Solutions to Problems Chapter 1

l. Derive the wave equation for the electric and magnetic fields starting from Maxwell's equations in a homogeneous isotropic source free region (see Chapter 3). How does this change if the material is anisotropic?

A1. This is an exercise in vector manipulations.   By taking the curl of both sides of $\vec{\nabla} \times \vec{E} = \partial \vec{B} / \partial t$ we get $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu \partial (\vec{\nabla} \times \vec{H}) / \partial t$, where we have used the constitutive relation $\vec{B} = \mu \vec{H}$ and assumed μ to be a space- and time-independent scalar. Now, employing $\vec{\nabla} \times \vec{H} = \vec{J} = \vec{J}_i + \vec{J}_d = \vec{J}_i + \partial \vec{D} / \partial t$,   the   above   equation   becomes $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu \varepsilon \partial^2 \vec{E} / \partial t^2 - \mu \partial \vec{J}_i / \partial t$, where we have used the constitutive relation $\vec{D} = \varepsilon \vec{E}$ and assumed ε to be a space- and time-independent scalar.   Thereafter, by using the vector relationship $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \bullet \vec{E}) - \nabla^2 \vec{E}$,   we   get $\nabla^2 \vec{E} - \mu \varepsilon \partial^2 \vec{E} / \partial t^2 = \mu \partial \vec{J}_i / \partial t + \vec{\nabla}(\vec{\nabla} \bullet \vec{E})$. Finally by using $\vec{\nabla} \bullet \vec{D} = \rho$ and the constitutive relation for $\vec{D}$ above, we obtain $\nabla^2 \vec{E} - \mu \varepsilon \partial^2 \vec{E} / \partial t^2 = \mu \partial \vec{J}_i / \partial t + (1 / \varepsilon) \vec{\nabla} \rho$.
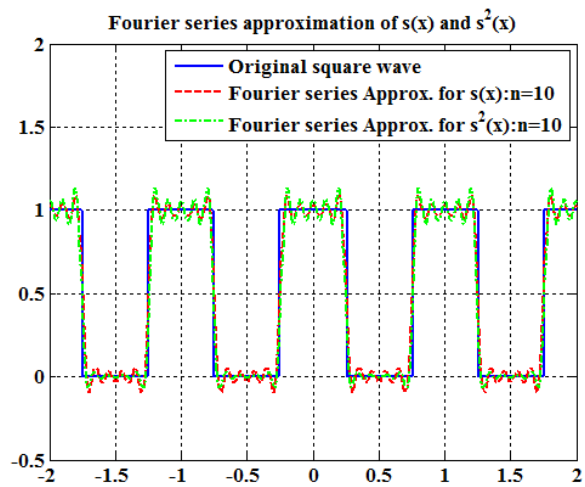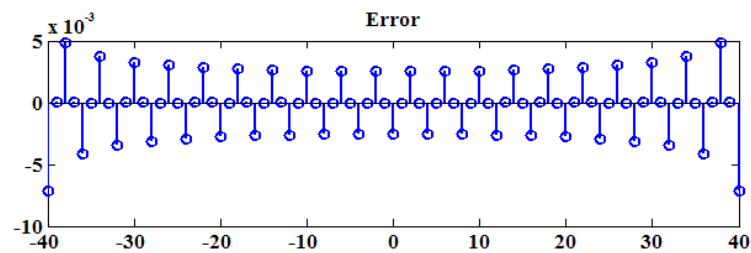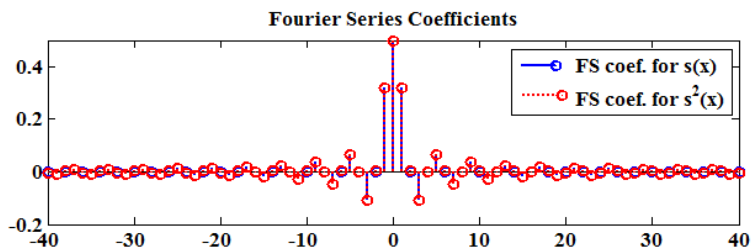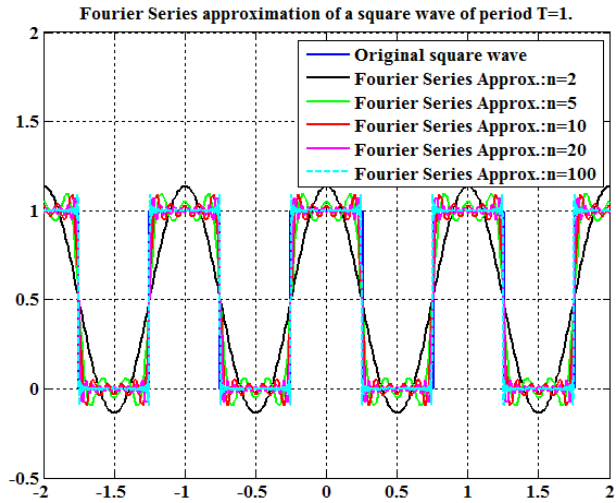
   If the material is anisotropic, the wave equation becomes a little more complicated. For instance, in space free of all independent current sources, $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu \partial^2 \left(\overline{\overline{\varepsilon}} \vec{E} \right) / \partial t^2$. Assuming monochromatic plane wave dependence of the electric field of the form $\exp j(\omega_0 t - \vec{k}_0 \cdot \vec{R})$, the wave equation above becomes $\vec{k}_0 \times \vec{k}_0 \times \vec{E} + \omega_0^2 \mu \overline{\overline{\varepsilon}} \vec{E} = 0$.
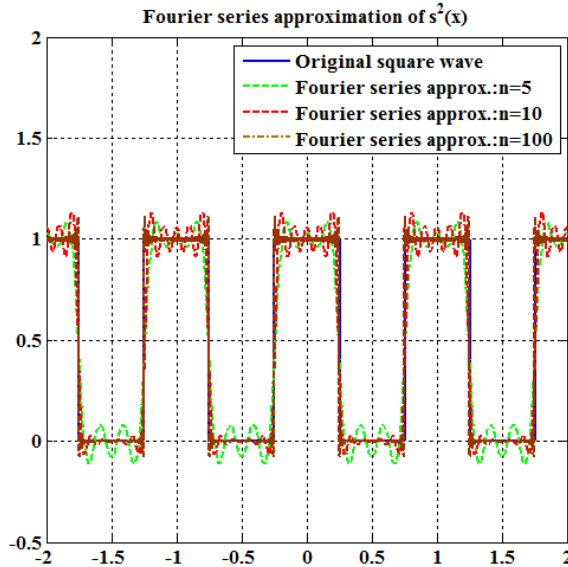
2. Find from first principles the Fourier series coefficients for a periodic square wave $s(x)$ of unit amplitude and 50% duty cycle. Now find the Fourier series coefficients of $s^2(x)$ (a) from first principles and (b) using the Laurent rule. Plot $s^2(x)$ vs x by employing the Fourier series coefficients you found using (b). Use 5, 10 and 100 Fourier coefficients. Describe the general trend(s).

A2. Assume $s(x) = \begin{cases} 1 \ \forall -0.25 \le x < 0.25 \\ 0 \ \forall 0.25 \le x < 0.75 \end{cases}$; then the Fourier coefficients are given by

$F_n = \int_{-0.25}^{0.25} e^{-jn2\pi x} dx = \dfrac{1}{2} \dfrac{\sin(n\pi/2)}{(n\pi/2)}$. Since in this case $s^2(x) = s(x)$, its Fourier coefficients, call

that H_n, must be the same. Laurent's rule states that $H_n = \displaystyle\sum_{-\infty}^{\infty} F_{n-m} F_m$.

Fourier Series approximation of a square wave of period T=1.

Legend:
- Original square wave
- Fourier Series Approx.:n=2
- Fourier Series Approx.:n=5
- Fourier Series Approx.:n=10
- Fourier Series Approx.:n=20
- Fourier Series Approx.:n=100

Fourier Series Coefficients

Legend:
- FS coef. for $s(x)$
- FS coef. for $s^2(x)$

Error

x 10^{-3}

Fourier series approximation of $s(x)$ and $s^2(x)$

Legend:
- Original square wave
- Fourier series Approx. for $s(x)$:n=10
- Fourier series Approx. for $s^2(x)$:n=10

Fourier series approximation of $s^2(x)$

3. Find the two-dimensional Fourier transform of a rectangle (*rect*) function of unit height and width a in each dimension.

A3. The Fourier transform in one transverse dimension (x) is

$$\int_{-\infty}^{\infty} rect(x/a)\exp jk_x x dx = \int_{-a/2}^{a/2} 1\cdot \exp jk_x x dx = \frac{\exp jk_x a/2 - \exp - jk_x a/2}{jk_x} = a\frac{\sin k_x a/2}{k_x a/2}.$$ The

result in the y-dimension is similar, so that $\Im_{xy}[rect(x/a, y/a) = a^2 \left(\dfrac{\sin k_x a/2}{k_x a/2}\right)\left(\dfrac{\sin k_y a/2}{k_y a/2}\right).$

4. Show that the two-dimensional Fourier transform of a Gaussian function of width w is another Gaussian function. Functions like this are called self- Fourier transformable. Find its width in the spatial frequency domain. Can you think of any other functions that are self-Fourier transformable?

A4. Assume a two-dimensional Gaussian of the form $f(x,y) = \exp-(x^2 + y^2) = \exp- r^2$. The

2D Fourier transform is $F(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\exp j(k_x x + k_y y)dxdy = F_1(k_x)F_2(k_y)$, where

$$F_1(k_x) = \int_{-\infty}^{\infty} \exp- x^2 \bullet \exp jk_x x dx = \exp- k_x^2/4 \int_{-\infty}^{\infty} \exp-(x - jk_x/2)^2 dx = \exp- k_x^2/4 \int_{-\infty}^{\infty} \exp- x'^2 dx'$$

The integral is equal to $\sqrt{\pi}$, so that $F_1(k_x) = \sqrt{\pi}\exp- k_x^2/4$. Similarly $F_2(k_y) = \sqrt{\pi}\exp- k_y^2/4$, hence, $F(k_x, k_y) = \pi\exp-(k_x^2 + k_y^2)/4$, which is also a Gaussian. Other examples of self-Fourier-transformable function is $f(x,y) = \sec h(x) \bullet \sec h(y)$ and $f(x,y) = comb(x) \bullet comb(y)$ where the *comb* is a periodic sequence of delta functions.

5. Find the Hankel transform of (a) a circular function defined as $g(\rho) = circ(\rho/\rho_0)$, which has a value of 1 within a circle of radius ρ_0 and is 0 otherwise; (b) a Gaussian function given by $g(\rho) = \exp-(\rho/\rho_0)^2$.

A5. The 2-D Fourier transform, or the Hankel transform, is defined by transforming the 2-D Fourier transform integral $G(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \exp j(k_x x + k_y y) dx dy$ to polar coordinates by substituting $x = \rho \cos\theta, y = \rho \sin\theta; k_x = k_\rho \cos\phi, k_y = k_\rho \sin\phi$. This transforms the above integral to $\overline{G}(k_\rho, \phi) = \int_0^{\infty} \int_0^{2\pi} \rho \overline{g}(\rho) \exp jk_\rho \rho \cos(\theta - \phi) d\theta d\rho = 2\pi \int_0^{\infty} \rho \overline{g}(\rho) J_0(k_\rho \rho) d\rho = \overline{G}(k_\rho)$, where we have used the integral definition of the Bessel function J_0.

   (a) If $\overline{g}(\rho) = circ(\rho/\rho_0), \overline{G}(k_\rho) = 2\pi \int_0^{\rho_0} \rho J_0(k_\rho \rho) d\rho = \dfrac{2\pi\rho_0}{k_\rho} J_1(k_\rho \rho_0)$, using the integral definition of the Bessel function J_1.

   (b) If $g(\rho) = \exp-(\rho/\rho_0)^2$,

$$\overline{G}(k_\rho) = 2\pi \int_0^{\infty} \rho \exp(-\rho/\rho_0)^2 J_0(k_\rho \rho) d\rho = \pi\rho_0^2 \exp(-k_\rho^2 \rho_0^2/4),$$ using integral tables. All integral tables and integral definitions are taken from Gradshteyn and Ryzhik, Table of Integrals, Series and Products, Academic, New York (1980).

6. Find the DFT of a square wave function using a software of your choice. Comment on the nature of the spectrum numerically computed as the width of the square wave changes.
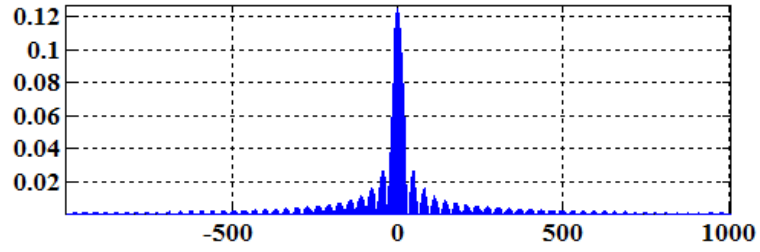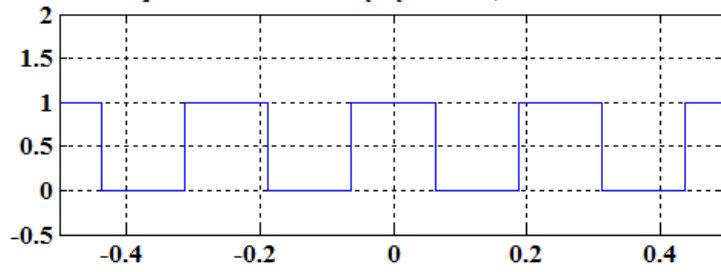
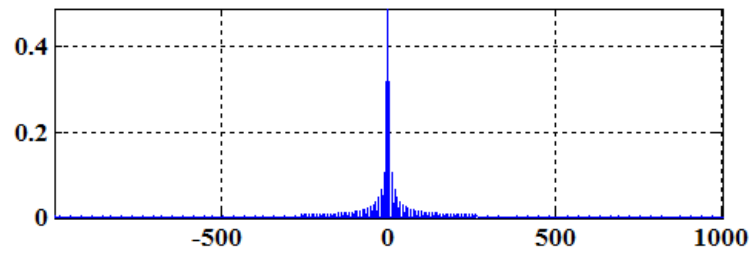A6.
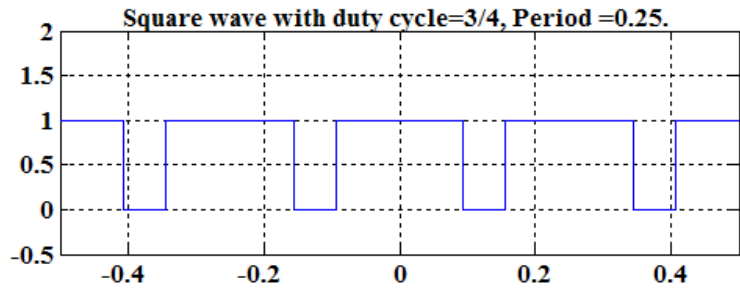

Square wave with duty cycle=1/32, Period =0.25.
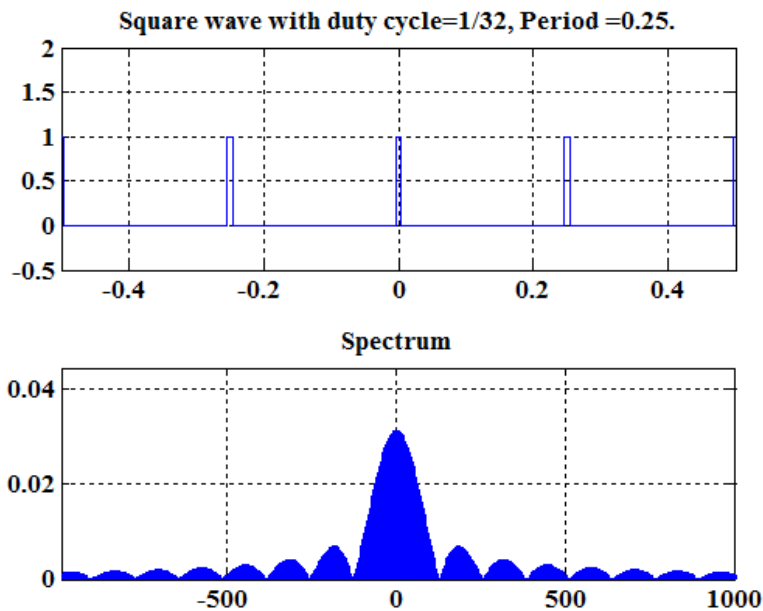
Spectrum

Square wave with duty cycle=1/8, Period =0.25.
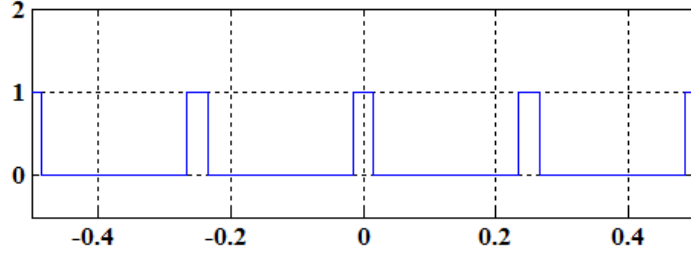
Spectrum

Square wave with duty cycle=1/2, Period =0.25.

Spectrum

Square wave with duty cycle=3/4, Period =0.25.

Spectrum

Square wave with duty cycle=1, Period =0.25.

Spectrum

Square wave with duty cycle 1/2^10, Period=0.25
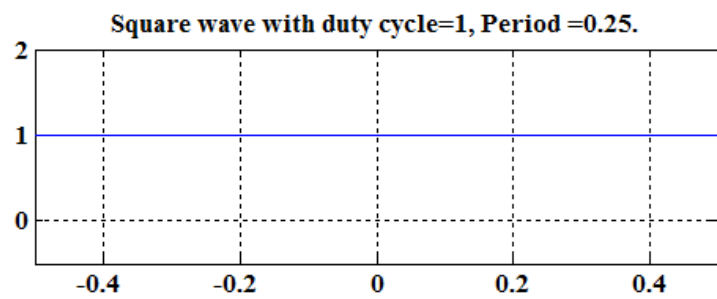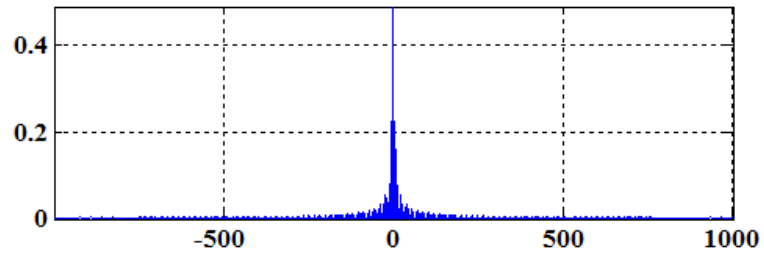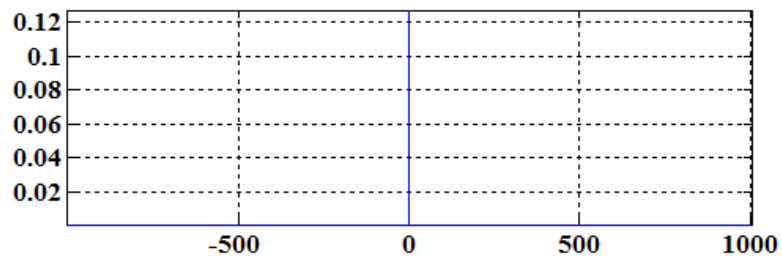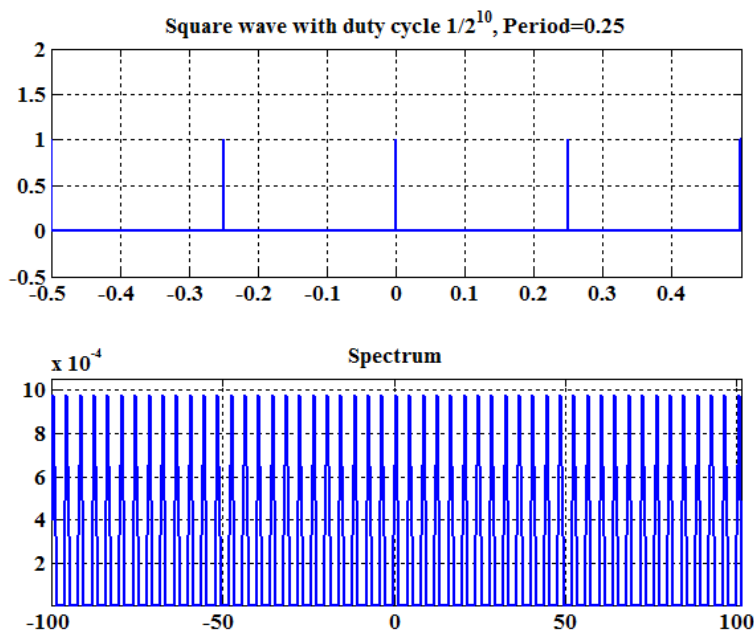

Spectrum

7. Find $\sin \underline{A}$ where $\underline{A}$ is a matrix given by $\begin{pmatrix} 1 & 20 & 0 \\ -1 & 7 & 1 \\ 3 & 0 & -2 \end{pmatrix}$ using the Cayley Hamilton theorem

[**7**].

A7. The eigenvalues of the matrix A are $\lambda_{i=1,2,3} = 1,2,3$, respectively. Now by the Cayley-Hamilton theorem, $\sin A = a_0 I + a_1 A + a_2 A^2$ and $\sin \underline{\lambda_i} = a_0 + a_1 \lambda_i + a_2 \lambda_i^2$, $i = 1,2,3$. Therefore

$$\sin 1 = a_0 + a_1 + a_2,$$
$$\sin 2 = a_0 + 2a_1 + 4a_2,$$
$$\sin 3 = a_0 + 3a_1 + 9a_2,$$

from which it follows that

$$a_0 = 3\sin 1 - 3\sin 2 + \sin 3,$$
$$a_1 = -(5/2)\sin 1 + 4\sin 2 - (3/2)\sin 3,$$
$$a_2 = (1/2)\sin 1 - \sin 2 + (1/2)\sin 3.$$

Substituting these values into $\sin A = a_0 I + a_1 A + a_2 A^2$ finally gives

$$\sin \begin{pmatrix} 1 & 20 & 0 \\ -1 & 7 & 1 \\ 3 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -9 & 30 & 10 \\ 0 & 0 & 0 \\ -9 & 30 & 10 \end{pmatrix} \sin 1 + \begin{pmatrix} 20 & -80 & -20 \\ 1 & -4 & -1 \\ 15 & -60 & -15 \end{pmatrix} \sin 2 + \begin{pmatrix} -10 & 50 & 10 \\ -1 & 5 & 1 \\ -6 & 30 & 6 \end{pmatrix} \sin 3 .$$

Solutions to Problems Chapter 2

l. Assume a Gaussian beam in air with plane wavefronts and waist w_0 at a distance d_0 from a converging lens of focal length f.

(a) Using the laws of q-transformation, find the distance behind the lens where the Gaussian beam focuses, i.e., again has plane wavefronts.

(b) Using the beam propagation method, simulate the propagation of the beam through air and through a lens.

(c) By setting $d_0 = f$, determine the profile of the beam a distance f behind the lens.

(d) By setting $d_0 = 2f$, determine the profile of the beam distances f and $2f$ behind the lens.

A1. (a) Using laws of q-transformation, the q after the lens and after a distance of propagation z

is $\quad q(z) = \dfrac{f(q_0 + d_0)}{f - (q_0 + d_0)} + z$, $q_0 = jz_R = jk_0 w_0^2/2$. $\quad$ So $q(z) = f\dfrac{[d_0(f - d_0) - z_R^2] + jz_R f}{(f - d_0)^2 + z_R^2} + z$.

When the Gaussian beam focuses, its q becomes purely imaginary again; hence its real part

becomes equal to zero. Using the above relations, this occurs at $z = \dfrac{f(z_R^2 + d_0^2 - fd_0)}{(f - d_0)^2 + z_R^2}$. The

imaginary part yields the new beam waist with a Raleigh range $z_R' = \dfrac{f^2 z_R}{(f - d_0)^2 + z_R^2}$.

(b) Assume initial Gaussian beam with $w_0 = 0.1mm$, wavelength $\lambda = 0.5\mu m$.
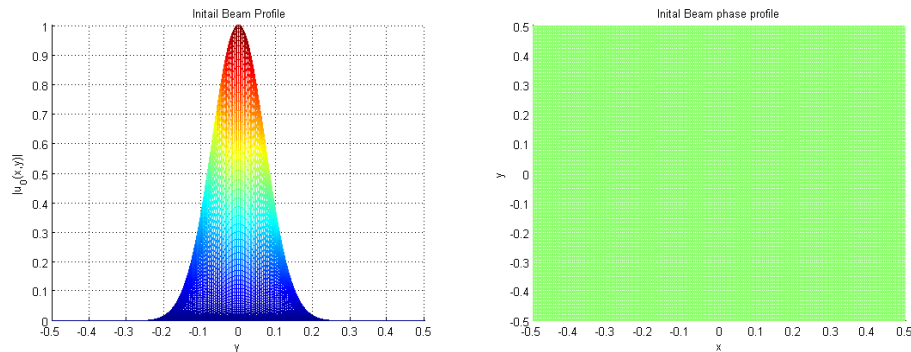
Beam propagates by the Rayleigh range $z_R = 63.8mm = f$, the focal length of the lens.
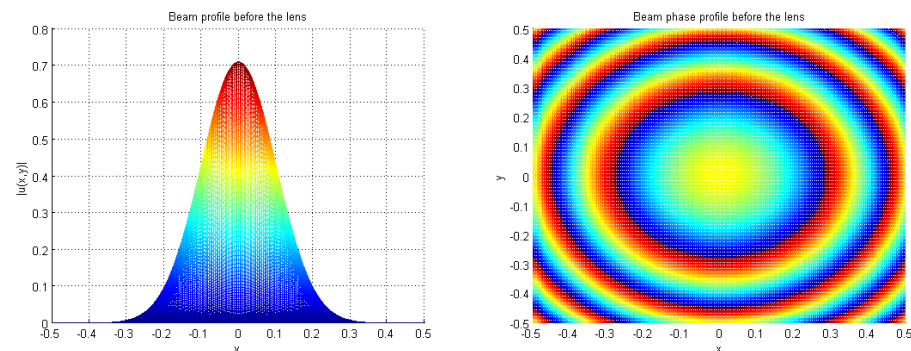
Immediately after lens, the phase should be equal in magnitude to the incident phase but opposite in sign.