

Chapter 2

Problem 2.1

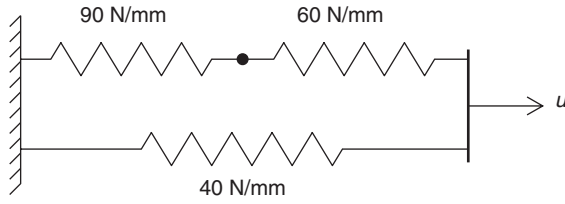


Figure S2.1

Referring to Figure S2.1 the springs with stiffness 60 N/mm and 90 N/mm are placed in series and have an effective stiffness given by

$$k_1 = \frac{1}{1/60 + 1/90} = 36 \text{ N/mm}$$

This combination is now placed in parallel with the spring of stiffness 40 N/mm giving a final effective stiffness of

$$k_{\text{eff}} = k_1 + 40 = 76 \text{ N/mm}$$

Problem 2.2

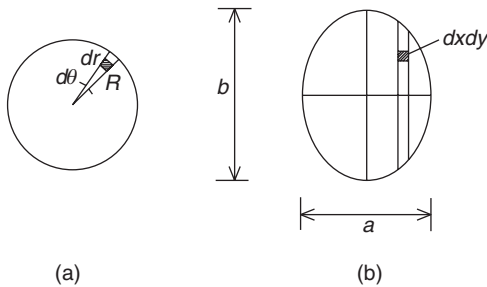


Figure S2.2

The infinitesimal area shown in Figure S2.2(a) is equal to $r d\theta dr$. When the circular disc moves in the x direction with acceleration \ddot{u}_x the inertia force on the infinitesimal area is $\gamma r d\theta dr \ddot{u}_x$, where γ is the mass per unit area. The resultant inertia force on the disc acting in the negative x direction is given by

$$I_x = \int_0^R \int_0^{2\pi} \gamma \ddot{u}_x r d\theta dr = \gamma \pi R^2 \ddot{u}_x = M \ddot{u}_x$$

where M is the total mass of the disc. The resultant moment of the inertia forces about the centre of the disc, which is also the centre of mass is given by

$$M_x = \int_0^R \int_0^{2\pi} \gamma \ddot{u}_x r^2 \sin \theta d\theta dr = 0$$

In a similar manner we get

$$I_y = M \ddot{u}_y$$

For an angular acceleration $\ddot{\theta}$ about the center of mass the inertia force on the infinitesimal element is directed along the tangent and is $\gamma r^2 \ddot{\theta} d\theta dr$. The x component of this force is $\gamma r^2 \ddot{\theta} d\theta dr \sin \theta$. It is easily seen that the resultant of all x direction forces is zero. In a similar manner the resultant y direction force is zero. However, a clockwise moment about the center of the disc exists and is given by

$$M_\theta = \int_0^R \int_0^{2\pi} \gamma \ddot{\theta} r^3 d\theta dr = \gamma \pi R^2 \frac{R^2}{2} \ddot{\theta} = M \frac{R^2}{2} \ddot{\theta}$$

The elliptical plate shown in Figure S2.2(c) is divided into the infinitesimal elements as shown. The mass of an element is $\gamma dx dy$ and the inertia force acting on it when the disc undergoes translation in the x direction with acceleration \ddot{u}_x is $\gamma \ddot{u}_x dx dy$. The resultant inertia force in the negative x direction is given by

$$\begin{aligned} I_x &= \int_{-a/2}^{a/2} \int_{-b/2\sqrt{1-4x^2/a^2}}^{b/2\sqrt{1-4x^2/a^2}} \gamma \ddot{u}_x dy dx \\ &= \gamma \ddot{u}_x \int_{-a/2}^{a/2} b \sqrt{1-4x^2/a^2} dx \\ &= \frac{\pi \gamma ab}{4} = M \ddot{u}_x \end{aligned}$$

The moment of the x direction inertia force on an element is $\gamma \ddot{u}_x y dx dy$. The resultant moment obtained over the area is zero. The inertia force produced by an acceleration in the y direction is obtained in a similar manner and is $M \ddot{u}_y$ directed in the negative y direction.

An angular acceleration $\ddot{\theta}$ produces a clockwise moment equal to $\gamma r^2 \ddot{\theta} dx dy = \gamma (x^2 + y^2) \ddot{\theta} dx dy$. Integration over the area yields the resultant moment, which is clockwise

$$\begin{aligned} I_\theta &= \int_{-a/2}^{a/2} \int_{-b/2\sqrt{1-4x^2/a^2}}^{b/2\sqrt{1-4x^2/a^2}} \gamma \ddot{\theta} (x^2 + y^2) dy dx \\ &= \gamma \frac{\pi ab}{4} \frac{a^2 + b^2}{16} \ddot{\theta} = M \frac{a^2 + b^2}{16} \ddot{\theta} \end{aligned}$$

The x and y direction inertia forces produced on the infinitesimal element are $-\gamma \ddot{\theta} \sin \theta dx dy$ and $\gamma \ddot{\theta} \cos \theta dx dy$, respectively. When summed over the area the net forces produced by these are easily shown to be zero.

Problem 2.3

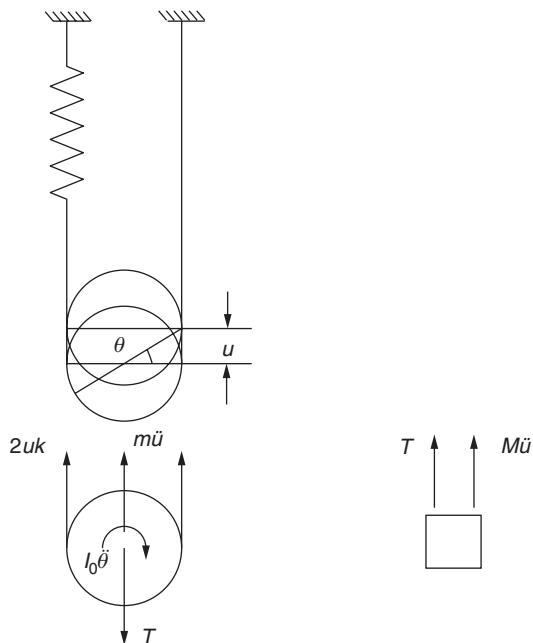


Figure S2.3

As the center of the pulley moves a distance u , the spring is stretched by $2u$ and the pulley rotates through u/R . Referring to the free body diagrams shown in Figure S2.3, the conditions of equilibrium are

$$\begin{aligned} 2uk \times 2R + m\ddot{u}R + I_0\ddot{\theta} - TR &= 0 \\ T + M\ddot{u} &= 0 \end{aligned}$$

Substituting $I_0 = mR^2/2$ and eliminating T gives

$$\left(M + \frac{3m}{2}\right)\ddot{u} + 4ku = 0$$

Problem 2.4

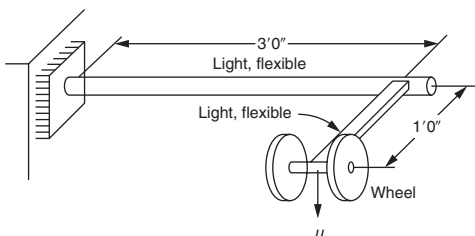


Figure S2.4

$$\begin{aligned} I_b &= \frac{1}{12} \times 1 \times \left(\frac{1}{4}\right)^3 = \frac{1}{768} \\ I_s &= \frac{\pi}{64}, \quad J_s = \frac{\pi}{32} \end{aligned}$$

The displacement at location A , shown in Figure S2.4, due to a unit load is given by

$$\begin{aligned} \Delta_A &= \frac{L_b^3}{3EI_b} + \frac{L_s^3}{3EI_s} + \frac{L_b^2 L_s}{GJ_s} \\ &= \frac{12^3 \times 768}{3 \times 30 \times 10^6} + \frac{36^3 \times 64}{3 \times 30 \times 10^6 \times \pi} \\ &\quad + \frac{12^2 \times 36 \times 32}{12 \times 10^6 \times \pi} \\ &= 0.01475 + 0.01056 + 0.0044 \\ &= 0.0297 \text{ in.} \end{aligned}$$

Equivalent spring stiffness = $\frac{1}{0.0297} = 33.67 \frac{\text{lb.}}{\text{in.}}$

Mass = $\frac{10}{386.4} = 0.02588 \frac{\text{lb.s}^2}{\text{in.}}$

The equation of motion is given by

$$0.02588\ddot{u} + 33.67u = 0$$

or

$$\ddot{u} + 1300u = 0$$

Problem 2.5

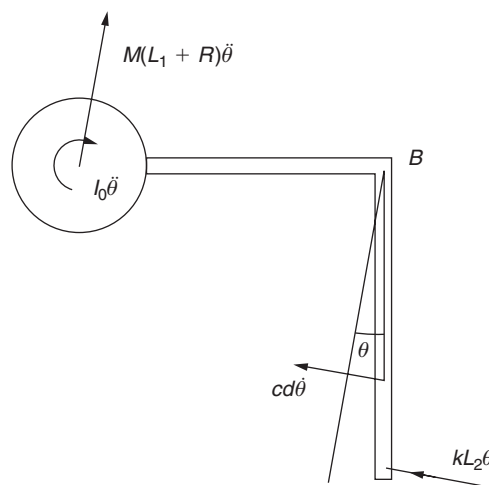


Figure S2.5

Taking moments about B shown in Figure S2.5, and noting that $I_0 = MR^2/2$

$$M(L_1 + R)^2 \ddot{\theta} + \frac{MR^2}{2} \ddot{\theta} + cd^2 \dot{\theta} + kL_2^2 \theta = 0$$

or

$$M \left(L_1^2 + \frac{3}{2}R^2 + 2L_1R \right) \ddot{\theta} + cd^2 \dot{\theta} + kL_2^2 \theta = 0$$

Problem 2.6

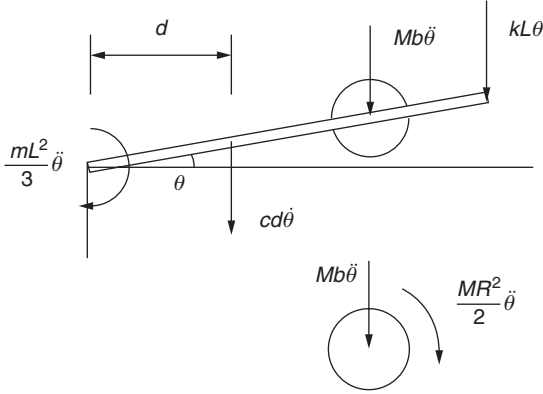


Figure S2.6

Taking moments about A shown in Figure S2.6

$$\left(\frac{mL^2}{3} + Mb^2\right)\ddot{\theta} + cd^2\dot{\theta} + kL^2\theta = 0$$

When the flywheel is braked, an additional rotational inertia term will exist. Moment equilibrium gives

$$\left[\frac{mL^2}{3} + M\left(b^2 + \frac{R^2}{2}\right)\right]\ddot{\theta} + cd^2\dot{\theta} + kL^2\theta = 0$$

Problem 2.7

Boundary conditions

$$\left. \begin{array}{l} \psi = 0 \\ \frac{d\psi}{dx} = 0 \end{array} \right\} x = 0$$

$$\left. \begin{array}{l} \psi = 0 \\ EI \frac{d^2\psi}{dx^2} = 0 \end{array} \right\} x = L$$

where

$$\frac{d\psi}{dx} = \frac{d\psi}{d\xi} \times \frac{1}{L} = \frac{1}{L} (8\xi^3 - 15\xi^2 + 6\xi)$$

$$\frac{d^2\psi}{dx^2} = \frac{1}{L^2} (24\xi^2 - 30\xi + 6)$$

All four boundary conditions are satisfied

$$m^* = \int_0^L \bar{m} \{\psi(x)\}^2 dx$$

$$= \bar{m}L \int_0^1 \{\psi(\xi)\}^2 d\xi = 0.03016\bar{m}L$$

$$k^* = \int_0^L EI \left\{ \frac{d^2\psi(x)}{dx^2} \right\}^2 dx$$

$$= \frac{EI}{L^3} \int_0^1 \left\{ \frac{d^2\psi(\xi)}{d\xi^2} \right\}^2 d\xi = 7.2 \frac{EI}{L^3}$$

Problem 2.8

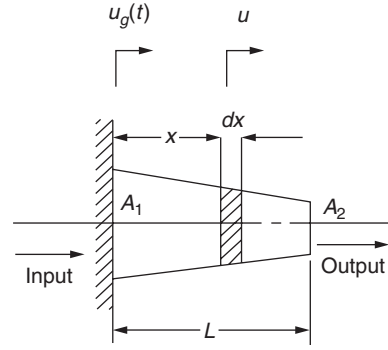


Figure S2.8

$$\frac{d\xi}{dx} = \frac{1}{L}, \quad A(\xi) = A_1 + (A_2 - A_1)\xi$$

$$\frac{d\psi}{dx} = (2 - 2\xi) \frac{1}{L}, \quad \bar{m}(\xi) = \rho A(\xi)$$

where ρ is the mass density.

$$m^* = \int_0^L \bar{m}(x) \{\psi(x)\}^2 dx$$

$$= \int_0^1 \rho L \{A_1 + (A_2 - A_1)\xi\} (4\xi^2 + \xi^4 - 4\xi^3) d\xi$$

$$= \frac{\rho L}{30} (5A_1 + 11A_2)$$

$$k^* = \int_0^L EA(x) \left\{ \frac{d\psi(x)}{dx} \right\}^2 dx$$

$$= \frac{E}{L} \int_0^1 \{A_1 + (A_2 - A_1)\xi\} (4 + 4\xi^2 - 8\xi) d\xi$$

$$= \frac{E}{L} \left(A_1 + \frac{1}{3}A_2 \right)$$

$$p^* = - \int_0^L \bar{m}(x) \ddot{u}_g \psi(x) dx$$

$$= - \int_0^1 \rho L \ddot{u}_g \{A_1 + (A_2 - A_1)\xi\} (2\xi - \xi^2) d\xi$$

$$= -\rho L \ddot{u}_g \left(\frac{1}{4}A_1 + \frac{5}{12}A_2 \right)$$

Problem 2.9

Refer to Figure S2.9 where the inertia force on an infinitesimal element is shown as $\bar{m} \frac{\partial^2 u^t}{\partial x^2} dx$. The total displacement $u^t = u + u_g$, where u is the displacement relative to the ground and u_g is the displacement of the ground. The inertia force

thus becomes $\bar{m}(\ddot{z}(t)\psi(x) + \ddot{u}_g) dx$. The first term within the parentheses produces the generalized mass m^* while the second term produces the generalized force p^* as follows

$$m^* = \int_0^L \bar{m}[\psi(x)]^2 dx = \frac{33}{140} \bar{m}L$$

$$p^* = -\bar{m}\ddot{u}_g \int_0^L \psi(x) dx = -\frac{3}{8} \bar{m}L\ddot{u}_g$$

The generalized stiffness is given by

$$k^* = \int_0^L EI [\psi''(x)]^2 dx = \frac{3EI}{L^3}$$

The geometric stiffness produced by the axial force $S(x) = (L-x)\bar{m}g$ is obtained from

$$k_G = \int_0^L S(x) [\psi'(x)]^2 dx = \frac{3}{8} \bar{m}g$$

The equation of motion including the gravity effect is

$$\frac{33}{140} \bar{m}L\ddot{z} + \left(\frac{3EI}{L^3} - \frac{3}{8} \bar{m}g \right) z = -\frac{3}{8} \bar{m}Lg$$

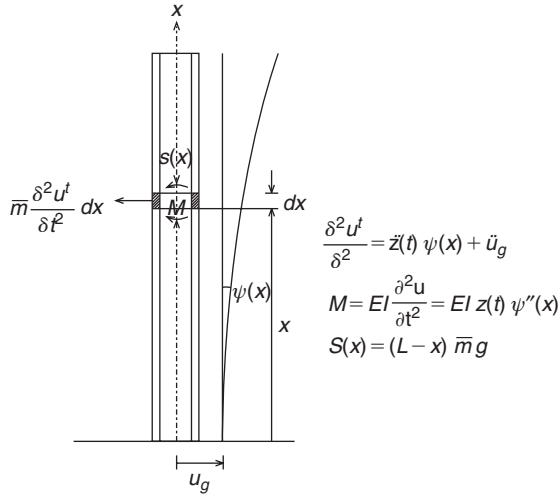


Figure S2.9

Problem 2.10

$$u = z\psi(x)$$

$$\psi = \sin \frac{\pi x}{L}, \quad \frac{d^2\psi}{dx^2} = -\frac{\pi^2}{L^2} \sin \frac{\pi x}{L}$$

$$m^* = \int_0^L \bar{m} \sin^2 \frac{\pi x}{L} dx = \frac{\bar{m}L}{2}$$

$$k^* = \int_0^L EI \frac{\pi^4}{L^4} \sin^2 \frac{\pi x}{L} dx = \frac{\pi^4 EI}{2L^3}$$

$$p^* = \int_0^L F \delta(x-vt) \sin \frac{\pi x}{L} dx$$

$$= F \sin \frac{\pi vt}{L}$$

Equation of motion:

$$\frac{\bar{m}L}{2} \ddot{z} + \frac{\pi^4 EI}{2L^3} z = F \sin \frac{\pi vt}{L}$$

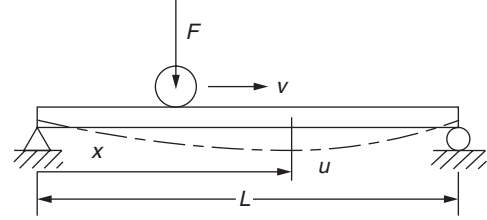


Figure S2.10

Problem 2.11

The mass matrix \mathbf{M} and the stiffness matrix \mathbf{K} are given by

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ kip s}^2/\text{in.}$$

$$\mathbf{K} = \begin{bmatrix} 1000 & -500 & 0 \\ -500 & 750 & -250 \\ 0 & -250 & 250 \end{bmatrix} \text{ kips/in.}$$

With $\psi^T = [1 \ 2 \ 3]$, the generalized mass m^* and generalized stiffness k^* are obtained from

$$m^* = \psi^T \mathbf{M} \psi = 19 \text{ kip s}^2/\text{in.}$$

$$k^* = \psi^T \mathbf{K} \psi = 1250 \text{ kips/in.}$$

The generalized force is obtained from

$$p^* = -\psi^T \mathbf{M} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ddot{u}_g = -1170 \sin(6\pi t) \text{ kips}$$

The equation of motion is given by

$$19\ddot{z} + 1250z = -1170 \sin(6\pi t)$$

Problem 2.12

Taking moments about A shown in Figure S2.12

$$ml^2\ddot{\theta} + 2ka^2\theta + mgl \sin \theta = 0$$

For small vibrations $\sin \theta \approx \theta$, hence

$$ml^2\ddot{\theta} + (2ka^2 + mgl) \theta = 0$$

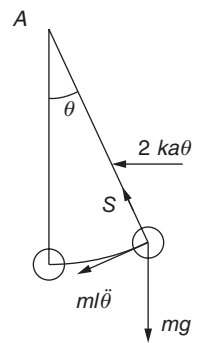


Figure S2.12

Chapter 3

Problem 3.1

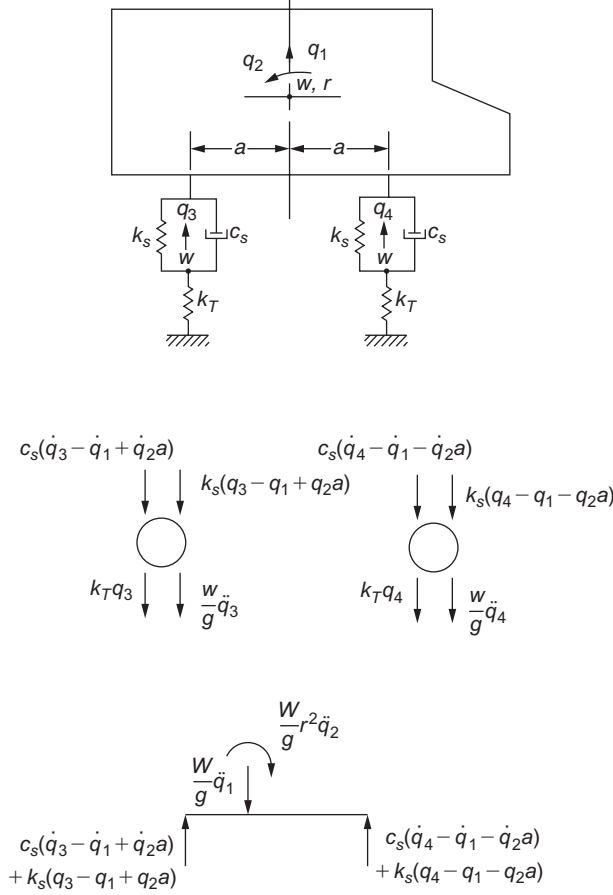


Figure S3.1

The degrees of freedom are identified in Figure S3.1. The free-body diagrams are also shown. For force balance on each of the front and rear axles

$$\begin{aligned} \frac{w}{g}\ddot{q}_4 + c_S(\dot{q}_4 - \dot{q}_1 - \dot{q}_2a) \\ + k_S(q_4 - q_1 - q_2a) + k_Tq_4 = 0 \\ \frac{w}{g}\ddot{q}_3 + c_S(\dot{q}_3 - \dot{q}_1 + \dot{q}_2a) \\ + k_S(q_3 - q_1 + q_2a) + k_Tq_3 = 0 \end{aligned}$$

For equilibrium of vertical forces on the vehicle body

$$\begin{aligned} \frac{W}{g}\ddot{q}_1 + c_S(\dot{q}_3 + \dot{q}_4 - 2\dot{q}_1) \\ + k_S(q_3 + q_4 - 2q_1) = 0 \end{aligned}$$

Taking moments about the mass center of the vehicle body

$$\begin{aligned} \frac{W}{g}r^2\ddot{q}_2 + c_Sa(\dot{q}_3 - \dot{q}_4 + 2a\dot{q}_2) \\ + k_Sa(q_3 - q_4 + 2aq_2) = 0 \end{aligned}$$

Collectively, the four equations can be expressed in the matrix form

$$\mathbf{M}^*\ddot{\mathbf{u}} + \mathbf{C}^*\dot{\mathbf{u}} + \mathbf{K}^*\mathbf{u} = 0$$

$$\mathbf{u}^T = [q_1 \quad aq_2 \quad q_3 \quad q_4]$$

$$\mathbf{M}^* = \begin{bmatrix} \frac{W}{g} & & & \\ & \frac{W}{g}\frac{r^2}{a^2} & & \\ & & \frac{w}{g} & \\ & & & \frac{w}{g} \end{bmatrix}$$

$$\mathbf{C}^* = c_S \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 1 & -1 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K}^* = k_S \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 1 & -1 \\ -1 & 1 & 1 + \frac{k_T}{k_S} & 0 \\ -1 & -1 & 0 & 1 + \frac{k_T}{k_S} \end{bmatrix}$$

Problem 3.2

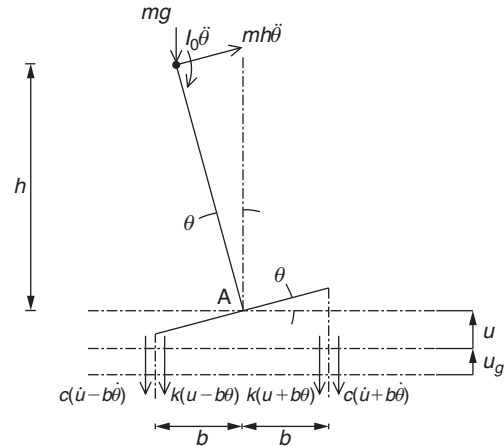


Figure S3.2

Coordinates \$u^t\$ and \$\theta\$ measure the vertical motion and rotation of the base. The forces on the structure are identified in Figure S3.2. Assuming that \$u^t\$ is measured from the position of equilibrium under gravity load, the condition of vertical force balance gives

$$m\ddot{u}^t + 2c\dot{u} + 2ku = 0 \quad (1)$$

or

$$m\ddot{u} + 2c\dot{u} + 2ku = -m\ddot{u}_g$$

Taking moments about \$A\$ and assuming that \$\theta\$ is small

$$(I_0 + mh^2)\ddot{\theta} + 2cb^2\dot{\theta} + 2kb^2\theta - mgh\theta = 0 \quad (2)$$