Chapter 2

Problem 2.1

Let $g(t) \rightleftharpoons G(f)$. Then

$$g(at) \rightleftharpoons \frac{1}{|a|}G(\frac{f}{a}) \tag{1}$$

To prove this property, we note that

$$F[g(at)] = \int_{-\infty}^{\infty} g(at)\exp(-\mathrm{j}2\pi ft)\mathrm{d}t$$

Set $\tau = at$. There are two cases that can arise, depending on whether the scaling factor a is positive or negative. If $a > 0$, we get

$$F[g(at)] = \frac{1}{a}\int_{-\infty}^{\infty} g(\tau)\exp\left[-\mathrm{j}2\pi\left(\frac{f}{a}\right)\tau\right]\mathrm{d}\tau$$
$$= \frac{1}{a}G\left(\frac{f}{a}\right)$$

On the other hand, if $a < 0$, the limits of integration are interchanged so that we have the multiplying factor $-(1/a)$ or, equivalently, $1/|a|$. This completes the proof of Eq. (1).

Note that the function $g(at)$ represents $g(t)$ compressed in time by a factor a, whereas the function $G(f/a)$ represents $G(f)$ expanded in frequency by the same factor a. Thus, the scaling property states that the compression of a function $g(t)$ in the time domain is equivalent to the expansion of its Fourier transform $G(f)$ in the frequency domain by the same factor, or vice versa.

For the special case when $a = -1$, we readily find from Eq. (1) that

$$g(-t) \rightleftharpoons G(-f) \tag{2}$$

In words, if a function $g(t)$ has a Fourier transform given by $G(f)$, then the Fourier transform of $g(-t)$ is $G(-f)$.

Problem 2.2

(a) This property follows from the relation defining the inverse Fourier Transform by writing it in the form:

$$g(-t) = \int_{-\infty}^{\infty} G(f) \exp(-j2\pi ft) \mathrm{d}f$$

and then interchanging t and f.
(b) To prove this property, we take the Fourier transform of $g(t - t_0)$ and then set $\tau = (t - t_0)$ to obtain

$$F[g(t - t_0)] = \exp(-\mathrm{j}2\pi f t_0) \int_{-\infty}^{\infty} g(\tau) \exp(-\mathrm{j}2\pi f\tau) \mathrm{d}\tau$$

The time-shifting property states that if a function $g(t)$ is shifted in the positive direction by an amount t_0, the effect is equivalent to multiplying its Fourier transform $G(f)$ by the factor $\exp(-\mathrm{j}2\pi f t_0)$. This means that the amplitude of $G(f)$ is unaffected by the time shift, but its phase is changed by the linear factor $-2\pi f t_0$.
(c) We note that

$$\cos(2\pi f_c t) = \frac{1}{2}[\exp(\mathrm{j}2\pi f_c t) + \exp(-\mathrm{j}2\pi f_c t)]$$

and

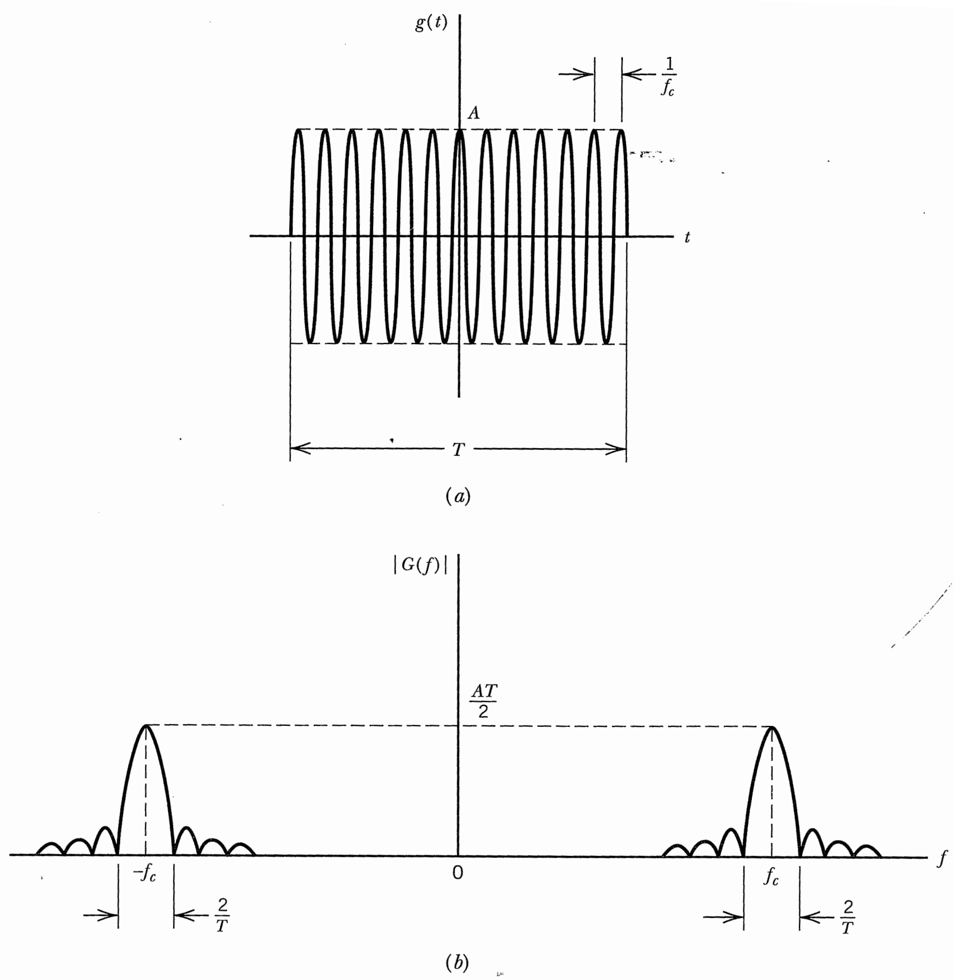
$$A\mathrm{rect}\left(\frac{t}{T}\right) \rightleftharpoons AT\,\mathrm{sinc}(fT) \tag{1}$$

Therefore, applying the frequency-shifting property to the Fourier-transform pair of Eq. (1), we get the desired result

$$G(f) = \frac{AT}{2}\left\{\mathrm{sinc}[T(f - f_c)] + \mathrm{sinc}[T(f + f_c)]\right\}$$

In the special case of $f_c \gg \dfrac{1}{T}$, we may use the approximate result

$$G(f) \simeq \begin{cases} \frac{AT}{2}\,\text{sinc}[T(f-f_c)], & f>0 \\ 0, & f=0 \\ \frac{AT}{2}\,\text{sinc}[T(f+f_c)], & f<0 \end{cases}$$



(a) RF pulse. (b) Amplitude spectrum.

Problem 2.3

(a) To prove this property, we first denote the Fourier transform of the product $g_1(t)g_2(t)$ by $G_{12}(f)$, so that we may write

$$g_1(t)g_2(t) \rightleftharpoons G_{12}(f)$$

where

$$G_{12}(f) = \int_{-\infty}^{\infty} g_1(t)g_2(t)\exp(-\mathrm{j}2\pi ft)\mathrm{d}t$$

For $g_2(t)$, we next substitute the inverse Fourier transform

$$g_2(t) = \int_{-\infty}^{\infty} G_2(f')\exp(-\mathrm{j}2\pi f't)\mathrm{d}f'$$

in the integral defining $G_{12}(f)$ to obtain

$$G_{12}(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(t)G_2(f')\exp[-\mathrm{j}2\pi(f-f')t]\mathrm{d}f'\mathrm{d}t$$

Define $\lambda = f - f'$. Then, interchanging the order of integration, we obtain

$$G_{12}(f) = \int_{-\infty}^{\infty} d\lambda G_2(f-\lambda)\int_{-\infty}^{\infty} g_1(t)\exp(-\mathrm{j}2\pi\lambda t)\mathrm{d}t$$

The inner integral is recognized simply as $G_1(\lambda)$, so we may write

$$G_{12}(f) = \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)\mathrm{d}\lambda$$

which is the desired result. This integral is known as the convolution integral expressed in the frequency domain, and the function $G_{12}(f)$ is referred to as the convolution of $G_1(f)$ and $G_2(f)$. We conclude that the multiplication of two signals in the time domain is transformed into the convolution of their individual Fourier transforms in the frequency domain. This property is known as the multiplication theorem.

In a discussion of convolution, the following shorthand notation is frequently used

$$G_{12}(f) = G_1(f) \star G_2(f)$$

Accordingly, we may use the following symbolic form:

$$g_1(t)g_2(t) \rightleftharpoons G_1(f) \star G_2(f) \tag{1}$$

Note that convolution is commutative, that is,

$$G_1(f) \star G_2(f) = G_2(f) \star G_1(f)$$

which follows directly from Eq. (1).
(b) This result follows directly by combining duality property and time-domain multiplication property. We may thus state that the convolution of two signals in the time domain is transformed into the multiplication of their individual transforms in the frequency domain. This property is known as the convolution theorem. Its use permits us to exchange a convolution operation for a transform multiplication, an operation that is ordinarily easier to manipulate. Accordingly, we may write:

$$g_1(t) \star g_2(t) = G_1(f)G_2(f)$$

where $\star$ denotes convolution.
(c)

According to the convolution theorem:

$$\int_{-\infty}^{t} g_1(\tau)g_2(\tau - t)\,\mathrm{d}\,\tau \rightleftharpoons G_1(f)G_2(f)$$

Suppose we complex conjugate the $G_2(f)$, yielding $G_2^*(f)$, by definition if $g_2(t) \rightleftharpoons G_2(f)$, then $g_2^*(t) \rightleftharpoons G_2^*(f)$. Therefore, we may go on to write:

$$\int_{-\infty}^{\infty} g_1(\tau)g_2^*(\tau - t)\,\mathrm{d}\,\tau \rightleftharpoons G_1(f)G_2^*(f)$$

which confirms the correlation theorem, entry 13 in Table 2.1.

Problem 2.4

This result follows from noting that the energy intensity $|g(t)|^2$ may be expressed as the product of two time functions, namely, $g(t)$ and its complex conjugate $g^*(t)$. The Fourier transform of $g^*(t)$ is equal to $G^*(-f)$, by virtue of complex conjugation property. Then, applying the multiplication theorem to the product $g(t)g^*(t)$ and evaluating the result for $f = 0$, we obtain the relation

$$\int_{-\infty}^{\infty} g(t)g^*(t)\mathrm{d}t = \int_{-\infty}^{\infty} G(\lambda)G^*(\lambda)\mathrm{d}\lambda$$

which is equivalent to

$$\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |G(\lambda)|^2 \mathrm{d}\lambda$$

Let $\epsilon_g(f)$ denote the squared amplitude spectrum of the signal $g(t)$, as shown by

$$\epsilon_g(f) = |G(f)|^2$$

The quantity $\epsilon_g(f)$ is referred to as the energy spectral density of the signal $g(t)$. To explain this meaning of the definition, suppose $g(t)$ denotes the voltage of a source connected to a 1-ohm load resistor. Then the quantity

$$\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{d}t$$

equals the total energy delivered by the source. According to Rayleigh's theorem, this energy equals the total area under the $\epsilon_g(f)$ curve. If follows, therefore, that the function $\epsilon_g(f)$ is a measure of the density of the energy contained in $g(t)$ in joules per Hertz. Note that since in the special case of a real-valued signal the amplitude spectrum is an even function of f, the energy spectral density of such a signal is symmetrical about the vertical axis passing through the origin.

Problem 2.5

By the differentiation property:

$$\begin{aligned} F\left(\frac{\mathrm{d}g(t)}{\mathrm{d}t}\right) &= j2\pi f G(f) \\ &= \tfrac{1}{\tau}\left[H(f)\exp(j2\pi f\tau) - H(f)\exp(-j2\pi f\tau)\right] \\ &= \tfrac{2j}{\tau} H(f)\sin(2\pi f\tau) \end{aligned}$$

But $H(f) = \tau \exp(-\pi f^2 \tau^2)$

$$\therefore G(f) = \tfrac{1}{\pi f} \exp(-\pi f^2 \tau^2) \sin(2\pi f T)$$

$$= \exp(-\pi f^2 \tau^2) \tfrac{\sin(2\pi f T)}{\pi f}$$

$$= 2T \exp(-\pi f^2 \tau^2) \text{sinc}(2\pi f T)$$

$$\lim_{\tau \to 0} G(f) = 2T \text{sinc}(2\pi f T)$$

Problem 2.6

(a)

$$\begin{aligned}
G(f) &= \int_{-\infty}^{\infty} g(t) \exp(-\mathrm{j}2\pi f t) \mathrm{d}t \\
&= \int_{-\infty}^{0} g(t) \exp(-\mathrm{j}2\pi f t) \mathrm{d}t + \int_{0}^{\infty} g(t) \exp(-\mathrm{j}2\pi f t) \mathrm{d}t \\
&= \int_{-\infty}^{0} g(t) \cos(2\pi f t) \mathrm{d}t - \int_{-\infty}^{0} \mathrm{j} g(t) \sin(2\pi f t) \mathrm{d}t \\
&+ \int_{0}^{\infty} g(t) \cos(2\pi f t) \mathrm{d}t - \int_{0}^{\infty} \mathrm{j} g(t) \sin(2\pi f t) \mathrm{d}t
\end{aligned}$$

If $g(t)$ is even, then $g(t) = -g(t)$. Hence

$$\int_{-\infty}^{0} g(t) \cos(2\pi f t) \mathrm{d}t = \int_{0}^{\infty} g(t) \cos(2\pi f t) \mathrm{d}t$$

$$\int_{-\infty}^{0} g(t) \sin(2\pi f t) \mathrm{d}t = -\int_{0}^{\infty} g(t) \sin(2\pi f t) \mathrm{d}t$$

and $G(f) = 2\int_{0}^{\infty} g(t) \cos(2\pi f t) \mathrm{d}t$, which is purely real.

If, on the other hand, $g(t)$ is odd, then $g(t) = -g(-t)$. Hence

$$\int_{-\infty}^{0} g(t) \sin(2\pi f t) \mathrm{d}t = \int_{0}^{\infty} g(t) \sin(2\pi f t) \mathrm{d}t$$

$$\int_{-\infty}^{0} g(t)\cos(2\pi ft)\mathrm{d}t = -\int_{0}^{\infty} g(t)\cos(2\pi ft)\mathrm{d}t$$

and $G(f) = -2\mathrm{j}\int_{0}^{\infty} g(t)\sin(2\pi ft)\mathrm{d}t$, which is purely imaginary.

(b) The Fourier transform of $g(t)$ is defined by

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-\mathrm{j}2\pi ft)\mathrm{d}t$$

Differentiating both sides of this relation n times with respect to f:

$$\frac{\mathrm{d}^n G(f)}{\mathrm{d}f^n} = (-\mathrm{j}2\pi)^n \int_{-\infty}^{\infty} t^n g(t)\exp(-\mathrm{j}2\pi ft)\mathrm{d}t \tag{1}$$

That is,

$$t^n g(t) \rightleftharpoons \left(\frac{j}{2\pi}\right)^n \frac{\mathrm{d}^n G(f)}{\mathrm{d}f^n}$$

(c) Putting $f = 0$ in Eq. (1), we get

$$\int_{-\infty}^{\infty} t^n g(t)\mathrm{d}t = \left(\frac{j}{2\pi}\right)^n G^{(n)}(0)$$

where $G^{(n)}(f) = \dfrac{\mathrm{d}^n G(f)}{\mathrm{d}f^n}$

(d) Since $g_2^*(t) \rightleftharpoons G_2^*(-f)$, it follows that

$$g_1(t)g_2^*(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda - f)\mathrm{d}\lambda$$

From this result we deduce the Fourier transform

$$\begin{aligned} F[g_1(t)g_2^*(t)] &= \int_{-\infty}^{\infty} g_1(t)g_2^*(t)\exp(-\mathrm{j}2\pi ft)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda - f)\mathrm{d}\lambda \end{aligned} \tag{2}$$

Setting $f = 0$ in Eq. (2), we get the desired relation

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\mathrm{d}t = \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda)\mathrm{d}\lambda$$

Problem 2.7

The Hilbert transform of $\frac{\mathrm{d}}{\mathrm{d}\,t}g(t)$ is, by definition:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}g(t)\right)\frac{1}{t-\tau}\,\mathrm{d}\,\tau$$

Interchange t with τ and vice versa:

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}g(t)\right)\frac{1}{\tau-t}\,\mathrm{d}\,t\\ &=-\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}g(t)\right)\frac{1}{t-\tau}\,\mathrm{d}\,t\end{aligned}$$

which is immediately recognized as inverse Hilbert transform of $\frac{\mathrm{d}}{\mathrm{d}\,t}g(t)$.

Conversely, we may say that $\frac{\mathrm{d}}{\mathrm{d}\,t}\hat{g}(t)$ is the Hilbert transform of $\frac{\mathrm{d}}{\mathrm{d}\,t}g(t)$, where $\hat{g}(t)$ is the Hilbert transform of $g(t)$.

Problem 2.8

We are given

$$\delta_{T_0}(t)=\sum_{m=-\infty}^{\infty}\delta(t-mT_0)$$

a) Taking the Fourier transform of both sides:

$$F\left[\delta_{T_0}(t)\right]=\sum_{m=-\infty}^{\infty}F\left[\delta(t-mT_0)\right]$$

Since $F\left[\delta(t)\right]=1$, then

$$\begin{aligned}F\left[\delta(t-mT_0)\right]&=\exp\left(-\mathrm{j}\,2\pi fT_0\right)\\ &=\cos(2\pi mfT_0)-\mathrm{j}\sin(2\pi mfT_0)\\ &=\cos(2\pi mfT_0)\end{aligned}$$

where the imaginary term is zero. Moreover,

$$\sum_{m=-\infty}^{\infty} \cos(2\pi m f T_0) = \delta(f) \qquad \text{for} \ \ -\frac{1}{2} \leq f \leq \frac{1}{2}$$

Therefore,

$$F\left[\delta_1(f)\right] \quad \text{for} \ \ T_0 = 1 \text{ and } f_0 = 1$$

Now introducing the scaling factor T_0 into play, we have

$$F\left[\delta_{T_0}(t)\right] = \frac{1}{T_0}\delta_{f_0}(f) = f_0\delta_{f_0}(f)$$

Equivalently, we may write

$$\delta_{T_0}(t) \rightleftharpoons \frac{1}{T_0}\delta_{f_0}(f)$$

b) from (2.34) in the text, we have

$$\begin{aligned}\delta(f) &= \sum_{m=-\infty}^{\infty} \cos(2\pi m f) \qquad -\frac{1}{2} \leq f \leq \frac{1}{2} \\ &= \sum_{m=-\infty}^{\infty} \exp(\mathrm{j}\, 2\pi m f)\end{aligned}$$

Introducing the scaling factor $f_0 = 1/T_0$, we may go on to write

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) = f_0 \sum_{n=-\infty}^{\infty} \exp(\mathrm{j}\, 2\pi n f_0 t)$$

According to the duality theorem, we have

$$T_0 \sum_{m=-\infty}^{\infty} \exp(\mathrm{j}\, 2\pi m f T_0) = \sum_{n=-\infty}^{\infty} \delta(f - n f_0) \tag{1}$$

b) Putting $T_0 = 1$, (1) reduces to

$$\sum_{m=-\infty}^{\infty} \exp(\mathrm{j}\, 2\pi m f) = \sum_{n=-\infty}^{\infty} \delta(f - n)$$

Problem 2.9

We are given

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_0)\delta(t - nT_0).$$

Transforming this expression into the frequency domain, we get

$$\begin{aligned} X(f) &= F[x(t)] \\ &= \sum_{n=-\infty}^{\infty} x(nT_0)\exp(-\mathrm{j}\,2\pi n f_0) \qquad \text{where } f_0 = \frac{1}{T_0}. \end{aligned}$$

Therefore, the Fourier transform of the output $y(t)$ is

$$\begin{aligned} Y(f) &= H(f)X(f) \\ &= H(f)\sum_{n=-\infty}^{\infty} x(nT_0)\exp(-\mathrm{j}\,2\pi n f_0). \end{aligned}$$

Thus, applying Parseval's theorem in Table 2.1, average power is given by

$$\begin{aligned} P_{av} &= |Y(f)|^2 \\ &= \sum_{n=-\infty}^{\infty} |x(nT_0)|^2 \cdot |H(f)|^2 \end{aligned}$$

(**Correction: the m in the first summation in the text should be n.**)

Problem 2.10

The system is said to be stable if the output signal is bounded for all bounded input signals; we refer to this as the bounded input-bounded output (BIBO) stability criterion, which is well suited for the analysis of linear time invariant systems. Let the input signal $x(t)$ be bounded, as shown by

$$|x(t)| \le M$$

where M is a positive real finite number. Substituting $x(t)$ in

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau$$

we get

$$|y(t)| \leq M \int_{-\infty}^{\infty} |h(\tau)| \mathrm{d}\tau$$

It follows, therefore, that for a linear time-invariant system to be stable, the impulse response $h(t)$ must be absolutely integrable. That is, the necessary and sufficient condition for BIBO stability is

$$\int_{-\infty}^{\infty} |h(\tau)| \mathrm{d}\tau \leq \infty$$

Problem 2.11

Property 1
The signal $g(t)$ and its Hilbert transform $\hat{g}(t)$ have the same amplitude spectrum.

To prove this property, we observe that the Fourier transform of $\hat{g}(t)$ is equal to $-j\text{sgn}(f)$ times the Fourier transform of $g(t)$, and since the magnitude of $-sgn(f)$ is equal to one for all f, the $g(t)$ and $\hat{g}(t)$ will both have the same amplitude spectrum. As a corollary to this property, we may state that if a signal $f(t)$ is band limited, then its Hilbert transform $\hat{g}(t)$ will also be band limited.

Property 2
If $\hat{g}(t)$ is the Hilbert transform of $g(t)$, then the Hilbert transform of $\hat{g}(t)$ is $-g(t)$.

To prove this property, we note that the process of Hilbert transformation is equivalent to passing $g(t)$ through a two-port device with a transfer function equal to $-j\text{sgn}(f)$. A double Hilbert transformation is therefore equivalent to passing $g(t)$ through a cascade of two such devices. The overall transfer function of such a cascade is equal to

$$[-\mathrm{j}\,\mathrm{sgn}(f)]^2 = -1 \quad \text{for all } f$$

The resulting output is thus $-g(t)$; that is, the Hilbert transform of $\hat{g}(t)$ is equal to $-g(t)$. This results is subject to the requirement that $G(0) = 0$, where $G(0)$ is the Fourier transform of $g(t)$ evaluated at $f = 0$.

Property 3
A signal $g(t)$ and its Hilbert transform $\hat{g}(t)$ are orthogonal.

To prove this property, we use a special case of the multiplication theorem. In particular, for a signal $g(t)$ multiplied by its Hilbert transform $\hat{g}(t)$ we may write

$$\int_{-\infty}^{\infty} g(t)\hat{g}(t)\,\mathrm{d}\,t = \int_{-\infty}^{\infty} G(f)\hat{G}(-f)\,\mathrm{d}\,f \tag{1}$$

which can be rewritten as:

$$\begin{aligned} \int_{-\infty}^{\infty} g(t)\hat{g}(t)\,\mathrm{d}\,t &= \mathrm{j}\int_{-\infty}^{\infty} \mathrm{sgn}(f)G(f)G(-f)\,\mathrm{d}\,f \\ &= \mathrm{j}\int_{-\infty}^{\infty} \mathrm{sgn}(f)G(f)G^{*}(f)\,\mathrm{d}\,f \\ &= \mathrm{j}\int_{-\infty}^{\infty} \mathrm{sgn}(f)\,|G(f)|^{2}\,\mathrm{d}\,f \end{aligned} \tag{2}$$

where the second line, we have used the fact that for a real-valued signal $G(-f) = G^{*}(f)$. The integrand in the right-hand side of (2) is an odd function of f, being the product of the odd function $\mathrm{sgn}(f)$ and the even function $|G(f)|^{2}$. Hence, the integral is zero, yielding the final result

$$\int_{-\infty}^{\infty} g(t)\hat{g}(t)\,\mathrm{d}\,t = 0 \tag{3}$$

This shows that the energy signal $g(t)$ and its Hilbert transform $\hat{g}(t)$ are orthogonal over the entire interval $(-\infty, \infty)$

Problem 2.12

a)

$$g(t) = \frac{\sin(t)}{t}$$

The Hilbert transform of $g(t)$ is

$$\begin{aligned}
\hat{g}(t) =& \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau} \,\mathrm{d}\,\tau \\
=& \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\tau)}{\tau\,(t-\tau)} \,\mathrm{d}\,\tau \\
=& \frac{1}{\pi t} \int_{-\infty}^{\infty} \left(\frac{1}{\tau} + \frac{1}{t-\tau}\right) \sin(\tau) \,\mathrm{d} \\
=& \frac{1}{\pi t} \int_{-\infty}^{\infty} \frac{\sin(\tau)}{\tau} \,\mathrm{d}\,\tau + \frac{1}{\pi t} \int_{-\infty}^{\infty} \frac{\sin(\tau)}{t-\tau} \,\mathrm{d}\,\tau
\end{aligned}$$

It is well known that:

$$\int_{-\infty}^{\infty} \operatorname{sinc}(y) \,\mathrm{d}\,y = \pi$$

therefore

$$\frac{1}{\pi t} \int_{-\infty}^{\infty} \frac{\sin(\tau)}{\tau} \,\mathrm{d}\,\tau = \frac{1}{t}$$

Likewise, it should be noted that that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{sin(\tau)}{t-\tau} \,\mathrm{d}\,\tau = & \int_{-\infty}^{\infty} \frac{\sin(t-\tau)}{\tau} \,\mathrm{d}\,tau \\
& = \sin(t) \int_{-\infty}^{\infty} \frac{\cos(\tau)}{\tau} \,\mathrm{d}\,\tau - \cos(t) \int_{-\infty}^{\infty} \frac{\sin(\tau)}{\tau} \,\mathrm{d}\,\tau \\
& = -\,\pi \cos(t)
\end{aligned}$$

We thus obtain the Hilbert transform

$$\hat{g}(t) = \frac{1}{t}\,(1 - \cos(t))$$

b)

$$g(t) = \text{rect}(t)$$

The Hilbert transform of $g(t)$ is

$$\hat{g}(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{1}{t-\tau} \, \mathrm{d}\,\tau$$

where P denotes the "principal value of". When $t < -1/2$ the singularity in the integrand is below the range of integration and the significant value of $t - \tau$ are negative. We then have

$$\begin{aligned}\hat{g}(t) &= -\frac{1}{\pi}\left[\ln(t-\tau)\right]_{-1/2}^{1/2} \\ &= -\frac{1}{\pi}\ln\left(\frac{t-1/2}{t+1/2}\right), \quad t < -\frac{1}{2}\end{aligned} \tag{1}$$

When $t > 1/2$, the singularity is above the range of integration, and the significant values of $t - \tau$ are positive. The corresponding value of $\hat{g}(t)$ is

$$\begin{aligned}\hat{g}(t) &= -\frac{1}{\pi}\left[\ln(t-\tau)\right]_{-1/2}^{1/2} \\ &= -\frac{1}{\pi}\ln\left(\frac{t-1/2}{t+1/2}\right), \quad t > \frac{1}{2}\end{aligned} \tag{2}$$

For the case when $-\frac{1}{2} < t < \frac{1}{2}$, we write

$$\begin{aligned}\hat{g}(t) =& \frac{1}{\pi}\lim_{\epsilon\to 0}\left[\int_{-1/2}^{t-\epsilon}\frac{\mathrm{d}\,\tau}{t-\tau} + \int_{t+\epsilon}^{1/2}\frac{\mathrm{d}\,\tau}{t-\tau}\right] \\ =& \frac{1}{\pi}\lim_{\epsilon\to 0}\left\{\left[-\ln(t-\tau)\right]_{-1/2}^{t-\epsilon} + \left[-\ln(t-\tau)\right]_{t+\epsilon}^{1/2}\right\} \\ =& \frac{1}{\pi}\lim_{\epsilon\to 0}\left[-\ln\left(\frac{\epsilon}{t+1/2}\right) - \ln\left(-\frac{t-1/2}{\epsilon}\right)\right] \\ =& -\frac{1}{\pi}\ln\left|\frac{\frac{1}{2}-t}{\frac{1}{2}-t}\right|, \quad -\frac{1}{2} < t < \frac{1}{2}\end{aligned} \tag{2}$$

We may finally combine the results of Eqs (1) and (2) by expressing the Hilbert transform $\hat{g}(t)$, for all t as follows

$$\hat{g}(t) = -\frac{1}{\pi}\ln\left|\frac{t-1/2}{t+1/2}\right|$$

c)

$$g(t) = \delta(t)$$

The Hilbert transform of the delta function $\delta(t)$ is

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\delta(\tau)}{t-\tau}\,\mathrm{d}\,\tau$$

Using the shifting property of the delta function, we get the desired result

$$\hat{g}(t) = \frac{1}{\pi t}$$

d)

$$g(t) = \frac{1}{1+t^2}$$

The Hilbert transform of $g(t)$ is

$$\begin{aligned}\hat{g}(t) =& \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{(1+\tau^2)\,(t-\tau)}\,\mathrm{d}\,\tau \\ =& \frac{1}{1+t^2}\int_{-\infty}^{\infty}\left(\frac{t+\tau}{1+\tau^2}+\frac{1}{t-\tau}\right)\mathrm{d}\,\tau\end{aligned} \tag{3}$$

But (see appendix 6)

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{1+\tau^2}\,\mathrm{d}\,\tau =& \pi \\ \int_{-\infty}^{\infty}\frac{\tau}{1+\tau^2}\,\mathrm{d} =& 0 \\ \int_{-\infty}^{\infty}\frac{1}{t-\tau}\,\mathrm{d} =& 0\end{aligned}$$

Therefore, Eq. (3) reduces to

$$\hat{g}(t) = \frac{t}{1+t^2}$$

Problem 2.13

Consider the Fourier-transform pair

$$\exp(t)u(-t) \rightleftharpoons \frac{1}{1 - \mathrm{j}2\pi f} \tag{1}$$

where

$$\exp(t)u(-t) = \begin{cases} 0, & t > 0 \\ \frac{1}{2}, & t = 0 \\ \exp(t), & t < 0 \end{cases}$$

Applying the duality property of the Fourier transform to Eq. (1):

$$\frac{1}{1 - \mathrm{j}2\pi t} \rightleftharpoons \exp(f)u(-f) = G(f) \tag{2}$$

where

$$G(f) = \begin{cases} \exp(-f), & f > 0 \\ \frac{1}{2}, & f = 0 \\ 0, & f < 0 \end{cases}$$

From Eq. (2), it follows therefore that the inverse Fourier transform of $G(f)$ is the complex time-function:

$$g(t) = \frac{1}{1 - \mathrm{j}2\pi t} \tag{1}$$

The real and imaginary parts of $g(t)$ are as follows:

$$\mathrm{Re}[g(t)] = \frac{1}{1 + (2\pi t)^2}$$

$$\mathrm{Im}[g(t)] = \frac{2\pi t}{1 + (2\pi t)^2}$$

From Table 2.3 of the textbook, we see that these two real-time functions are the Hilbert transform of each other, except for the minor change of $2\pi t$ being replaced simply by t.

Problem 2.14

To solve the problem in the time domain, we have to be careful: rigorously, speaking, the Hilbert transform of $g(t)$ is defined by the principal value of the integral in (2.54) in the text, as shown by:

$$\hat{g}(t) = \lim_{\Delta t \to 0} \frac{1}{\pi} \left[\int_{-\infty}^{\tau - \Delta t} \frac{g(\tau)}{t - \tau} \, \mathrm{d}\, \tau + \int_{\tau + \Delta t}^{\infty} \frac{g(\tau)}{t - \tau} \, \mathrm{d}\, \tau \right]$$

However, an easier way of solving the problem is to work in the frequency domain in two steps:

Step 1)

Differentiation of $\hat{g}(t)$, the Hilbert transform of the function $g(t)$, requires multiplying the Fourier transform of the differential $\mathrm{d}\, \hat{g}(t) / \mathrm{d}\, t$, as shown by

$$\mathbf{F} \left[\frac{\mathrm{d}}{\mathrm{d}\, t} \hat{g}(t) \right] = \mathrm{j}\, 2\pi f \hat{G}(f) \tag{1}$$

Where $\hat{g}(t) \rightleftharpoons \hat{G}(f)$

Performing the Hilbert transformation in the frequency domain, we write

$$\hat{G}(f) = -\,\mathrm{j}\, \mathrm{sgn}(f) G(f) \tag{2}$$

Where $g(t) \rightleftharpoons G(f)$, and $\mathrm{sgn}(f)$ is the signum function

In an inverse manner to (2), we may write

$$G(f) = \mathrm{j}\, \mathrm{sgn}(f) \hat{G}(f) \tag{3}$$

Multiplying both sides of Equation (3) by $\mathrm{j}\, 2\pi f$, we get

$$\mathrm{j}\, 2\pi f G(F) = \mathrm{j}\, 2\pi f \left(\mathrm{j}\, \mathrm{sgn}(f) \hat{G}(f) \right) \tag{4}$$

Translating the terms in Equation (4) into the time domain, we proceed with the second step.

Step 2)

It may be noted that the left side of Equation (4) is equivalent to

$$\mathrm{j}\, 2\pi f G(f) = \mathbf{F} \left[\frac{\mathrm{d}}{\mathrm{d}\, t} g(t) \right] \tag{5}$$

Likewise, on the right of Equation (4) we have

$$\mathrm{j}\,2\pi f\left(\mathrm{j}\,\mathrm{sgn}(f)\hat{G}(f)\right) = \mathbf{F}\left[\frac{\mathrm{d}}{\mathrm{d}\,t}\hat{g}(t)\right] \tag{6}$$

Finally, in light of Equations (5), and (6), we immediately make the statement: The differential $\frac{\mathrm{d}}{\mathrm{d}\,t}\hat{g}(t)$ is the Hilbert transform of $\frac{\mathrm{d}}{\mathrm{d}\,t}g(t)$.

Hence the problem is solved

Problem 2.15

a) The Hilbert transform of the integral $\int_{-\infty}^{\infty} g(t)\,\mathrm{d}\,t$ is defined by

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t-\tau}\left[\int_{-\infty}^{\infty} g(t)\,\mathrm{d}\,t\right]\mathrm{d}\,\tau$$

which, in general, is not equal to the integral $\int_{-\infty}^{\infty}\hat{g}(t)\,\mathrm{d}\,t$, because the integral $\int_{-\infty}^{\infty} g(t)\,\mathrm{d}\,t$ is the total area under the function $g(t)$. In other words, the integral under the square brackets is independent of time t.

b) However, if we are permitted to write

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{g(t)}{t-\tau}\,\mathrm{d}\,t\,\mathrm{d}\,\tau &= \int_{-\infty}^{\infty}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t-\tau}\,\mathrm{d}\,\tau\right]\mathrm{d}\,t\\ &= \int_{-\infty}^{\infty}\hat{g}(t)\,\mathrm{d}\,t\end{aligned}$$

then the integral $\int_{-\infty}^{\infty}\hat{g}(t)\,\mathrm{d}\,t$ is equal to the Hilbert transform of the integral $\int_{-\infty}^{\infty} g(t)\,\mathrm{d}\,t$. Under this approach, the total area under $g(t)$ is the same as that under $\hat{g}(t)$.

Problem 2.16

(a) $g(t) = \text{sinc}(t) = \dfrac{\sin(\pi t)}{\pi t}$

We note that

$$\hat{g}(t) = \frac{1 - \cos(\pi t)}{\pi t}$$

Therefore

$$\begin{aligned} g_+(t) &= g(t) + j\hat{g}(t) \\ &= \frac{\sin(\pi t)}{\pi t} + \mathrm{j}\frac{1 - \cos(\pi t)}{\pi t} \\ &= \frac{\mathrm{j}}{\pi t}[1 - \cos(\pi t) - \mathrm{j}\sin(\pi t)] \\ &= \frac{\mathrm{j}}{\pi t}[1 - \exp(\mathrm{j}\pi t)] \end{aligned}$$

(b)

$$\begin{aligned} g(t) &= [1 + k\cos(2\pi f_m t)]\cos(2\pi f_c t)] \\ &= \cos(2\pi f_c t) + \frac{k}{2}\cos[2\pi(f_c + f_m)t] + \frac{k}{2}\cos[2\pi(f_c - f_m)t] \end{aligned}$$

Since the Hilbert transform of $\cos(2\pi f t)$ is equal to $\sin(2\pi f t)$, it follows that

$$\hat{g}(t) = \sin(2\pi f_c t) + \frac{k}{2}\sin[2\pi(f_c + f_m)t] + \frac{k}{2}\sin[2\pi(f_c - f_m)t]$$

where it is assumed that $f_c > f_m$. Therefore,

$$\begin{aligned} g_+(t) &= \exp(\mathrm{j}2\pi f_c t) + \frac{k}{2}\exp[\mathrm{j}2\pi(f_c + f_m)t] + \frac{k}{2}\exp[\mathrm{j}2\pi(f_c - f_m)t] \\ &= [1 + \frac{k}{2}\exp(\mathrm{j}2\pi f_m t) + \frac{k}{2}\exp(-\mathrm{j}2\pi f_m t)]\exp(\mathrm{j}2\pi f_c t) \\ &= [1 + k\cos(2\pi f_m t)]\exp(\mathrm{j}2\pi f_c t) \end{aligned}$$