) slope =  $m = \frac{(6x - x^2) - 8}{x - 2} = \frac{(x - 2)(4 - x)}{x - 2} = (4 - x), x \neq 2$

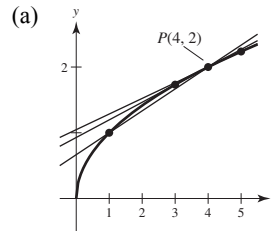
For  $x = 3, m = 4 - 3 = 1$

For  $x = 2.5, m = 4 - 2.5 = 1.5 = \frac{3}{2}$

For  $x = 1.5, m = 4 - 1.5 = 2.5 = \frac{5}{2}$

- (c) At  $P(2, 8)$ , the slope is 2. You can improve your approximation by considering values of  $x$  close to 2.

7.  $f(x) = \sqrt{x}$



(b) slope =  $m = \frac{\sqrt{x} - 2}{x - 4}$   
 $= \frac{\sqrt{x} - 2}{(\sqrt{x} + 2)(\sqrt{x} - 2)}$   
 $= \frac{1}{\sqrt{x} + 2}, x \neq 4$

$x = 1: m = \frac{1}{\sqrt{1} + 2} = \frac{1}{3}$

$x = 3: m = \frac{1}{\sqrt{3} + 2} \approx 0.2679$

$x = 5: m = \frac{1}{\sqrt{5} + 2} \approx 0.2361$

- (c) At  $P(4, 2)$  the slope is  $\frac{1}{\sqrt{4} + 2} = \frac{1}{4} = 0.25$ .

You can improve your approximation of the slope at  $x = 4$  by considering  $x$ -values very close to 4.

9. (a)  $\text{Area} \approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$   
 $\text{Area} \approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$   
 (b) You could improve the approximation by using more rectangles.

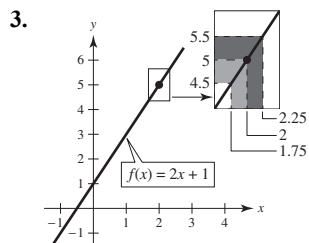
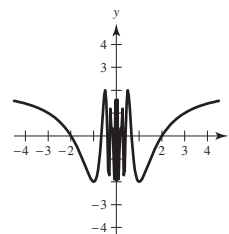
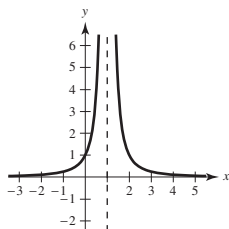
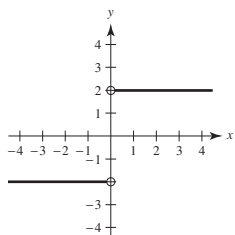
10. Answers will vary. *Sample answer:*

The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

11. (a)  $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$   
 (b)  $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$   
 $\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$   
 (c) Increase the number of line segments.

### Section 1.2 Finding Limits Graphically and Numerically

1. As the graph of the function approaches 8 on the horizontal axis, the graph approaches 25 on the vertical axis.  
 2. (i) The values of  $f$  approach different numbers as  $x$  approaches  $c$  from different sides of  $c$ :  
 (ii) The values of  $f$  increase without bound as  $x$  approaches  $c$ :  
 (iii) The values of  $f$  oscillate between two fixed numbers as  $x$  approaches  $c$ :



4. No. For example, consider Example 2 from this section.

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$$\lim_{x \rightarrow 2} f(x) = 1, \text{ but } f(2) = 0$$

5.

$x$	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	0.3448	0.3344	0.3334	?	0.3332	0.3322	0.3226

$$\lim_{x \rightarrow 4} \frac{x-4}{x^2-5x-4} \approx 0.3333 \quad \left( \text{Actual limit is } \frac{1}{3} \right)$$

6.

$x$	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	0.1695	0.1669	0.1667	?	0.1666	0.1664	0.1639

$$\lim_{x \rightarrow 3} \frac{x-3}{x^2-9} \approx 0.1667 \quad \left( \text{Actual limit is } \frac{1}{6} \right)$$

7.

$x$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} \approx 0.5000 \quad \left( \text{Actual limit is } \frac{1}{2} \right)$$

8.

$x$	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	-0.0641	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

$$\lim_{x \rightarrow 3} \frac{[1/(x+1)] - (1/4)}{x-3} \approx -0.0625 \quad \left( \text{Actual limit is } -\frac{1}{16} \right)$$

9.

$x$	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \approx 1.0000 \quad (\text{Actual limit is } 1.) \quad (\text{Make sure you use radian mode.})$$

10.

$x$	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	-0.0005	-0.0050	-0.0500

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \approx 0.0000 \quad (\text{Actual limit is } 0.) \quad (\text{Make sure you use radian mode.})$$

11.

$x$	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$$\lim_{x \rightarrow 1} \frac{x-2}{x^2+x-6} \approx 0.2500 \quad \left( \text{Actual limit is } \frac{1}{4} \right)$$

12.

$x$	-4.1	-4.01	-4.001	-4	-3.999	-3.99	-3.9
$f(x)$	1.1111	1.0101	1.0010	?	0.9990	0.9901	0.9091

$$\lim_{x \rightarrow -4} \frac{x+4}{x^2+9x+20} \approx 1.0000 \quad (\text{Actual limit is } 1.)$$

13.

$x$	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.7340	0.6733	0.6673	0.6660	0.6600	0.6015

$$\lim_{x \rightarrow 1} \frac{x^4 - 1}{x^6 - 1} \approx 0.6666 \quad \left( \text{Actual limit is } \frac{2}{3} \right)$$

14.

$x$	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9
$f(x)$	27.91	27.0901	27.0090	?	26.9910	26.9101	26.11

$$\lim_{x \rightarrow -3} \frac{x^3 + 27}{x + 3} \approx 27.0000 \quad (\text{Actual limit is } 27.)$$

15.

$x$	-6.1	-6.01	-6.001	-6	-5.999	-5.99	-5.9
$f(x)$	-0.1248	-0.1250	-0.1250	?	-0.1250	-0.1250	-0.1252

$$\lim_{x \rightarrow -6} \frac{\sqrt{10-x} - 4}{x+6} \approx -0.1250 \quad \left( \text{Actual limit is } -\frac{1}{8} \right)$$

16.

$x$	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	0.1149	0.115	0.1111	?	0.1111	0.1107	0.1075

$$\lim_{x \rightarrow 2} \frac{x/(x+1) - 2/3}{x-2} \approx 0.1111 \quad \left( \text{Actual limit is } \frac{1}{9} \right)$$

17.

$x$	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.9867	1.9999	2.0000	2.0000	1.9999	1.9867

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{x} \approx 2.0000 \quad (\text{Actual limit is } 2.) \quad (\text{Make sure you use radian mode.})$$

18.

$x$	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.4950	0.5000	0.5000	0.5000	0.5000	0.4950

$$\lim_{x \rightarrow 0} \frac{\tan x}{\tan 2x} \approx 0.5000 \quad \left( \text{Actual limit is } \frac{1}{2} \right)$$

19.  $f(x) = \frac{2}{x^3}$

$x$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-2000	$-2 \times 10^6$	$-2 \times 10^9$	?	$2 \times 10^9$	$2 \times 10^6$	2000

As  $x$  approaches 0 from the left, the function decreases without bound. As  $x$  approaches 0 from the right, the function increases without bound.

20.  $f(x) = \frac{3|x|}{x^2}$

$x$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	30	300	3000	?	3000	300	30

As  $x$  approaches 0 from either side, the function increases without bound.

21.  $\lim_{x \rightarrow 3} (4 - x) = 1$

22.  $\lim_{x \rightarrow 0} \sec x = 1$

23.  $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (4 - x) = 2$

24.  $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x^2 + 3) = 4$

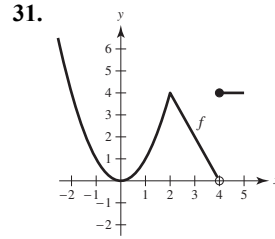
25.  $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$  does not exist.

For values of  $x$  to the left of 2,  $\frac{|x-2|}{x-2} = -1$ , whereas

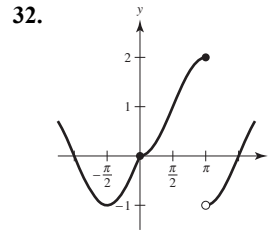
for values of  $x$  to the right of 2,  $\frac{|x-2|}{x-2} = 1$ .

26.  $\lim_{x \rightarrow 5} \frac{2}{x-5}$  does not exist because the function increases and decreases without bound as  $x$  approaches 5.

27.  $\lim_{x \rightarrow 0} \cos(1/x)$  does not exist because the function oscillates between  $-1$  and  $1$  as  $x$  approaches  $0$ .
28.  $\lim_{x \rightarrow \pi/2} \tan x$  does not exist because the function increases without bound as  $x$  approaches  $\frac{\pi}{2}$  from the left and decreases without bound as  $x$  approaches  $\frac{\pi}{2}$  from the right.
29. (a)  $f(1)$  exists. The black dot at  $(1, 2)$  indicates that  $f(1) = 2$ .
- (b)  $\lim_{x \rightarrow 1} f(x)$  does not exist. As  $x$  approaches  $1$  from the left,  $f(x)$  approaches  $3.5$ , whereas as  $x$  approaches  $1$  from the right,  $f(x)$  approaches  $1$ .
- (c)  $f(4)$  does not exist. The hollow circle at  $(4, 2)$  indicates that  $f$  is not defined at  $4$ .
- (d)  $\lim_{x \rightarrow 4} f(x)$  exists. As  $x$  approaches  $4$ ,  $f(x)$  approaches  $2$ :  $\lim_{x \rightarrow 4} f(x) = 2$ .
30. (a)  $f(-2)$  does not exist. The vertical dotted line indicates that  $f$  is not defined at  $-2$ .
- (b)  $\lim_{x \rightarrow -2} f(x)$  does not exist. As  $x$  approaches  $-2$ , the values of  $f(x)$  do not approach a specific number.
- (c)  $f(0)$  exists. The black dot at  $(0, 4)$  indicates that  $f(0) = 4$ .
- (d)  $\lim_{x \rightarrow 0} f(x)$  does not exist. As  $x$  approaches  $0$  from the left,  $f(x)$  approaches  $\frac{1}{2}$ , whereas as  $x$  approaches  $0$  from the right,  $f(x)$  approaches  $4$ .
- (e)  $f(2)$  does not exist. The hollow circle at  $(2, \frac{1}{2})$  indicates that  $f(2)$  is not defined.
- (f)  $\lim_{x \rightarrow 2} f(x)$  exists. As  $x$  approaches  $2$ ,  $f(x)$  approaches  $\frac{1}{2}$ :  $\lim_{x \rightarrow 2} f(x) = \frac{1}{2}$ .
- (g)  $f(4)$  exists. The black dot at  $(4, 2)$  indicates that  $f(4) = 2$ .
- (h)  $\lim_{x \rightarrow 4} f(x)$  does not exist. As  $x$  approaches  $4$ , the values of  $f(x)$  do not approach a specific number.

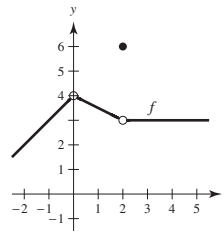


$\lim_{x \rightarrow c} f(x)$  exists for all values of  $c \neq 4$ .

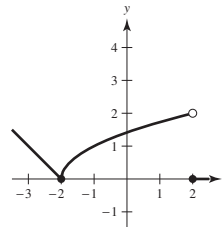


$\lim_{x \rightarrow c} f(x)$  exists for all values of  $c \neq \pi$ .

33. One possible answer is



34. One possible answer is



35. You need  $|f(x) - 3| = |(x + 1) - 3| = |x - 2| < 0.4$ .  
 So, take  $\delta = 0.4$ . If  $0 < |x - 2| < 0.4$ ,  
 then  $|x - 2| = |(x + 1) - 3| = |f(x) - 3| < 0.4$ ,  
 as desired.

36. You need  $|f(x) - 1| = \left| \frac{1}{x-1} - 1 \right| = \left| \frac{2-x}{x-1} \right| < 0.01$ . Let  $\delta = \frac{1}{101}$ . If  $0 < |x - 2| < \frac{1}{101}$ , then

$$\begin{aligned} -\frac{1}{101} < x - 2 < \frac{1}{101} &\Rightarrow 1 - \frac{1}{101} < x - 1 < 1 + \frac{1}{101} \\ &\Rightarrow \frac{100}{101} < x - 1 < \frac{102}{101} \\ &\Rightarrow |x - 1| > \frac{100}{101} \end{aligned}$$

and you have

$$|f(x) - 1| = \left| \frac{1}{x-1} - 1 \right| = \left| \frac{2-x}{x-1} \right| < \frac{1/101}{100/101} = \frac{1}{100} = 0.01.$$

37. You need to find  $\delta$  such that  $0 < |x - 1| < \delta$  implies

$$|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < 0.1. \text{ That is,}$$

$$-0.1 < \frac{1}{x} - 1 < 0.1$$

$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$

$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$

$$\frac{10}{9} > x > \frac{10}{11}$$

$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$

$$\frac{1}{9} > x - 1 > -\frac{1}{11}$$

So take  $\delta = \frac{1}{11}$ . Then  $0 < |x - 1| < \delta$  implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$

$$-\frac{1}{11} < x - 1 < \frac{1}{9}$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < 0.1.$$

38. You need to find  $\delta$  such that  $0 < |x - 1| < \delta$  implies

$$|f(x) - 1| = \left| 2 - \frac{1}{x} - 1 \right| = \left| 1 - \frac{1}{x} \right| < \varepsilon.$$

$$-\varepsilon < \frac{1}{x} - 1 < \varepsilon$$

$$1 - \varepsilon < \frac{1}{x} < 1 + \varepsilon$$

$$\frac{1}{1 - \varepsilon} > x > \frac{1}{1 + \varepsilon}$$

$$\frac{1}{1 - \varepsilon} - 1 > x - 1 > \frac{1}{1 + \varepsilon} - 1$$

$$\frac{\varepsilon}{1 - \varepsilon} > x - 1 > \frac{-\varepsilon}{1 + \varepsilon}$$

For  $\varepsilon = 0.05$ , take  $\delta = \frac{0.05}{1 - 0.05} \approx 0.05$ .

For  $\varepsilon = 0.01$ , take  $\delta = \frac{0.01}{1 - 0.01} \approx 0.01$ .

For  $\varepsilon = 0.005$ , take  $\delta = \frac{0.005}{1 - 0.005} \approx 0.005$ .

As  $\varepsilon$  decreases, so does  $\delta$ .

$$39. \lim_{x \rightarrow 2} (3x + 2) = 3(2) + 2 = 8 = L$$

$$(a) \quad |(3x + 2) - 8| < 0.01$$

$$|3x - 6| < 0.01$$

$$3|x - 2| < 0.01$$

$$0 < |x - 2| < \frac{0.01}{3} \approx 0.0033 = \delta$$

So, if  $0 < |x - 2| < \delta = \frac{0.01}{3}$ , you have

$$3|x - 2| < 0.01$$

$$|3x - 6| < 0.01$$

$$|(3x + 2) - 8| < 0.01$$

$$|f(x) - L| < 0.01.$$

$$(b) \quad |(3x + 2) - 8| < 0.005$$

$$|3x - 6| < 0.005$$

$$3|x - 2| < 0.005$$

$$0 < |x - 2| < \frac{0.005}{3} \approx 0.00167 = \delta$$

Finally, as in part (a), if  $0 < |x - 2| < \frac{0.005}{3}$ ,

you have  $|(3x + 2) - 8| < 0.005$ .

$$40. \lim_{x \rightarrow 6} \left(6 - \frac{x}{3}\right) = 6 - \frac{6}{3} = 4 = L$$

$$(a) \quad \left| \left(6 - \frac{x}{3}\right) - 4 \right| < 0.01$$

$$\left| 2 - \frac{x}{3} \right| < 0.01$$

$$\left| -\frac{1}{3}(x - 6) \right| < 0.01$$

$$|x - 6| < 0.03$$

$$0 < |x - 6| < 0.03 = \delta$$

So, if  $0 < |x - 6| < \delta = 0.03$ , you have

$$\left| -\frac{1}{3}(x - 6) \right| < 0.01$$

$$\left| 2 - \frac{x}{3} \right| < 0.01$$

$$\left| \left(6 - \frac{x}{3}\right) - 4 \right| < 0.01$$

$$|f(x) - L| < 0.01.$$

$$(b) \quad \left| \left(6 - \frac{x}{3}\right) - 4 \right| < 0.005$$

$$\left| 2 - \frac{x}{3} \right| < 0.005$$

$$\left| -\frac{1}{3}(x - 6) \right| < 0.005$$

$$|x - 6| < 0.015$$

$$0 < |x - 6| < 0.015 = \delta$$

As in part (a), if  $0 < |x - 6| < 0.015$ , you have

$$\left| \left(6 - \frac{x}{3}\right) - 4 \right| < 0.005.$$

$$41. \lim_{x \rightarrow 2} (x^2 - 3) = 2^2 - 3 = 1 = L$$

$$(a) \quad |(x^2 - 3) - 1| < 0.01$$

$$|x^2 - 4| < 0.01$$

$$|(x + 2)(x - 2)| < 0.01$$

$$|x + 2||x - 2| < 0.01$$

$$|x - 2| < \frac{0.01}{|x + 2|}$$

If you assume  $1 < x < 3$ , then

$$\delta \approx 0.01/5 = 0.002.$$

So, if  $0 < |x - 2| < \delta \approx 0.002$ , you have

$$|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x + 2|}(0.01)$$

$$|x + 2||x - 2| < 0.01$$

$$|x^2 - 4| < 0.01$$

$$|(x^2 - 3) - 1| < 0.01$$

$$|f(x) - L| < 0.01.$$

$$(b) \quad |(x^2 - 3) - 1| < 0.005$$

$$|x^2 - 4| < 0.005$$

$$|(x + 2)(x - 2)| < 0.005$$

$$|x + 2||x - 2| < 0.005$$

$$|x - 2| < \frac{0.005}{|x + 2|}$$

If you assume  $1 < x < 3$ , then

$$\delta = \frac{0.005}{5} = 0.001.$$

Finally, as in part (a), if  $0 < |x - 2| < 0.001$ ,

you have  $|(x^2 - 3) - 1| < 0.005$ .

$$42. \lim_{x \rightarrow 4} (x^2 + 6) = 4^2 + 6 = 22 = L$$

$$\begin{aligned} \text{(a)} \quad & |(x^2 + 6) - 22| < 0.01 \\ & |x^2 - 16| < 0.01 \\ & |(x + 4)(x - 4)| < 0.01 \\ & |x + 4||x - 4| < 0.01 \\ & |x - 4| < \frac{0.01}{|x + 4|} \end{aligned}$$

If you assume  $3 < x < 5$ , then

$$\delta = \frac{0.01}{9} \approx 0.00111.$$

So, if  $0 < |x - 4| < \delta \approx \frac{0.01}{9}$ , you have

$$\begin{aligned} |x - 4| &< \frac{0.01}{9} < \frac{0.01}{|x + 4|} \\ |(x + 4)(x - 4)| &< 0.01 \\ |x^2 - 16| &< 0.01 \\ |(x^2 + 6) - 22| &< 0.01 \\ |f(x) - L| &< 0.01. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & |(x^2 + 6) - 22| < 0.005 \\ & |x^2 - 16| < 0.005 \\ & |(x - 4)(x + 4)| < 0.005 \\ & |x - 4||x + 4| < 0.005 \\ & |x - 4| < \frac{0.005}{|x + 4|} \end{aligned}$$

If you assume  $3 < x < 5$ , then

$$\delta = \frac{0.005}{9} \approx 0.00056.$$

Finally, as in part (a), if  $0 < |x - 4| < \frac{0.005}{9}$ ,

you have  $|(x^2 + 6) - 22| < 0.005$ .

$$43. \lim_{x \rightarrow 4} (x^2 - x) = 16 - 4 = 12 = L$$

$$\begin{aligned} \text{(a)} \quad & |(x^2 - x) - 12| < 0.01 \\ & |(x - 4)(x + 3)| < 0.01 \\ & |x - 4||x + 3| < 0.01 \\ & |x - 4| < \frac{0.01}{|x + 3|} \end{aligned}$$

If you assume  $3 < x < 5$ , then

$$\delta = \frac{0.01}{8} = 0.00125.$$

So, if  $0 < |x - 4| < \frac{0.01}{8}$ , you have

$$\begin{aligned} |x - 4| &< \frac{0.01}{|x + 3|} \\ |x - 4||x + 3| &< 0.01 \\ |x^2 - x - 12| &< 0.01 \\ |(x^2 - x) - 12| &< 0.01 \\ |f(x) - L| &< 0.01 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & |(x^2 - x) - 12| < 0.005 \\ & |(x - 4)(x + 3)| < 0.005 \\ & |x - 4||x + 3| < 0.005 \\ & |x - 4| < \frac{0.005}{|x + 3|} \end{aligned}$$

If you assume  $3 < x < 5$ , then

$$\delta = \frac{0.005}{8} = 0.000625.$$

Finally, as in part (a), if  $0 < |x - 4| < \frac{0.005}{8}$ ,

you have  $|(x^2 - x) - 12| < 0.005$ .

$$44. \lim_{x \rightarrow 3} x^2 = 3^2 = 9 = L$$

$$(a) \quad |x^2 - 9| < 0.01$$

$$|(x-3)(x+3)| < 0.01$$

$$|x-3||x+3| < 0.01$$

$$|x-3| < \frac{0.01}{|x+3|}$$

If you assume  $2 < x < 4$ , then

$$\delta = \frac{0.01}{7} \approx 0.0014.$$

So, if  $0 < |x-3| < \frac{0.01}{7}$ , you have

$$|x-3| < \frac{0.01}{|x+3|}$$

$$|x-3||x+3| < 0.01$$

$$|x^2 - 9| < 0.01$$

$$|f(x) - L| < 0.01$$

$$(b) \quad |x^2 - 9| < 0.005$$

$$|(x-3)(x+3)| < 0.005$$

$$|x-3||x+3| < 0.005$$

$$|x-3| < \frac{0.005}{|x+3|}$$

If you assume  $2 < x < 4$ , then

$$\delta = \frac{0.005}{7} \approx 0.00071.$$

Finally, as in part (a), if  $0 < |x-3| < \frac{0.005}{7}$ , you

have  $|x^2 - 9| < 0.005$ .

$$45. \lim_{x \rightarrow 4} (x+2) = 4+2 = 6$$

Given  $\varepsilon > 0$ :

$$|(x+2) - 6| < \varepsilon$$

$$|x-4| < \varepsilon = \delta$$

So, let  $\delta = \varepsilon$ . So, if  $0 < |x-4| < \delta = \varepsilon$ , you have

$$|x-4| < \varepsilon$$

$$|(x+2) - 6| < \varepsilon$$

$$|f(x) - L| < \varepsilon.$$

$$46. \lim_{x \rightarrow -2} (4x+5) = 4(-2) + 5 = -3$$

Given  $\varepsilon > 0$ :

$$|(4x+5) - (-3)| < \varepsilon$$

$$|4x+8| < \varepsilon$$

$$4|x+2| < \varepsilon$$

$$|x+2| < \frac{\varepsilon}{4} = \delta$$

So, let  $\delta = \frac{\varepsilon}{4}$ .

So, if  $0 < |x+2| < \delta = \frac{\varepsilon}{4}$ , you have

$$|x+2| < \frac{\varepsilon}{4}$$

$$|4x+8| < \varepsilon$$

$$|(4x+5) - (-3)| < \varepsilon$$

$$|f(x) - L| < \varepsilon.$$

$$47. \lim_{x \rightarrow -4} \left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$$

Given  $\varepsilon > 0$ :

$$\left|\left(\frac{1}{2}x - 1\right) - (-3)\right| < \varepsilon$$

$$\left|\frac{1}{2}x + 2\right| < \varepsilon$$

$$\frac{1}{2}|x - (-4)| < \varepsilon$$

$$|x - (-4)| < 2\varepsilon$$

So, let  $\delta = 2\varepsilon$ .

So, if  $0 < |x - (-4)| < \delta = 2\varepsilon$ , you have

$$|x - (-4)| < 2\varepsilon$$

$$\left|\frac{1}{2}x + 2\right| < \varepsilon$$

$$\left|\left(\frac{1}{2}x - 1\right) + 3\right| < \varepsilon$$

$$|f(x) - L| < \varepsilon.$$

$$48. \lim_{x \rightarrow 3} \left( \frac{3}{4}x + 1 \right) = \frac{3}{4}(3) + 1 = \frac{13}{4}$$

Given  $\varepsilon > 0$ :

$$\left| \left( \frac{3}{4}x + 1 \right) - \frac{13}{4} \right| < \varepsilon$$

$$\left| \frac{3}{4}x - \frac{9}{4} \right| < \varepsilon$$

$$\frac{3}{4}|x - 3| < \varepsilon$$

$$|x - 3| < \frac{4}{3}\varepsilon$$

So, let  $\delta = \frac{4}{3}\varepsilon$ .

So, if  $0 < |x - 3| < \delta = \frac{4}{3}\varepsilon$ , you have

$$|x - 3| < \frac{4}{3}\varepsilon$$

$$\frac{3}{4}|x - 3| < \varepsilon$$

$$\left| \frac{3}{4}x - \frac{9}{4} \right| < \varepsilon$$

$$\left| \left( \frac{3}{4}x + 1 \right) - \frac{13}{4} \right| < \varepsilon$$

$$|f(x) - L| < \varepsilon.$$

$$49. \lim_{x \rightarrow 6} 3 = 3$$

Given  $\varepsilon > 0$ :

$$|3 - 3| < \varepsilon$$

$$0 < \varepsilon$$

So, any  $\delta > 0$  will work.

So, for any  $\delta > 0$ , you have

$$|3 - 3| < \varepsilon$$

$$|f(x) - L| < \varepsilon.$$

$$50. \lim_{x \rightarrow 2} (-1) = -1$$

$$\text{Given } \varepsilon > 0: |-1 - (-1)| < \varepsilon$$

$$0 < \varepsilon$$

So, any  $\delta > 0$  will work.

So, for any  $\delta > 0$ , you have

$$|(-1) - (-1)| < \varepsilon$$

$$|f(x) - L| < \varepsilon.$$

$$51. \lim_{x \rightarrow 0} \sqrt[3]{x} = 0$$

$$\text{Given } \varepsilon > 0: \left| \sqrt[3]{x} - 0 \right| < \varepsilon$$

$$\left| \sqrt[3]{x} \right| < \varepsilon$$

$$|x| < \varepsilon^3 = \delta$$

So, let  $\delta = \varepsilon^3$ .

So, for  $0 < |x - 0| < \delta = \varepsilon^3$ , you have

$$|x| < \varepsilon^3$$

$$\left| \sqrt[3]{x} \right| < \varepsilon$$

$$\left| \sqrt[3]{x} - 0 \right| < \varepsilon$$

$$|f(x) - L| < \varepsilon.$$

$$52. \lim_{x \rightarrow 4} \sqrt{x} = \sqrt{4} = 2$$

$$\text{Given } \varepsilon > 0: \left| \sqrt{x} - 2 \right| < \varepsilon$$

$$\left| \sqrt{x} - 2 \right| \left| \sqrt{x} + 2 \right| < \varepsilon \left| \sqrt{x} + 2 \right|$$

$$|x - 4| < \varepsilon \left| \sqrt{x} + 2 \right|$$

Assuming  $1 < x < 9$ , you can choose  $\delta = 3\varepsilon$ . Then,

$$0 < |x - 4| < \delta = 3\varepsilon \Rightarrow |x - 4| < \varepsilon \left| \sqrt{x} + 2 \right|$$

$$\Rightarrow \left| \sqrt{x} - 2 \right| < \varepsilon.$$

$$53. \lim_{x \rightarrow -5} |x - 5| = |(-5) - 5| = |-10| = 10$$

$$\text{Given } \varepsilon > 0: \left| |x - 5| - 10 \right| < \varepsilon$$

$$\left| -(x - 5) - 10 \right| < \varepsilon \quad (x - 5 < 0)$$

$$|-x - 5| < \varepsilon$$

$$|x - (-5)| < \varepsilon$$

So, let  $\delta = \varepsilon$ .

So for  $|x - (-5)| < \delta = \varepsilon$ , you have

$$|-(x + 5)| < \varepsilon$$

$$\left| -(x - 5) - 10 \right| < \varepsilon$$

$$\left| |x - 5| - 10 \right| < \varepsilon \quad (\text{because } x - 5 < 0)$$

$$|f(x) - L| < \varepsilon.$$

$$54. \lim_{x \rightarrow 3} |x - 3| = |3 - 3| = 0$$

$$\text{Given } \varepsilon > 0: \begin{aligned} ||x - 3| - 0| &< \varepsilon \\ |x - 3| &< \varepsilon \end{aligned}$$

So, let  $\delta = \varepsilon$ .

So, for  $0 < |x - 3| < \delta = \varepsilon$ , you have

$$\begin{aligned} |x - 3| &< \varepsilon \\ ||x - 3| - 0| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

$$55. \lim_{x \rightarrow 1} (x^2 + 1) = 1^2 + 1 = 2$$

$$\text{Given } \varepsilon > 0: \begin{aligned} |(x^2 + 1) - 2| &< \varepsilon \\ |x^2 - 1| &< \varepsilon \\ |(x + 1)(x - 1)| &< \varepsilon \\ |x - 1| &< \frac{\varepsilon}{|x + 1|} \end{aligned}$$

If you assume  $0 < x < 2$ , then  $\delta = \varepsilon/3$ .

So for  $0 < |x - 1| < \delta = \frac{\varepsilon}{3}$ , you have

$$\begin{aligned} |x - 1| &< \frac{1}{3}\varepsilon < \frac{1}{|x + 1|}\varepsilon \\ |x^2 - 1| &< \varepsilon \\ |(x^2 + 1) - 2| &< \varepsilon \\ |f(x) - 2| &< \varepsilon. \end{aligned}$$

$$56. \lim_{x \rightarrow -4} (x^2 + 4x) = (-4)^2 + 4(-4) = 0$$

$$\text{Given } \varepsilon > 0: \begin{aligned} |(x^2 + 4x) - 0| &< \varepsilon \\ |x(x + 4)| &< \varepsilon \\ |x + 4| &< \frac{\varepsilon}{|x|} \end{aligned}$$

If you assume  $-5 < x < -3$ , then  $\delta = \frac{\varepsilon}{5}$ .

So for  $0 < |x - (-4)| < \delta = \frac{\varepsilon}{5}$ , you have

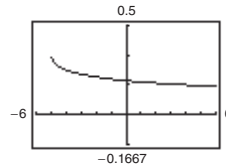
$$\begin{aligned} |x + 4| &< \frac{\varepsilon}{5} < \frac{1}{|x|}\varepsilon \\ |x(x + 4)| &< \varepsilon \\ |(x^2 + 4x) - 0| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

$$57. \lim_{x \rightarrow \pi} f(x) = \lim_{x \rightarrow \pi} 4 = 4$$

$$58. \lim_{x \rightarrow \pi} f(x) = \lim_{x \rightarrow \pi} x = \pi$$

$$59. f(x) = \frac{\sqrt{x + 5} - 3}{x - 4}$$

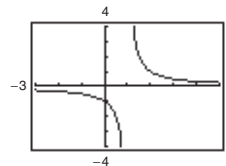
$$\lim_{x \rightarrow 4} f(x) = \frac{1}{6}$$



The domain is  $[-5, 4) \cup (4, \infty)$ . The graphing utility does not show the hole at  $(4, \frac{1}{6})$ .

$$60. f(x) = \frac{x - 3}{x^2 - 4x + 3}$$

$$\lim_{x \rightarrow 3} f(x) = \frac{1}{2}$$

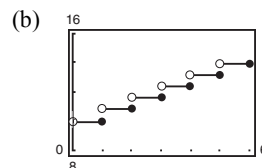


The domain is all  $x \neq 1, 3$ . The graphing utility does not show the hole at  $(3, \frac{1}{2})$ .

$$61. C(t) = 9.99 - 0.79[1 - t], t > 0$$

$$\begin{aligned} \text{(a) } C(10.75) &= 9.99 - 0.79[1 - 10.75] \\ &= 9.99 - 0.79(-10) \\ &= \$17.89 \end{aligned}$$

$C(10.75)$  represents the cost of a 10-minute, 45-second call.

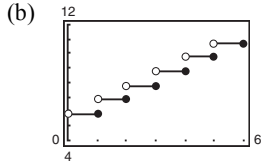


(c) The limit does not exist because the limits from the left and right are not equal.

62.  $C(t) = 5.79 - 0.99[1 - t]$ ,  $t > 0$

(a)  $C(10.75) = 5.79 - 0.99[1 - 10.75]$   
 $= 5.79 - 0.99(-10)$   
 $= \$15.69$

$C(10.75)$  represents the cost of a 10-minute, 45-second call.



(c) The limit does not exist because the limits from the left and right are not equal.

63. Choosing a smaller positive value of  $\delta$  will still satisfy the inequality  $|f(x) - L| < \epsilon$ .

64. In the definition of  $\lim_{x \rightarrow c} f(x)$ ,  $f$  must be defined on both sides of  $c$ , but does not have to be defined at  $c$  itself. The value of  $f$  at  $c$  has no bearing on the limit as  $x$  approaches  $c$ .

65. No. The fact that  $f(2) = 4$  has no bearing on the existence of the limit of  $f(x)$  as  $x$  approaches 2.

66. No. The fact that  $\lim_{x \rightarrow 2} f(x) = 4$  has no bearing on the value of  $f$  at 2.

67. (a)  $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) When  $C = 5.5$ :  $r = \frac{5.5}{2\pi} \approx 0.87535 \text{ cm}$

When  $C = 6.5$ :  $r = \frac{6.5}{2\pi} \approx 1.03451 \text{ cm}$

So  $0.87535 < r < 1.03451$ .

(c)  $\lim_{x \rightarrow 3/\pi} (2\pi r) = 6$ ;  $\epsilon = 0.5$ ;  $\delta \approx 0.0796$

70.  $f(x) = \frac{|x + 1| - |x - 1|}{x}$

$x$	-1	-0.5	-0.1	0	0.1	0.5	1.0
$f(x)$	2	2	2	Undef.	2	2	2

$$\lim_{x \rightarrow 0} f(x) = 2$$

Note that for

$$-1 < x < 1, x \neq 0, f(x) = \frac{(x + 1) + (x - 1)}{x} = 2.$$

68.  $V = \frac{4}{3}\pi r^3, V = 2.48$

(a)  $2.48 = \frac{4}{3}\pi r^3$

$$r^3 = \frac{1.86}{\pi}$$

$$r \approx 0.8397 \text{ in.}$$

(b)  $2.45 \leq V \leq 2.51$

$$2.45 \leq \frac{4}{3}\pi r^3 \leq 2.51$$

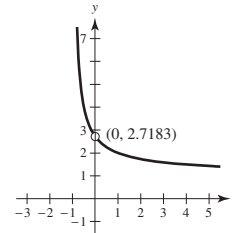
$$0.5849 \leq r^3 \leq 0.5992$$

$$0.8363 \leq r \leq 0.8431$$

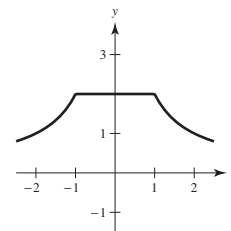
(c) For  $\epsilon = 2.51 - 2.48 = 0.03$ ,  $\delta \approx 0.003$

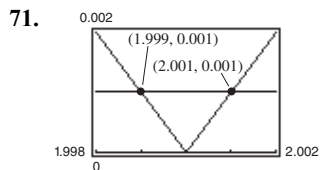
69.  $f(x) = (1 + x)^{1/x}$

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} = e \approx 2.71828$$



$x$	$f(x)$	$x$	$f(x)$
-0.1	2.867972	0.1	2.593742
-0.01	2.731999	0.01	2.704814
-0.001	2.719642	0.001	2.716942
-0.0001	2.718418	0.0001	2.718146
-0.00001	2.718295	0.00001	2.718268
-0.000001	2.718283	0.000001	2.718280





Using the zoom and trace feature,  $\delta = 0.001$ . So  $(2 - \delta, 2 + \delta) = (1.999, 2.001)$ .

**Note:**  $\frac{x^2 - 4}{x - 2} = x + 2$  for  $x \neq 2$ .

72. (a)  $\lim_{x \rightarrow c} f(x)$  exists for all  $c \neq -3$ .  
 (b)  $\lim_{x \rightarrow c} f(x)$  exists for all  $c \neq -2, 0$ .
73. False. The existence or nonexistence of  $f(x)$  at  $x = c$  has no bearing on the existence of the limit of  $f(x)$  as  $x \rightarrow c$ .

74. True

75. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$$f(2) = 0$$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x - 4) = 2 \neq 0$$

81. If  $\lim_{x \rightarrow c} f(x) = L_1$  and  $\lim_{x \rightarrow c} f(x) = L_2$ , then for every  $\varepsilon > 0$ , there exists  $\delta_1 > 0$  and  $\delta_2 > 0$  such that  $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$  and  $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$ . Let  $\delta$  equal the smaller of  $\delta_1$  and  $\delta_2$ . Then for  $|x - c| < \delta$ , you have  $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$ . Therefore,  $|L_1 - L_2| < 2\varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, it follows that  $L_1 = L_2$ .

82.  $f(x) = mx + b$ ,  $m \neq 0$ . Let  $\varepsilon > 0$  be given.

$$\text{Take } \delta = \frac{\varepsilon}{|m|}.$$

$$\text{If } 0 < |x - c| < \delta = \frac{\varepsilon}{|m|}, \text{ then}$$

$$|m||x - c| < \varepsilon$$

$$|mx - mc| < \varepsilon$$

$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that  $\lim_{x \rightarrow c} (mx + b) = mc + b$ .

76. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x - 4) = 2 \text{ and } f(2) = 0 \neq 2$$

77.  $f(x) = \sqrt{x}$

$$\lim_{x \rightarrow 0.25} \sqrt{x} = 0.5 \text{ is true.}$$

As  $x$  approaches  $0.25 = \frac{1}{4}$  from either side,

$$f(x) = \sqrt{x} \text{ approaches } \frac{1}{2} = 0.5.$$

78.  $f(x) = \sqrt{x}$

$$\lim_{x \rightarrow 0} \sqrt{x} = 0 \text{ is false.}$$

$f(x) = \sqrt{x}$  is not defined on an open interval containing 0 because the domain of  $f$  is  $x \geq 0$ .

79. Using a graphing utility, you see that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2, \text{ etc.}$$

$$\text{So, } \lim_{x \rightarrow 0} \frac{\sin nx}{x} = n.$$

80. Using a graphing utility, you see that

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\tan 2x}{x} = 2, \text{ etc.}$$

$$\text{So, } \lim_{x \rightarrow 0} \frac{\tan(nx)}{x} = n.$$

83.  $\lim_{x \rightarrow c} [f(x) - L] = 0$  means that for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $0 < |x - c| < \delta$ , then

$$|(f(x) - L) - 0| < \varepsilon.$$

This means the same as  $|f(x) - L| < \varepsilon$  when

$$0 < |x - c| < \delta.$$

So,  $\lim_{x \rightarrow c} f(x) = L$ .

$$\begin{aligned}
 84. (a) \quad (3x+1)(3x-1)x^2 + 0.01 &= (9x^2-1)x^2 + \frac{1}{100} \\
 &= 9x^4 - x^2 + \frac{1}{100} \\
 &= \frac{1}{100}(10x^2-1)(90x^2-1)
 \end{aligned}$$

So,  $(3x+1)(3x-1)x^2 + 0.01 > 0$  if

$$10x^2 - 1 < 0 \text{ and } 90x^2 - 1 < 0.$$

$$\text{Let } (a, b) = \left(-\frac{1}{\sqrt{90}}, \frac{1}{\sqrt{90}}\right).$$

For all  $x \neq 0$  in  $(a, b)$ , the graph is positive.

You can verify this with a graphing utility.

(b) You are given  $\lim_{x \rightarrow c} g(x) = L > 0$ . Let  $\varepsilon = \frac{1}{2}L$ .

There exists  $\delta > 0$  such that  $0 < |x - c| < \delta$

implies that  $|g(x) - L| < \varepsilon = \frac{L}{2}$ . That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$

$$\frac{L}{2} < g(x) < \frac{3L}{2}$$

For  $x$  in the interval  $(c - \delta, c + \delta)$ ,  $x \neq c$ , you

have  $g(x) > \frac{L}{2} > 0$ , as desired.

85. The radius  $OP$  has a length equal to the altitude  $z$  of the triangle plus  $\frac{h}{2}$ . So,  $z = 1 - \frac{h}{2}$ .

$$\text{Area triangle} = \frac{1}{2}b\left(1 - \frac{h}{2}\right)$$

$$\text{Area rectangle} = bh$$

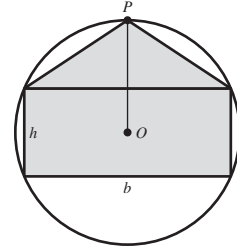
Because these are equal,

$$\frac{1}{2}b\left(1 - \frac{h}{2}\right) = bh$$

$$1 - \frac{h}{2} = 2h$$

$$\frac{5}{2}h = 1$$

$$h = \frac{2}{5}.$$



86. Consider a cross section of the cone, where  $EF$  is a diagonal of the inscribed cube.  $AD = 3$ ,  $BC = 2$ .

Let  $x$  be the length of a side of the cube.

Then  $EF = x\sqrt{2}$ .

By similar triangles,

$$\frac{EF}{BC} = \frac{AG}{AD}$$

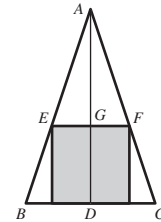
$$\frac{x\sqrt{2}}{2} = \frac{3-x}{3}$$

Solving for  $x$ ,

$$3\sqrt{2}x = 6 - 2x$$

$$(3\sqrt{2} + 2)x = 6$$

$$x = \frac{6}{3\sqrt{2} + 2} = \frac{9\sqrt{2} - 6}{7} \approx 0.96.$$



## Section 1.3 Evaluating Limits Analytically

1. For polynomial functions  $p(x)$ , substitute  $c$  for  $x$ , and simplify.

2. An indeterminate form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as  $0/0$ . That is,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$$

for which  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$

3. If a function  $f$  is squeezed between two functions  $h$  and  $g$ ,  $h(x) \leq f(x) \leq g(x)$ , and  $h$  and  $g$  have the same limit

$L$  as  $x \rightarrow c$ , then  $\lim_{x \rightarrow c} f(x)$  exists and equals  $L$

$$4. \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

$$5. \lim_{x \rightarrow 2} x^3 = 2^3 = 8$$

$$6. \lim_{x \rightarrow -3} x^4 = (-3)^4 = 81$$

$$7. \lim_{x \rightarrow -3} (2x + 5) = 2(-3) + 5 = -1$$

$$8. \lim_{x \rightarrow 9} (4x - 1) = 4(9) - 1 = 36 - 1 = 35$$

$$9. \lim_{x \rightarrow -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$$

$$10. \lim_{x \rightarrow 2} (-x^3 + 1) = (-2)^3 + 1 = -8 + 1 = -7$$

$$11. \lim_{x \rightarrow -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 \\ = 18 - 12 + 1 = 7$$

$$12. \lim_{x \rightarrow 1} (2x^3 - 6x + 5) = 2(1)^3 - 6(1) + 5 \\ = 2 - 6 + 5 = 1$$

$$13. \lim_{x \rightarrow 3} \sqrt{x + 8} = \sqrt{3 + 8} = \sqrt{11}$$

$$14. \lim_{x \rightarrow 2} \sqrt[3]{12x + 3} = \sqrt[3]{12(2) + 3} \\ = \sqrt[3]{24 + 3} = \sqrt[3]{27} = 3$$

$$15. \lim_{x \rightarrow -4} (1 - x)^3 = [1 - (-4)]^3 = 5^3 = 125$$

$$16. \lim_{x \rightarrow 0} (3x - 2)^4 = (3(0) - 2)^4 = (-2)^4 = 16$$

$$17. \lim_{x \rightarrow 2} \frac{3}{2x + 1} = \frac{3}{2(2) + 1} = \frac{3}{5}$$

$$18. \lim_{x \rightarrow -5} \frac{5}{x + 3} = \frac{5}{-5 + 3} = -\frac{5}{2}$$

$$19. \lim_{x \rightarrow 1} \frac{x}{x^2 + 4} = \frac{1}{1^2 + 4} = \frac{1}{5}$$

$$20. \lim_{x \rightarrow 1} \frac{3x + 5}{x + 1} = \frac{3(1) + 5}{1 + 1} = \frac{3 + 5}{2} = \frac{8}{2} = 4$$

$$21. \lim_{x \rightarrow 7} \frac{3x}{\sqrt{x + 2}} = \frac{3(7)}{\sqrt{7 + 2}} = \frac{21}{3} = 7$$

$$22. \lim_{x \rightarrow 3} \frac{\sqrt{x + 6}}{x + 2} = \frac{\sqrt{3 + 6}}{3 + 2} = \frac{\sqrt{9}}{5} = \frac{3}{5}$$

$$23. (a) \lim_{x \rightarrow 1} f(x) = 5 - 1 = 4$$

$$(b) \lim_{x \rightarrow 4} g(x) = 4^3 = 64$$

$$(c) \lim_{x \rightarrow 1} g(f(x)) = g(f(1)) = g(4) = 64$$

$$37. \lim_{x \rightarrow c} f(x) = \frac{2}{5}, \lim_{x \rightarrow c} g(x) = 2$$

$$(a) \lim_{x \rightarrow c} [5g(x)] = 5 \lim_{x \rightarrow c} g(x) = 5(2) = 10$$

$$(b) \lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = \frac{2}{5} + 2 = \frac{12}{5}$$

$$(c) \lim_{x \rightarrow c} [f(x) + g(x)] = \left[ \lim_{x \rightarrow c} f(x) \right] + \left[ \lim_{x \rightarrow c} g(x) \right] = \frac{2}{5}(2) = \frac{4}{5}$$

$$(d) \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{2/5}{2} = \frac{1}{5}$$

$$24. (a) \lim_{x \rightarrow -3} f(x) = (-3) + 7 = 4$$

$$(b) \lim_{x \rightarrow 4} g(x) = 4^2 = 16$$

$$(c) \lim_{x \rightarrow -3} g(f(x)) = g(4) = 16$$

$$25. (a) \lim_{x \rightarrow 1} f(x) = 4 - 1 = 3$$

$$(b) \lim_{x \rightarrow 3} g(x) = \sqrt{3 + 1} = 2$$

$$(c) \lim_{x \rightarrow 1} g(f(x)) = g(3) = 2$$

$$26. (a) \lim_{x \rightarrow 4} f(x) = 2(4^2) - 3(4) + 1 = 21$$

$$(b) \lim_{x \rightarrow 21} g(x) = \sqrt[3]{21 + 6} = 3$$

$$(c) \lim_{x \rightarrow 4} g(f(x)) = g(21) = 3$$

$$27. \lim_{x \rightarrow \pi/2} \sin x = \sin \frac{\pi}{2} = 1$$

$$28. \lim_{x \rightarrow \pi} \tan x = \tan \pi = 0$$

$$29. \lim_{x \rightarrow 1} \cos \frac{\pi x}{3} = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$30. \lim_{x \rightarrow 2} \sin \frac{\pi x}{12} = \sin \frac{\pi(2)}{12} = \sin \frac{\pi}{6} = \frac{1}{2}$$

$$31. \lim_{x \rightarrow 0} \sec 2x = \sec 0 = 1$$

$$32. \lim_{x \rightarrow \pi} \cos 3x = \cos 3\pi = -1$$

$$33. \lim_{x \rightarrow 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$$

$$34. \lim_{x \rightarrow 5\pi/3} \cos x = \cos \frac{5\pi}{3} = \frac{1}{2}$$

$$35. \lim_{x \rightarrow 3} \tan \left( \frac{\pi x}{4} \right) = \tan \frac{3\pi}{4} = -1$$

$$36. \lim_{x \rightarrow 7} \sec \left( \frac{\pi x}{6} \right) = \sec \frac{7\pi}{6} = \frac{-2\sqrt{3}}{3}$$

$$38. \lim_{x \rightarrow c} f(x) = 2, \lim_{x \rightarrow c} g(x) = \frac{3}{4}$$

$$(a) \lim_{x \rightarrow c} [4f(x)] = 4 \lim_{x \rightarrow c} f(x) = 4(2) = 8$$

$$(b) \lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = 2 + \frac{3}{4} = \frac{11}{4}$$

$$(c) \lim_{x \rightarrow c} [f(x)g(x)] = \left[ \lim_{x \rightarrow c} f(x) \right] \left[ \lim_{x \rightarrow c} g(x) \right] = 2 \left( \frac{3}{4} \right) = \frac{3}{2}$$

$$(d) \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{2}{(3/4)} = \frac{8}{3}$$

$$39. \lim_{x \rightarrow c} f(x) = 16$$

$$(a) \lim_{x \rightarrow c} [f(x)]^2 = \left[ \lim_{x \rightarrow c} f(x) \right]^2 = (16)^2 = 256$$

$$(b) \lim_{x \rightarrow c} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow c} f(x)} = \sqrt{16} = 4$$

$$(c) \lim_{x \rightarrow c} [3f(x)] = 3 \left[ \lim_{x \rightarrow c} f(x) \right] = 3(16) = 48$$

$$(d) \lim_{x \rightarrow c} [f(x)]^{3/2} = \left[ \lim_{x \rightarrow c} f(x) \right]^{3/2} = (16)^{3/2} = 64$$

$$40. \lim_{x \rightarrow c} f(x) = 27$$

$$(a) \lim_{x \rightarrow c} \sqrt[3]{f(x)} = \sqrt[3]{\lim_{x \rightarrow c} f(x)} = \sqrt[3]{27} = 3$$

$$(b) \lim_{x \rightarrow c} \frac{f(x)}{18} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} 18} = \frac{27}{18} = \frac{3}{2}$$

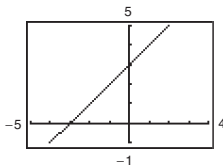
$$(c) \lim_{x \rightarrow c} [f(x)]^2 = \left[ \lim_{x \rightarrow c} f(x) \right]^2 = (27)^2 = 729$$

$$(d) \lim_{x \rightarrow c} [f(x)]^{2/3} = \left[ \lim_{x \rightarrow c} f(x) \right]^{2/3} = (27)^{2/3} = 9$$

$$41. f(x) = \frac{x^2 + 3x}{x} = \frac{x(x + 3)}{x} \text{ and } g(x) = x + 3$$

agree except at  $x = 0$ .

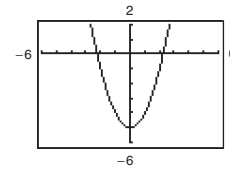
$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (x + 3) = 0 + 3 = 3$$



$$42. f(x) = \frac{x^4 - 5x^2}{x^2} = \frac{x^2(x^2 - 5)}{x^2} \text{ and } g(x) = x^2 - 5$$

agree except at  $x = 0$ .

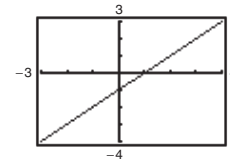
$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (x^2 - 5) = 0^2 - 5 = -5$$



$$43. f(x) = \frac{x^2 - 1}{x + 1} = \frac{(x + 1)(x - 1)}{x + 1} \text{ and } g(x) = x - 1$$

agree except at  $x = -1$ .

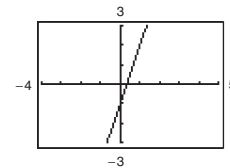
$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} (x - 1) = -1 - 1 = -2$$



$$44. f(x) = \frac{3x^2 + 5x - 2}{x + 2} = \frac{(x + 2)(3x - 1)}{x + 2} \text{ and } g(x) = 3x - 1$$

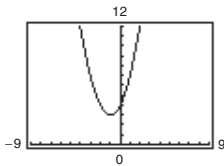
agree except at  $x = -2$ .

$$\begin{aligned} \lim_{x \rightarrow -2} f(x) &= \lim_{x \rightarrow -2} g(x) = \lim_{x \rightarrow -2} (3x - 1) \\ &= 3(-2) - 1 = -7 \end{aligned}$$



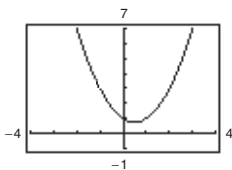
45.  $f(x) = \frac{x^3 - 8}{x - 2}$  and  $g(x) = x^2 + 2x + 4$  agree except at  $x = 2$ .

$$\begin{aligned}\lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} g(x) = \lim_{x \rightarrow 2} (x^2 + 2x + 4) \\ &= 2^2 + 2(2) + 4 = 12\end{aligned}$$



46.  $f(x) = \frac{x^3 + 1}{x + 1}$  and  $g(x) = x^2 - x + 1$  agree except at  $x = -1$ .

$$\begin{aligned}\lim_{x \rightarrow -1} f(x) &= \lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} (x^2 - x + 1) \\ &= (-1)^2 - (-1) + 1 = 3\end{aligned}$$



$$\begin{aligned}52. \lim_{x \rightarrow 2} \frac{x^2 + 2x - 8}{x^2 - x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 4)}{(x - 2)(x + 1)} \\ &= \lim_{x \rightarrow 2} \frac{x + 4}{x + 1} = \frac{2 + 4}{2 + 1} = \frac{6}{3} = 2\end{aligned}$$

$$\begin{aligned}53. \lim_{x \rightarrow 4} \frac{\sqrt{x + 5} - 3}{x - 4} &= \lim_{x \rightarrow 4} \frac{\sqrt{x + 5} - 3}{x - 4} \cdot \frac{\sqrt{x + 5} + 3}{\sqrt{x + 5} + 3} \\ &= \lim_{x \rightarrow 4} \frac{(x + 5) - 9}{(x - 4)(\sqrt{x + 5} + 3)} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x + 5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}\end{aligned}$$

$$\begin{aligned}54. \lim_{x \rightarrow 3} \frac{\sqrt{x + 1} - 2}{x - 3} &= \lim_{x \rightarrow 3} \frac{\sqrt{x + 1} - 2}{x - 3} \cdot \frac{\sqrt{x + 1} + 2}{\sqrt{x + 1} + 2} = \lim_{x \rightarrow 3} \frac{x - 3}{(x - 3)(\sqrt{x + 1} + 2)} \\ &= \lim_{x \rightarrow 3} \frac{1}{\sqrt{x + 1} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}\end{aligned}$$

$$\begin{aligned}55. \lim_{x \rightarrow 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x + 5} + \sqrt{5}}{\sqrt{x + 5} + \sqrt{5}} \\ &= \lim_{x \rightarrow 0} \frac{(x + 5) - 5}{x(\sqrt{x + 5} + \sqrt{5})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x + 5} + \sqrt{5}} = \frac{1}{\sqrt{5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}\end{aligned}$$

$$\begin{aligned}56. \lim_{x \rightarrow 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x} \cdot \frac{\sqrt{2 + x} + \sqrt{2}}{\sqrt{2 + x} + \sqrt{2}} \\ &= \lim_{x \rightarrow 0} \frac{2 + x - 2}{(\sqrt{2 + x} + \sqrt{2})x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{2 + x} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}\end{aligned}$$

$$47. \lim_{x \rightarrow 0} \frac{x}{x^2 - x} = \lim_{x \rightarrow 0} \frac{x}{x(x - 1)} = \lim_{x \rightarrow 0} \frac{1}{x - 1} = \frac{1}{0 - 1} = -1$$

$$48. \lim_{x \rightarrow 0} \frac{7x^3 - x^2}{x} = \lim_{x \rightarrow 0} (7x^2 - x) = 0 - 0 = 0$$

$$\begin{aligned}49. \lim_{x \rightarrow 4} \frac{x - 4}{x^2 - 16} &= \lim_{x \rightarrow 4} \frac{x - 4}{(x + 4)(x - 4)} \\ &= \lim_{x \rightarrow 4} \frac{1}{x + 4} = \frac{1}{4 + 4} = \frac{1}{8}\end{aligned}$$

$$\begin{aligned}50. \lim_{x \rightarrow 5} \frac{5 - x}{x^2 - 25} &= \lim_{x \rightarrow 5} \frac{-(x - 5)}{(x - 5)(x + 5)} \\ &= \lim_{x \rightarrow 5} \frac{-1}{x + 5} = \frac{-1}{5 + 5} = -\frac{1}{10}\end{aligned}$$

$$\begin{aligned}51. \lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x^2 - 9} &= \lim_{x \rightarrow -3} \frac{(x + 3)(x - 2)}{(x + 3)(x - 3)} \\ &= \lim_{x \rightarrow -3} \frac{x - 2}{x - 3} = \frac{-3 - 2}{-3 - 3} = \frac{-5}{-6} = \frac{5}{6}\end{aligned}$$

$$57. \lim_{x \rightarrow 0} \frac{\frac{1}{3+x} - \frac{1}{3}}{x} = \lim_{x \rightarrow 0} \frac{3 - (3+x)}{(3+x)3(x)} = \lim_{x \rightarrow 0} \frac{-x}{(3+x)3(x)} = \lim_{x \rightarrow 0} \frac{-1}{(3+x)3} = \frac{-1}{(3)3} = -\frac{1}{9}$$

$$58. \lim_{x \rightarrow 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \lim_{x \rightarrow 0} \frac{\frac{4 - (x+4)}{4(x+4)}}{x} \\ = \lim_{x \rightarrow 0} \frac{-1}{4(x+4)} = \frac{-1}{4(4)} = -\frac{1}{16}$$

$$59. \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2 = 2$$

$$60. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x$$

$$61. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x} \\ = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2$$

$$62. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\ = \lim_{\Delta x \rightarrow 0} \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

$$63. \lim_{x \rightarrow 0} \frac{\sin x}{5x} = \lim_{x \rightarrow 0} \left[ \left( \frac{\sin x}{x} \right) \left( \frac{1}{5} \right) \right] = (1) \left( \frac{1}{5} \right) = \frac{1}{5}$$

$$67. \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} \left[ \frac{\sin x}{x} \sin x \right] = (1) \sin 0 = 0$$

$$64. \lim_{x \rightarrow 0} \frac{3(1 - \cos x)}{x} = \lim_{x \rightarrow 0} \left[ 3 \left( \frac{1 - \cos x}{x} \right) \right] = (3)(0) = 0$$

$$68. \lim_{x \rightarrow 0} \frac{\tan^2 x}{x} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \cos^2 x} = \lim_{x \rightarrow 0} \left[ \frac{\sin x}{x} \cdot \frac{\sin x}{\cos^2 x} \right] \\ = (1)(0) = 0$$

$$65. \lim_{x \rightarrow 0} \frac{(\sin x)(1 - \cos x)}{x^2} = \lim_{x \rightarrow 0} \left[ \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x} \right] \\ = (1)(0) = 0$$

$$69. \lim_{h \rightarrow 0} \frac{(1 - \cos h)^2}{h} = \lim_{h \rightarrow 0} \left[ \frac{1 - \cos h}{h} (1 - \cos h) \right] \\ = (0)(0) = 0$$

$$66. \lim_{\theta \rightarrow 0} \frac{\cos \theta \tan \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$70. \lim_{\phi \rightarrow \pi} \phi \sec \phi = \pi(-1) = -\pi$$

$$71. \lim_{x \rightarrow 0} \frac{6 - 6 \cos x}{3} = \frac{6 - 6 \cos 0}{3} = \frac{6 - 6}{3} = 0$$

$$72. \lim_{x \rightarrow 0} \frac{\cos x - \sin x - 1}{2x} = \lim_{x \rightarrow 0} \frac{-\sin x}{2x} + \lim_{x \rightarrow 0} \frac{\cos x - 1}{2x} \\ = -\frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} - \frac{1}{2} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \\ = -\frac{1}{2}(1) - \frac{1}{2}(0) = -\frac{1}{2}$$

$$73. \lim_{t \rightarrow 0} \frac{\sin 3t}{2t} = \lim_{t \rightarrow 0} \left( \frac{\sin 3t}{3t} \right) \left( \frac{3}{2} \right) = (1) \left( \frac{3}{2} \right) = \frac{3}{2}$$

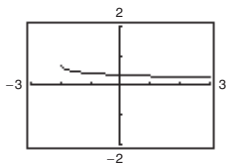
$$74. \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \rightarrow 0} \left[ 2 \left( \frac{\sin 2x}{2x} \right) \left( \frac{1}{3} \right) \left( \frac{3x}{\sin 3x} \right) \right]$$

$$= 2(1) \left( \frac{1}{3} \right) (1) = \frac{2}{3}$$

$$75. f(x) = \frac{\sqrt{x+2} - \sqrt{2}}{x}$$

$x$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	?	0.354	0.353	0.349

It appears that the limit is 0.354.



The graph has a hole at  $x = 0$ .

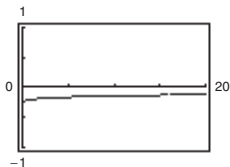
$$\text{Analytically, } \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}}$$

$$= \lim_{x \rightarrow 0} \frac{x+2-2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$$

$$76. f(x) = \frac{4 - \sqrt{x}}{x - 16}$$

$x$	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	-0.1252	-0.125	-0.125	?	-0.125	-0.125	-0.1248

It appears that the limit is -0.125.



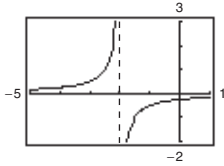
The graph has a hole at  $x = 16$ .

$$\text{Analytically, } \lim_{x \rightarrow 16} \frac{4 - \sqrt{x}}{x - 16} = \lim_{x \rightarrow 16} \frac{(4 - \sqrt{x})}{(\sqrt{x} + 4)(\sqrt{x} - 4)} = \lim_{x \rightarrow 16} \frac{-1}{\sqrt{x} + 4} = -\frac{1}{8}.$$

$$77. f(x) = \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$$

$x$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	?	-0.250	-0.249	-0.238

It appears that the limit is  $-0.250$ .



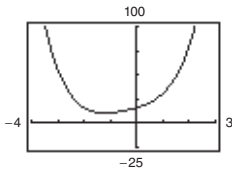
The graph has a hole at  $x = 0$ .

$$\text{Analytically, } \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{2 - (2+x)}{2(2+x)} \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \frac{-x}{2(2+x)} \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \frac{-1}{2(2+x)} = -\frac{1}{4}.$$

$$78. f(x) = \frac{x^5 - 32}{x - 2}$$

$x$	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41

It appears that the limit is 80.



The graph has a hole at  $x = 2$ .

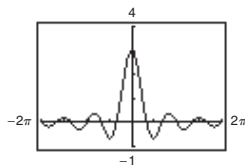
$$\text{Analytically, } \lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} = \lim_{x \rightarrow 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$$

(Hint: Use long division to factor  $x^5 - 32$ .)

$$79. f(t) = \frac{\sin 3t}{t}$$

$t$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

It appears that the limit is 3.



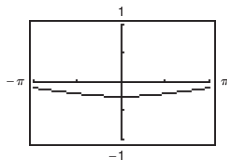
The graph has a hole at  $t = 0$ .

$$\text{Analytically, } \lim_{t \rightarrow 0} \frac{\sin 3t}{t} = \lim_{t \rightarrow 0} 3 \left( \frac{\sin 3t}{3t} \right) = 3(1) = 3.$$

80.  $f(x) = \frac{\cos x - 1}{2x^2}$

$x$	-1	-0.1	-0.01	0.01	0.1	1
$f(x)$	-0.2298	-0.2498	-0.25	-0.25	-0.2498	-0.2298

It appears that the limit is -0.25.



The graph has a hole at  $x = 0$ .

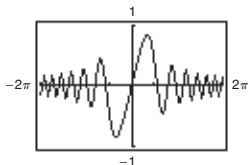
Analytically,  $\frac{\cos x - 1}{2x^2} \cdot \frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{2x^2(\cos x + 1)} = \frac{-\sin^2 x}{2x^2(\cos x + 1)} = \frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}$

$\lim_{x \rightarrow 0} \left[ \frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)} \right] = 1 \left( \frac{-1}{4} \right) = -\frac{1}{4} = -0.25$

81.  $f(x) = \frac{\sin x^2}{x}$

$x$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998

It appears that the limit is 0.



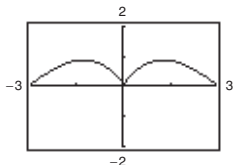
The graph has a hole at  $x = 0$ .

Analytically,  $\lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} x \left( \frac{\sin x^2}{x^2} \right) = 0(1) = 0$ .

82.  $f(x) = \frac{\sin x}{\sqrt[3]{x}}$

$x$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215

It appears that the limit is 0.



The graph has a hole at  $x = 0$ .

Analytically,  $\lim_{x \rightarrow 0} \frac{\sin x}{\sqrt[3]{x}} = \lim_{x \rightarrow 0} \sqrt[3]{x^2} \left( \frac{\sin x}{x} \right) = (0)(1) = 0$ .

83.  $f(x) = 3x - 2$

$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3(x + \Delta x) - 2 - (3x - 2)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3x + 3\Delta x - 2 - 3x + 2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3\Delta x}{\Delta x} = 3$

84.  $f(x) = -6x + 3$

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{[-6(x + \Delta x) + 3] - [-6x + 3]}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-6x - 6\Delta x + 3 + 6x - 3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-6\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} (-6) = -6\end{aligned}$$

85.  $f(x) = x^2 - 4x$

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - 4(x + \Delta x) - (x^2 - 4x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 4x - 4\Delta x - x^2 + 4x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x - 4)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 4) = 2x - 4\end{aligned}$$

86.  $f(x) = 3x^2 + 1$

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{[3(x + \Delta x)^2 + 1] - [3x^2 + 1]}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(3x^2 + 6x\Delta x + 1) - (3x^2 + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{6x\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 6x = 6x\end{aligned}$$

87.  $f(x) = 2\sqrt{x}$

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{2\sqrt{x + \Delta x} - 2\sqrt{x}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2(\sqrt{x + \Delta x} - \sqrt{x})}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x - x)}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \lim_{\Delta x \rightarrow 0} \frac{2\Delta x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}} = x^{-1/2}\end{aligned}$$

88.  $f(x) = \sqrt{x} - 5$

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x} - 5) - (\sqrt{x} - 5)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}\end{aligned}$$

89.  $f(x) = \frac{1}{x + 3}$

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x + 3} - \frac{1}{x + 3}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x + 3 - (x + \Delta x + 3)}{(x + \Delta x + 3)(x + 3)} \cdot \frac{1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x + 3)(x + 3)} = \frac{-1}{(x + 3)^2}\end{aligned}$$

90.  $f(x) = \frac{1}{x^2}$

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x + \Delta x)^2} - \frac{1}{x^2}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^2 - (x + \Delta x)^2}{x^2(x + \Delta x)^2 \Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 - [x^2 + 2x\Delta x + (\Delta x)^2]}{x^2(x + \Delta x)^2 \Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{x^2(x + \Delta x)^2 \Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x - \Delta x}{x^2(x + \Delta x)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3} \end{aligned}$$

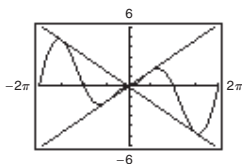
91.  $\lim_{x \rightarrow 0} (4 - x^2) \leq \lim_{x \rightarrow 0} f(x) \leq \lim_{x \rightarrow 0} (3 + x^2)$   
 $4 \leq \lim_{x \rightarrow 0} f(x) \leq 4$

Therefore,  $\lim_{x \rightarrow 0} f(x) = 4$ .

92.  $\lim_{x \rightarrow a} [b - |x - a|] \leq \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} [b + |x - a|]$   
 $b \leq \lim_{x \rightarrow a} f(x) \leq b$

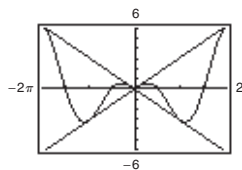
Therefore,  $\lim_{x \rightarrow a} f(x) = b$ .

93.  $f(x) = |x| \sin x$



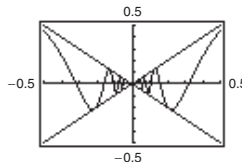
$\lim_{x \rightarrow 0} |x| \sin x = 0$

94.  $f(x) = |x| \cos x$



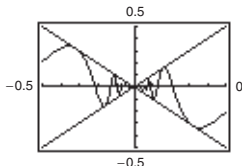
$\lim_{x \rightarrow 0} |x| \cos x = 0$

95.  $f(x) = x \sin \frac{1}{x}$



$\lim_{x \rightarrow 0} \left( x \sin \frac{1}{x} \right) = 0$

96.  $f(x) = x \cos \frac{1}{x}$



$\lim_{x \rightarrow 0} \left( x \cos \frac{1}{x} \right) = 0$

97. (a) Two functions  $f$  and  $g$  agree at all but one point (on an open interval) if  $f(x) = g(x)$  for all  $x$  in the interval except for  $x = c$ , where  $c$  is in the interval.

(b)  $f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1}$  and  $g(x) = x + 1$  agree at all points except  $x = 1$ .

(Other answers possible.)

98. Answers will vary. *Sample answers:*

(a) linear:  $f(x) = \frac{1}{2}x$ ;  $\lim_{x \rightarrow 8} \frac{1}{2}x = \frac{1}{2}(8) = 4$

(b) polynomial of degree 2:  $f(x) = x^2 - 60$ ;  $\lim_{x \rightarrow 8} (x^2 - 60) = 8^2 - 60 = 4$

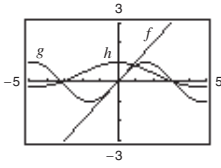
(c) rational:  $f(x) = \frac{x}{2x - 14}$ ;  $\lim_{x \rightarrow 8} \frac{x}{2x - 14} = \frac{8}{2(8) - 14} = \frac{8}{2} = 4$

(d) radical:  $f(x) = \sqrt{x + 8}$ ;  $\lim_{x \rightarrow 8} \sqrt{x + 8} = \sqrt{8 + 8} = \sqrt{16} = 4$

(e) cosine:  $f(x) = 4 \cos(\pi x)$ ;  $\lim_{x \rightarrow 8} 4 \cos(\pi x) = 4 \cos(8\pi) = 4(1) = 4$

(f) sine:  $f(x) = 4 \sin\left(\frac{\pi}{16}x\right)$ ;  $\lim_{x \rightarrow 8} 4 \sin\left(\frac{\pi}{16}x\right) = 4 \sin\left(\frac{\pi}{2}\right) = 4(1) = 4$

99.  $f(x) = x$ ,  $g(x) = \sin x$ ,  $h(x) = \frac{\sin x}{x}$



When the  $x$ -values are “close to” 0 the magnitude of  $f$  is approximately equal to the magnitude of  $g$ . So,  $|g|/|f| \approx 1$  when  $x$  is “close to” 0.

100. (a) Use the dividing out technique because the numerator and denominator have a common factor.  
 (b) Use the rationalizing technique because the numerator involves a radical expression.

102.  $s(t) = -16t^2 + 500 = 0$  when  $t = \sqrt{\frac{500}{16}} = \frac{5\sqrt{5}}{2}$  sec. The velocity at time  $a = \frac{5\sqrt{5}}{2}$  is

$$\begin{aligned} \lim_{t \rightarrow \left(\frac{5\sqrt{5}}{2}\right)} \frac{s\left(\frac{5\sqrt{5}}{2}\right) - s(t)}{\frac{5\sqrt{5}}{2} - t} &= \lim_{t \rightarrow \left(\frac{5\sqrt{5}}{2}\right)} \frac{0 - (-16t^2 + 500)}{\frac{5\sqrt{5}}{2} - t} \\ &= \lim_{t \rightarrow \left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t^2 - \frac{125}{4}\right)}{\frac{5\sqrt{5}}{2} - t} \\ &= \lim_{t \rightarrow \left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t + \frac{5\sqrt{5}}{2}\right)\left(t - \frac{5\sqrt{5}}{2}\right)}{\frac{5\sqrt{5}}{2} - t} \\ &= \lim_{t \rightarrow \frac{5\sqrt{5}}{2}} \left[-16\left(t + \frac{5\sqrt{5}}{2}\right)\right] = -80\sqrt{5} \text{ ft/sec} \\ &\approx -178.9 \text{ ft/sec.} \end{aligned}$$

The velocity of the paint can when it hits the ground is about 178.9 ft/sec.

103.  $s(t) = -4.9t^2 + 200$

$$\begin{aligned} \lim_{t \rightarrow 3} \frac{s(3) - s(t)}{3 - t} &= \lim_{t \rightarrow 3} \frac{-4.9(3)^2 + 200 - (-4.9t^2 + 200)}{3 - t} \\ &= \lim_{t \rightarrow 3} \frac{4.9(t^2 - 9)}{3 - t} \\ &= \lim_{t \rightarrow 3} \frac{4.9(t - 3)(t + 3)}{3 - t} \\ &= \lim_{t \rightarrow 3} [-4.9(t + 3)] \\ &= -29.4 \text{ m/sec} \end{aligned}$$

The object is falling about 29.4 m/sec.

101.  $s(t) = -16t^2 + 500$

$$\begin{aligned} \lim_{t \rightarrow 2} \frac{s(2) - s(t)}{2 - t} &= \lim_{t \rightarrow 2} \frac{-16(2)^2 + 500 - (-16t^2 + 500)}{2 - t} \\ &= \lim_{t \rightarrow 2} \frac{436 + 16t^2 - 500}{2 - t} \\ &= \lim_{t \rightarrow 2} \frac{16(t^2 - 4)}{2 - t} \\ &= \lim_{t \rightarrow 2} \frac{16(t - 2)(t + 2)}{2 - t} \\ &= \lim_{t \rightarrow 2} -16(t + 2) = -64 \text{ ft/sec} \end{aligned}$$

The paint can is falling at about 64 feet/second.

104.  $-4.9t^2 + 200 = 0$  when  $t = \sqrt{\frac{200}{4.9}} = \frac{20\sqrt{5}}{7}$  sec. The velocity at time  $a = \frac{20\sqrt{5}}{7}$  is

$$\begin{aligned}\lim_{t \rightarrow a} \frac{s(a) - s(t)}{a - t} &= \lim_{t \rightarrow a} \frac{0 - [-4.9t^2 + 200]}{a - t} \\ &= \lim_{t \rightarrow a} \frac{4.9(t + a)(t - a)}{a - t} \\ &= \lim_{t \rightarrow \frac{20\sqrt{5}}{7}} \left[ -4.9 \left( t + \frac{20\sqrt{5}}{7} \right) \right] = -28\sqrt{5} \text{ m/sec} \\ &\approx -62.6 \text{ m/sec.}\end{aligned}$$

The velocity of the object when it hits the ground is about 62.6 m/sec.

105. Let  $f(x) = 1/x$  and  $g(x) = -1/x$ .  $\lim_{x \rightarrow 0} f(x)$  and  $\lim_{x \rightarrow 0} g(x)$  do not exist. However,

$$\lim_{x \rightarrow 0} [f(x) + g(x)] = \lim_{x \rightarrow 0} \left[ \frac{1}{x} + \left( -\frac{1}{x} \right) \right] = \lim_{x \rightarrow 0} [0] = 0 \text{ and therefore does not exist.}$$

106. Suppose, on the contrary, that  $\lim_{x \rightarrow c} g(x)$  exists. Then,

because  $\lim_{x \rightarrow c} f(x)$  exists, so would  $\lim_{x \rightarrow c} [f(x) + g(x)]$ , which is a contradiction. So,  $\lim_{x \rightarrow c} g(x)$  does not exist.

107. Given  $f(x) = b$ , show that for every  $\varepsilon > 0$  there exists

a  $\delta > 0$  such that  $|f(x) - b| < \varepsilon$  whenever  $|x - c| < \delta$ . Because  $|f(x) - b| = |b - b| = 0 < \varepsilon$  for every  $\varepsilon > 0$ , any value of  $\delta > 0$  will work.

108. Given  $f(x) = x^n$ ,  $n$  is a positive integer, then

$$\begin{aligned}\lim_{x \rightarrow c} x^n &= \lim_{x \rightarrow c} (xx^{n-1}) \\ &= \left[ \lim_{x \rightarrow c} x \right] \left[ \lim_{x \rightarrow c} x^{n-1} \right] = c \left[ \lim_{x \rightarrow c} (xx^{n-2}) \right] \\ &= c \left[ \lim_{x \rightarrow c} x \right] \left[ \lim_{x \rightarrow c} x^{n-2} \right] = c(c) \lim_{x \rightarrow c} (xx^{n-3}) \\ &= \dots = c^n.\end{aligned}$$

109. If  $b = 0$ , the property is true because both sides are equal to 0. If  $b \neq 0$ , let  $\varepsilon > 0$  be given. Because

$\lim_{x \rightarrow c} f(x) = L$ , there exists  $\delta > 0$  such that  $|f(x) - L| < \varepsilon/|b|$  whenever  $0 < |x - c| < \delta$ . So, whenever  $0 < |x - c| < \delta$ , we have

$$|b||f(x) - L| < \varepsilon \quad \text{or} \quad |bf(x) - bL| < \varepsilon$$

which implies that  $\lim_{x \rightarrow c} [bf(x)] = bL$ .

110. Given  $\lim_{x \rightarrow c} f(x) = 0$ :

For every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$|f(x) - 0| < \varepsilon \text{ whenever } 0 < |x - c| < \delta.$$

Now  $|f(x) - 0| = |f(x)| = \left| \frac{f(x)}{1} - 0 \right| < \varepsilon$  for

$$|x - c| < \delta. \text{ Therefore, } \lim_{x \rightarrow c} |f(x)| = 0.$$

111.  $-M|f(x)| \leq f(x)g(x) \leq M|f(x)|$   
 $\lim_{x \rightarrow c} (-M|f(x)|) \leq \lim_{x \rightarrow c} [f(x)g(x)] \leq \lim_{x \rightarrow c} (M|f(x)|)$   
 $-M(0) \leq \lim_{x \rightarrow c} [f(x)g(x)] \leq M(0)$   
 $0 \leq \lim_{x \rightarrow c} [f(x)g(x)] \leq 0$

Therefore,  $\lim_{x \rightarrow c} [f(x)g(x)] = 0$ .

112. (a) If  $\lim_{x \rightarrow c} |f(x)| = 0$ , then  $\lim_{x \rightarrow c} [-|f(x)|] = 0$ .

$$-|f(x)| \leq f(x) \leq |f(x)|$$

$$\lim_{x \rightarrow c} [-|f(x)|] \leq \lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} |f(x)|$$

$$0 \leq \lim_{x \rightarrow c} f(x) \leq 0$$

Therefore,  $\lim_{x \rightarrow c} f(x) = 0$ .

- (b) Given  $\lim_{x \rightarrow c} f(x) = L$ :

For every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - c| < \delta. \text{ Since}$$

$$\left| |f(x)| - |L| \right| \leq |f(x) - L| < \varepsilon \text{ for } |x - c| < \delta,$$

then  $\lim_{x \rightarrow c} |f(x)| = |L|$ .