Contents

Chapter 1

The Real Number System

1.1 The Field Properties

EXERCISE SET 1.1-A

1. A0, A1, A2, M0, M1, M2, M3, D. **2.** A0, A1, A2, A3, A4, M0, M1, M2, M3, D; i.e., all but M4.

3. All field axioms. **4.** A0, A1, A2, M0, M1

5. A0, A1, A2, A3, A4, but not M0. Thus, M1 - M4 and D are not relevant.

6. A0, A1, A2, A3, A4, M1, D. **7.** A0, A1, A2, A3, A4, M0, D.

8. All field axioms. **9.** All field axioms. **10.** All field axioms except M4.

EXERCISE SET 1.1-B

1. $x \neq 0 \Rightarrow \exists u \in F \ni xu = ux = 1$. Then $xy = xz \Rightarrow u(xy) = u(xz) \Rightarrow (ux)y = (ux)z \Rightarrow 1y = 1z \Rightarrow y = z$.

2. Suppose $1, 1'$ both have property described in (M3); i.e., $\forall x \in F$, $x \cdot 1 = x$ and $x \cdot 1' = x$. Then $1' \cdot 1 = 1'$ and $1 \cdot 1' = 1$. But $1' \cdot 1 = 1 \cdot 1'$, so $1' = 1$.

3. Suppose $x \neq 0$ and u, u' both have property described in (M4); i.e., $xu = 1$, and $xu' = 1$. Then $u = u1$ $= u(xu') = (ux)u' = 1u' = u'$.

4. By (M3), (M1), (M4), & defn. of 1^{-1}, $1^{-1} = 1^{-1} \cdot 1 = 1 \cdot 1^{-1} = 1$.

5. Suppose $x \neq 0$. Then $xx^{-1} = 1 \neq 0$, so by (d), $x^{-1} \neq 0$.

6. Suppose $x, y \neq 0$. Then by (M4), $\exists\ x^{-1}, y^{-1}$. If $xy = 0$, then by (d), $x^{-1}(xy) = x^{-1}0 = 0$. But $x^{-1}(xy) = (x^{-1}x)y = 1y = y \neq 0$. Contradiction. $\therefore xy \neq 0$.
Now $(xy)(x^{-1}y^{-1}) = (xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = xx^{-1} = 1$. $\therefore$ by Thm.1.1.3(d), $(xy)^{-1} = x^{-1}y^{-1}$.

7. Using (h) and (M2), $(-x)y = [(-1)x]y = (-1)(xy) = -(xy)$. Similarly, $x(-y) = x[(-1)y] = [x(-1)]y$ $= [(-1)x]y = (-x)y$.

8. By (i), $(-1)(-1) = -(1 \cdot -1) = -(-1) = 1$ by (b).

9. Using (i), $(-x)(-y) = -[x(-y)] = -[-(xy)] = xy$ by (b).

10. By defn. of subtraction, $0 - x = 0 + (-x) = (-x) + 0 = -x$ by (A1) and (A3).

11. By Thm.1.1.4 (h), $-(x + y) = (-1)(x + y) = (-1)x + (-1)y = -x + (-y) = -x - y$ by Defn.1.1.5.

12. By Thm.1.1.4 (h), and (D), $-(x - y) = (-1)(x - y) = (-1)x + (-1)(-y) = -x + 1y = y + (-x) = y - x$ by defn. of subtraction.

13. By Defn.1.6, $x \neq 0 \Rightarrow 0 \div x = 0x^{-1} = x^{-1}0 = 0$ by Thm.1.1.4 (d).

14. $x \div 1 = x \cdot 1^{-1} = x \cdot 1 = x$; and $1 \div x = 1 \cdot x^{-1} = x^{-1} \cdot 1 = x^{-1}$.

15. By Thm.1.1.4, $(-x)(-x^{-1}) = xx^{-1} = 1$. Apply Thm.1.1.3 (d).

16. For $b, d \neq 0$, $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad}{bd} + \dfrac{bc}{bd}$ [by (e) and (M1)] $= (ad)(bd)^{-1} + (bc)(bd)^{-1}$ [by defn. of fraction] $= (ad + bc)(bd)^{-1}$ [by (D)] $= \dfrac{ad + bc}{bd}$ [by defn. of fraction].

17. Suppose $b, d \neq 0$. Then (supply reasons) $\frac{a}{b} \cdot \frac{c}{d} = (ab^{-1})(cd^{-1}) = [(ab^{-1})c]d^{-1} = [a(b^{-1}c)]d^{-1}$ $= [a(cb^{-1})]d^{-1} = [(ac)b^{-1}]d^{-1} = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1} = \frac{ac}{bd}$.

18. For $b \neq 0$, $-\dfrac{a}{b} = -\left[ab^{-1}\right] = (-a)b^{-1} = \dfrac{-a}{b}$ [by defn. of fraction, Thm.1.1.4 (i) and defn. of fraction] $= a(-(b^{-1})) = a(-b)^{-1}$ [by Thm.1.1.4 (i) and Thm.1.1.8 (c)] $= \dfrac{a}{-b}$ by defn. of fraction.

19. $a, b \neq 0 \Rightarrow \dfrac{a}{b} \cdot \dfrac{b}{a} = \dfrac{ab}{ba} = \dfrac{ab}{ab} = (ab)(ab)^{-1} = 1$. Apply Thm.1.1.3 (d).

20. Suppose $a \neq 0$. Then $a\left(\frac{-b}{a}\right) + b = a\left[(-b)a^{-1}\right] + b = [a(a^{-1})](-b) + b = 1(-b) + b = -b + b = 0$. Thus, $\frac{-b}{a}$ is a solution to the equation $ax + b = 0$. Suppose y is a solution to this equation. Then $ay = (ay + b) - b = 0 - b = -b$; i.e., $ay = -b$. $\therefore y = y(aa^{-1}) = ay(a^{-1}) = (-b)a^{-1} = \frac{-b}{a}$.

1.2 The Order Properties

EXERCISE SET 1.2-A

1. #3, #8

2. By Defn.1.2.4, $x > 0 \Leftrightarrow 0 < x \Leftrightarrow x - 0 \in \mathcal{P} \Leftrightarrow x \in \mathcal{P}$. Also, $x < 0 \Leftrightarrow 0 - x \in \mathcal{P} \Leftrightarrow -x \in \mathcal{P}$.

3. By (O3) one & only one is true: $y - x \in \mathcal{P}$, $x - y \in \mathcal{P}$, $y - x = 0$.

4. By (b), $x \leq y \Leftrightarrow [x < y \text{ or } x = y] \Leftrightarrow x \not> y$. Also, $x \geq y \Leftrightarrow [x > y \text{ or } x = y] \Leftrightarrow [y < x \text{ or } y = x] \Leftrightarrow x \not< y$.

5. x, y negative$\Rightarrow -x, -y \in \mathcal{P} \Rightarrow (-x)(-y) \in \mathcal{P} \Rightarrow xy \in \mathcal{P}$.

6. Suppose x positive and y negative. Then $x \in \mathcal{P}$, $-y \in \mathcal{P}$. By (O2), $-xy = x(-y) \in \mathcal{P}$; i.e., xy negative.

7. Suppose $xy > 0$. Then $x, y \neq 0$. If x, y do not have the same sign, then one must be positive and the other negative. Then, by (d), $xy < 0$. Contradiction.

8. Suppose $xy < 0$. By (O2), x and y are not both positive, and by (b) they are not both negative. Also, $x, y \neq 0$. $\therefore$ one of them is positive and the other is negative.

9. $1 = 1^2$ and $1 \neq 0$. Apply Thm.1.2.6 (c).

10. $x < y \Leftrightarrow y - x \in \mathcal{P} \Leftrightarrow (y + z) - (x + z) \in \mathcal{P} \Leftrightarrow x + z < y + z$. Also, $\Leftrightarrow (y - z) - (x - z) \in \mathcal{P} \Leftrightarrow x - z < y - z$.

11. Suppose $z < 0$. Then $x < y \Rightarrow y - x \in \mathcal{P}, - z \in \mathcal{P} \Rightarrow -z(y - x) \in \mathcal{P} \Rightarrow xz - yx \in \mathcal{P} \Rightarrow xy > xz$.

12. Suppose $x, y > 0$, and $x^2 < y^2$. Then $y^2 - x^2 > 0$, so $(y - x)(y + x) > 0$. By Thm.1.2.6 (a), $y - x$ and $y + x$ must have the same sign, so $y - x > 0$; i.e., $x < y$.

13. (a) $xx^{-1} = 1 > 0$, so by Thm.1.2.6 (e), x, x^{-1} have the same sign. To prove (b) and (c), apply part (a) to Thm.1.2.8 (c) and (d).

14. Suppose $0 < y^{-1} < x^{-1}$. Then, by the "$\Rightarrow$" part, $0 < (x^{-1})^{-1} < (y^{-1})^{-1}$; i.e., $0 < x < y$.

15. $x < y, u < v \Rightarrow (y - x), (v - u) \in \mathcal{P} \Rightarrow (y - x) + (v - u) \in \mathcal{P} \Rightarrow (y + v) - (x + u) \in \mathcal{P} \Rightarrow x + u < y + v$.

16. Suppose $0 < x < y$ and $0 < u < v$. By (a), $0 < v^{-1} < u^{-1}$, so by the first claim, $0 < xv^{-1} < yu^{-1}$; i.e., $0 < \frac{x}{v} < \frac{y}{u}$.

17. By (b), $x < y \Rightarrow x + x < y + x = x + y < y + y \Rightarrow 2x < x + y < 2y$, so by Cor.1.2.9 (b), $x < \frac{x+y}{2} < y$.

18. Let $x > 0$. Since $0 < \frac{1}{2} < 1$, $0 < \frac{x}{2} < x$ by Thm.1.2.8 (c). Thus, x cannot be the smallest positive element of F.

19. (O1) $a + b\sqrt{2}, c + d\sqrt{2} \in \mathcal{P}' \Rightarrow a > b\sqrt{2}, c > d\sqrt{2} \Rightarrow (a + c) > (b + d)\sqrt{2} \Rightarrow (a + c) + (b + d)\sqrt{2} \in \mathcal{P}'$ $\Rightarrow (a + b\sqrt{2}) + (c + d)\sqrt{2} \in \mathcal{P}'$.

(O2) $a > b\sqrt{2}, c > d\sqrt{2} \Rightarrow (a - b\sqrt{2})(c - d)\sqrt{2} > 0 \Rightarrow (ac + 2bd) > (ad + bc)\sqrt{2} \Rightarrow (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathcal{P}' \Rightarrow (a + b\sqrt{2})(c + d\sqrt{2}) \in \mathcal{P}'$.

(O3) Given $a, b \in \mathbb{Q}$, exactly one is true: $a > b\sqrt{2}$, $a = b\sqrt{2}$, $a < b\sqrt{2}$.
Case 1: $a > b\sqrt{2} \Rightarrow a + b\sqrt{2} \in \mathcal{P}'$.
Case 2: $a = b\sqrt{2} \Rightarrow a = b = 0$, since otherwise $\frac{a}{b} = \sqrt{2}$, which would tell us that $\sqrt{2}$ is rational. So, in this case $a + b\sqrt{2} = 0$.
Case 3: $a < b\sqrt{2} \Rightarrow (-a) > (-b)\sqrt{2} \Rightarrow (-a) + (-b)\sqrt{2} \in \mathcal{P}' \Rightarrow -(a + b\sqrt{2}) \in \mathcal{P}'$.

20. The field given in Ex.19 can be ordered by the usual set $\mathcal{P} = \{a + b\sqrt{2} : a + b\sqrt{2} > 0\}$ and the set $\mathcal{P}'$ given in that exercise.

21. If $\mathbb{C}$ were an ordered field, then by Thm.1.2.6 (c), $i^2 > 0$. But $i^2 = -1$, which is < 0 by Cor.1.2.7.

22. Consider the field F given in Ex.1.1-A.9. Since $1 > 0$, $4 = 1 + 1 + 1 + 1 > 0$. But $4 = -1$ in F. If F were an ordered field, this would violate the law of trichotomy since $-1 < 0$.

EXERCISE SET 1.2-B

1. (a) By (O3), $x \geq 0$ or $x < 0$. In the former case $|x| = x \geq 0$ and in the latter, $|x| = -x \geq 0$.

(d) $|x - y| = |-(y - x)| = |y - x|$ by (b).

(e) We have four cases:

(1) $x \geq 0, y \geq 0$. Then $xy \geq 0$ and $|xy| = xy = |x||y|$.

(2) $x \geq 0, y < 0$. Then $xy \leq 0$ and $|xy| = -xy = x(-y) = |x||y|$.

(3) $x < 0, y \geq 0$. Then $xy \leq 0$ and $|xy| = -xy = (-x)y = |x||y|$.

(4) $x < 0, y < 0$. Then $xy > 0$ and $|xy| = xy = (-x)(-y) = |x||y|$.

2. Let $a \geq 0$.
(a) Let $x \in F$. By the law of trichotomy we must have $x \geq 0$ or $x < 0$.
Case 1 ($x \geq 0$): Then $|x| = x$. Thus, $|x| < a \Leftrightarrow x < a \Leftrightarrow -a < x < a$.
Case 2 ($x < 0$): Then $|x| = -x$. Thus, $|x| < a \Leftrightarrow -x < a \Leftrightarrow -a < x < a$.

(b) By (a), $|x - y| < a \Leftrightarrow -a < x - y < a \Leftrightarrow y - a < x < y + a$ by Thm.1.2.8 (b).

3. (c) By (b), $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |x - y|$. Since $||x| - |y||$ = either $|x| - |y|$ or $|y| - |x|$, the desired result follows.

4. (c) Let $x < y$ in $(a, b]$, and $z \in [x, y]$. Then $a < x \leq z \leq y \leq b$, so $z \in (a, b]$. $\therefore [x, y] \subseteq (a, b]$.

(g) Let $x < y$ in $(a, +\infty)$, and $z \in [x, y]$. Then $a < x \leq z \leq y$, so $z \in (a, +\infty)$. $\therefore [x, y] \subseteq (a, +\infty)$.

5. Let $A = \cup\{[y, z] : y, z \in I\}$. Show $A = I$.
$x \in A \Rightarrow x \in [y, z]$ for some $y, z \in I \Rightarrow x \in I$ since I is an interval. Thus, $A \subseteq I$.
$x \in I \Rightarrow [x, x] \subseteq I \Rightarrow x \in A$. Thus, $I \subseteq A$.

6. Let $x, y \in F$. WLOG, $x \leq y$. Then, $|x - y| = y - x$, $\max\{x, y\} = y$, and $\min\{x, y\} = x$. Then $\dfrac{x + y + |x - y|}{2} = \dfrac{x + y + (y - x)}{2} = \dfrac{2y}{2} = y = \max\{x, y\}$, and $\dfrac{x + y - |x - y|}{2} = \dfrac{x + y - (y - x)}{2} = \dfrac{2x}{2}$ $= x = \min\{x, y\}$.

7. $x \leq y \Rightarrow \min\{x, y\} = x = -(-x) = -\max\{-x, -y\}$ since $-y \leq -x$.

8. Suppose $0 \leq x < y$. Then $0 < 1 + x < 1 + y$, so by Thm.1.2.10 (a), $0 < \dfrac{1}{1 + y} < \dfrac{1}{1 + x}$. Thus, by Thm.1.2.8 (c), $\dfrac{x}{1 + y} \leq \dfrac{x}{1 + x} < \dfrac{y}{1 + x}$, so $\dfrac{x}{1 + y} < \dfrac{y}{1 + x}$.

9. Multiply both sides of the given inequality by the lowest common denominator. Prove the resulting inequality and then divide both sides by the LCD.

1.3 Natural Numbers

EXERCISE SET 1.3

1. $1-1 \notin \mathbb{N}_F$; $1 \div 2 \notin \mathbb{N}_F$.

2. Let $n \in \mathbb{N}$. If $n > 1$, then by Thm.1.2.8 (c), $n^2 > n > 1 > 0$ so by Thm.1.2.10 (a), $0 < \frac{1}{n^2} < \frac{1}{n} < 1$. If $n = 1$, then $0 < \frac{1}{n^2} = \frac{1}{n} = 1$. Thus, $\forall\, n \in \mathbb{N}$, $0 < \frac{1}{n^2} \le \frac{1}{n} \le 1$.

In Exercises 3 - 19 we show only the induction step, $P(k) \Rightarrow P(k+1)$. Begin by assuming $P(k)$. Then,

3. $1+2+3+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) = (k+1)\left[\frac{k}{2}+1\right] = \frac{(k+1)(k+2)}{2} = \frac{(k+1)[(k+1)+1]}{2}$.

4. $1^2+2^2+\cdots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = (k+1)\left[\frac{2k^2+k}{6} + \frac{6(k+1)}{6}\right] = (k+1)\frac{k^2+7k+6}{6}$
$= \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(k+2)[2(k+1)+1]}{6} = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$

5. $1^3+2^3+\cdots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = (k+1)^2\left[\frac{k^2}{4} + (k+1)\right] = (k+1)^2\frac{k^2+4k+4}{4} = \frac{(k+1)^2(k+2)^2}{2}$.

6. $1+3+5+\cdots+[2(k+1)-1] = 1+3+5+\cdots+(2k-1)+[2(k+1)-1] = k^2+(2k+1) = (k+1)^2$.

7. $1+4+7+\cdots+[3(k+1)-2] = 1+4+7+\cdots+(3k-2)+[3k+1] = \frac{k(3k-1)}{2} + 3k+1 = \frac{3k^2+5k+2}{2}$
$= \frac{(k+1)(3k+2)}{2} = \frac{(k+1)(3(k+1)-1)}{2}$.

8. By $P(k)$, $\exists m \in \mathbb{N} \ni k(k+1)(k+2) = 3m$. Then, $(k+1)(k+2)(k+3) = k(k+1)(k+2)+3(k+1)(k+2)$
$= 3m + 3(k+1)(k+2) = 3[m+(k+1)(k+2)]$, which is divisible by 3.

9. By $P(k)$, $k^5 - k = 5m$ for some $m \in \mathbb{Z}$. Then $(k+1)^5-(k+1) = k^5+5k^4+10k^3+10k^2+5k+1-k-1$
$= (k^5-k)+(5k^4+10k^3+10k^2+5k) = 5[m+k^4+2k^3+2k^2+k]$.

10. $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{k+1}} = 1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} = \left(2-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}} = 2-\frac{1}{2^k}\left[1-\frac{1}{2}\right] = 2-\frac{1}{2^{k+1}}$.

11. $1+\frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^{k+1}} = 1+\frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^k}+\frac{1}{3^{k+1}} = \left[\frac{3}{2}-\frac{1}{2}\left(\frac{1}{3^k}\right)\right]+\frac{1}{3^{k+1}} = \frac{3}{2}+\frac{-3+2}{2\cdot 3^{k+1}} = \frac{3}{2}-\frac{1}{2}\frac{1}{3^{k+1}}$.

12. $a+ar+ar^2+\cdots+ar^{k+1} = \dfrac{a-ar^{k+1}}{1-r}+ar^{k+1} = \dfrac{a-ar^{k+1}+ar^{k+1}(1-r)}{1-r} = \dfrac{a-ar^{(k+1)+1}}{1-r}$.

13. $2^{k+1} = 2^k 2 \le (k+1)!2 \le (k+1)!(k+2) = (k+2)!$.

14. $(1+x)^{k+1} = (1+x)^k(1+x) = (1+x)^k + x(1+x)^k \ge 1+kx+x(1+x)^k \ge 1+kx+x = 1+(k+1)x$.

15. $(1+x)^{k+1} = (1+x)^k(1+x) \ge [1+kx+\frac{1}{2}k(k-1)x^2](1+x)$ since $x \ge 0$.
$= 1+kx+\frac{1}{2}k(k-1)x^2+x+kx^2+\frac{1}{2}k(k-1)x^3 \ge 1+(k+1)x+[\frac{1}{2}k(k-1)+k]x^2$
$= 1+(k+1)x+[\frac{1}{2}k^2+\frac{1}{2}k]x^2 = 1+(k+1)x+\frac{1}{2}(k+1)kx^2$.

16. By $P(k)$, $\exists\, m \in \mathbb{N} \ni 13^k - 6^k = 7m$. Then $13^{k+1}-6^{k+1} = 13\cdot 13^k - [13-7]6^k = 13[13^k-6^k]+7\cdot 6^k$
$= 13(7m)+7\cdot 6^k = 7[13m+6^k]$, which is divisible by 7.

17. Assume $2^{2k-1}+1 = 3m$ for some $m \in \mathbb{N}$. Then $2^{2(k+1)-1}+1 = 2^{2k+1}+1 = 2^2 2^{2k-1}+1 = 4(3m-1)+1$
$= 12m-4+1 = 3(4m-1)$.

18. Suppose $n_0 \in \mathbb{N}$ and $p(n)$ is a proposition satisfying the given hypotheses. Let $q(n)$ be the proposition $p(n_0-1+n)$. Then,
(i) $q(1) \equiv p(n_0)$, which is true by hypothesis.
(ii) Assume $k \in \mathbb{N}$ and $q(k)$ is true. Then $n_0-1+k = n_0+(k-1) \ge n_0$ and $p(n_0-1+k)$ is true.
$\therefore$ by hypothesis (2), $p(n_0-1+k+1)$ is true; i.e.,, $q(k+1)$ is true. Therefore, $q(k) \Rightarrow q(k+1)$.
By the principle of mathematical induction, $\forall n \in \mathbb{N}$, $q(n)$ is true. $\therefore \forall n \in \mathbb{N}$, $p(n_0-1+n)$ is true. That is, $\forall n \ge n_0$ in $\mathbb{N}$, $p(n)$ is true.

19. $x^{k+1}-y^{k+1} = x^k x - y^k y = x^k x - y^k x + y^k x - y^k y = x(x^k-y^k)+y^k(x-y)$
$= x(x-y)(x^{k-1}+x^{k-2}y+x^{k-3}y^2+\cdots+xy^{k-2}+y^{k-1})+y^k(x-y)$
$= (x-y)(x^k+x^{k-1}y+x^{k-2}y^2+\cdots+xy^{k-1}+y^k)$.

20. Let $m \in \mathbb{N}$ be fixed, and $\forall n \in \mathbb{N}$, let $p(n)$ denote $a^m a^n = a^{m+n}$. Then
(i) $p(1)$ is true since by Defn.1.3.12, $a^{m+1} = a^m a^1$.
(ii) Assume $p(k)$. Then $a^m a^{k+1} = a^m[a^k a] = [a^m a^k]a = a^{m+k}a = a^{m+k+1}$. $\therefore p(k+1)$.

21. Let $a, b \in \mathbb{R}$, let m be a fixed element of $\mathbb{N}$, and $\forall n \in \mathbb{N}$, let $p(n)$ denote $(a^m)^n = a^{mn}$. Then
(i) $p(1)$ is true since by Defn.1.3.12, $(a^m)^1 = a^m$.
(ii) Assume $p(n)$. Then $(a^m)^{n+1} = (a^m)^n a^m = a^{mn} a^m = a^{mn+m} a = a^{m(n+1)}$. $\therefore p(n+1)$.

22. Let $a, b \in \mathbb{R}$ and $\forall n \in \mathbb{N}$, let $p(n)$ denote $a^n b^n = (ab)^n$. Then
(i) $p(1)$ is true since $a^1 b^1 = ab = (ab)^1$.
(ii) Assume $p(k)$. Then $a^{k+1} b^{k+1} = a^k \cdot a \cdot b^k \cdot b = a^k b^k \cdot ab = (ab)^k (ab) = (ab)^{k+1}$. $\therefore p(k+1)$.

23. $\binom{n}{0} = \dfrac{n!}{0!\,n!} = \binom{n}{n} = 1.$ $\qquad \binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \binom{n}{n-k}.$

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n+1-k)!} + \frac{n!}{k!(n-k)!} = \frac{n!k + n!(n+1-k)}{k!(n+1-k)!} = \frac{n!(k+n+1-k)}{k!(n+1-k)!}$$
$$= \frac{n!(n+1)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}.$$

24. Let $a, b \in \mathbb{R}$, and $\forall n \in \mathbb{N}$, let $p(n)$ denote $(a+b)^n = \sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Then

(i) $p(1)$ is true since $\sum\limits_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 = a + b = (a+b)^1$.

(ii) Assume $p(n)$. Then $(a+b)^{n+1} = (a+b)^n (a+b) = a \sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k} b^k + b \sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k} b^k =$

$a^{n+1} + \sum\limits_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum\limits_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + b^{n+1} = a^{n+1} + \sum\limits_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + b \sum\limits_{k=1}^{n} \binom{n}{k-1} a^{n+1-k} b^k + b^{n+1}$

$= a^{n+1} + \sum\limits_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] a^{n+1-k} b^k + b^{n+1} = \sum\limits_{k=0}^{n+1} \binom{n+1}{k} a^{(n+1)-k} b^k$. (See Ex.23.)

1.4 Rational Numbers

EXERCISE SET 1.4

1. The sum of any two integers is an integer, so (A0) is true. The product of any two integers is an integer, so (M0) is true. (A1), (A2), (M1), (M2) and (D) are inherited from F.
(A3) 0 is an integer, and for all integers x, $x + 0 = x$.
(A4) For all integers x, $-x$ is an integer, and $x + (-x) = 0$.
(M3) 1 is an integer, and for all integers x, $x \cdot 1 = x$.
$\mathbb{Z}$ does not satisfy (M4): for example, $2 \in \mathbb{Z}$ but $\nexists y \in \mathbb{Z} \ni 2y = 1$.

2. Let F be any ordered field. First, $\mathbb{N} \subseteq F$ by defn. of $\mathbb{N}$. Then, $\forall m \in \mathbb{Z}$, $m \in \mathbb{N}$, $-m \in \mathbb{N}$, or $m = 0$; in any case, $m \in F$. $\therefore \mathbb{Z} \subseteq F$.
Finally, $\forall x \in \mathbb{Q}$, $\exists\, m, n \in \mathbb{Z} \ni x = mn^{-1}, n \neq 0$. Thus, since F is a field, $x \in F$. $\therefore \mathbb{Q} \subseteq F$.

3. Let $n \in \mathbb{Z}$. n not divisible by 2 $\Rightarrow \exists k \in \mathbb{Z} \ni n = 2k+1 \Rightarrow n^2 = 4k^2 + 4k + 1 \Rightarrow n^2$ not divisible by 2.

4. First, we prove the lemma: $\forall n \in \mathbb{Z}$, if n^2 is divisible by 3, then n is divisible by 3.
Proof: We prove the contrapositive. Suppose $n \in \mathbb{Z}$ is not divisible by 3. Then n is of the form $3m+1$ or $3m+2$, for some $m \in \mathbb{Z}$. But $(3m+1)^2 = 9m^2 + 6m + 1 = 3[3m^2 + 2m] + 1$, and $(3m+2)^2 = 9m^2 + 12m + 4 = 3[3m^2 + 4m] + 1$, neither of which is divisible by 3. $\therefore n^2$ is not divisible by 3.
Then, redo the proof of Thm.1.4.5, replacing 2 by 3 at appropriate places.

5. Suppose x is rational and y is irrational, and let $z = x + y$. Then $y = z - x$. If z is rational than so is y, since $\mathbb{Q}$ is a field, which would be a contradiction.

6. Suppose $x \neq 0$ is rational and y is irrational, and let $z = xy$. Then $y = zx^{-1}$. If z is rational than so is y, since $\mathbb{Q}$ is a field, which would be a contradiction.

7. Let x be irrational. Since $x + (-x) = 0$, Ex.5 says $-x$ cannot be rational. Since $x(x^{-1}) = 1$, Ex.6 says x^{-1} cannot be rational.

8. Let z be irrational. Then, by Ex.7, $-z$ is irrational. But $z + (-z) = 0$, which is rational.

9. $\sqrt{2} \cdot \sqrt{2} = 2$.

10. $\forall n \in \mathbb{N}$, $n + \sqrt{2}$ is irrational, by Ex.5. Moreover, $n + \sqrt{2} = m + \sqrt{2} \Rightarrow n = m$, so there are infinitely many such irrational numbers.

11. Repeat the proof of Ex.1.3.18, replacing $\mathbb{N}$ by $\{n \in \mathbb{Z} : n \geq n_0\}$.

12. Let $x, y \in \mathbb{Z}$. Then

Case 1 ($x, y \in \mathbb{N}$): Then $x + y$ and xy are unique elements of $\mathbb{N}$, hence of $\mathbb{Z}$.

Case 2 ($-x, -y \in \mathbb{N}$): Then $-x - y = -x + (-y)$ is a unique element of $\mathbb{N}$, so $x + y = -(-x - y) \in \mathbb{Z}$. Similarly, $xy = (-x)(-y) \in \mathbb{Z}$.

Case 3 ($-x \in \mathbb{N}, y \in \mathbb{N}$): Then $-(xy) = (-x)y$ is a unique element of $\mathbb{N}$, so $xy \in \mathbb{Z}$. To show $x + y \in \mathbb{Z}$ requires a subtler argument.

Subcase a ($-x \leq y$): By Thm.1.3.7, $y - (-x) \in \mathbb{N}$, so $x + y \in \mathbb{N}$, hence $x + y \in \mathbb{Z}$.

Subcase b ($-x > y$): By Thm.1.3.7, $-x - y \in \mathbb{N}$, so $x + y = -(-x - y) \in \mathbb{Z}$.

Case 4 ($x \in \mathbb{N}, -y \in \mathbb{N}$): Same as case 3, with x and y interchanged.

1.5 The Archimedean Property

EXERCISE SET 1.5

1. $x \in \mathbb{Q} \Rightarrow x = \frac{a}{b}$ for some $a \in \mathbb{Z}$, $b \in \mathbb{N}$. Then $|a| + 1 > \frac{|a|}{b} \geq \frac{a}{b} = x$ and $|a| + 1 \in \mathbb{N}$.

2. Assume (a) and let $x \in F$. Then $|x| + 1 > 0$. By (a), $\exists n \in \mathbb{N} \ni n > |x| + 1$. Then $n > x$. $\therefore$ A.P.

3. Assume F has A.P., and $a > 0$. Then $\forall x \in F$, $\frac{x}{a} \in F$ so by A.P. $\exists n \in \mathbb{N} \ni n > \frac{x}{a}$. Since $a > 0$, this means $na > x$.

4. Assume (c) and let $x > 0$. Then $\frac{1}{x} > 0$. By (c), $\exists n \in \mathbb{N} \ni \frac{1}{n} < \frac{1}{x}$. Then by Thm.1.2.10 (a), $n > x$.

5. Suppose $x \in$ Archimedean F. First prove existence. If x is an integer, take $n = x + 1$. If x is not an integer then $x > 0$ or $x < 0$. The first case is covered by Thm.1.5.3. Suppose $x < 0$. Then $-x > 0$ so by Thm.1.5.3, $\exists m \in \mathbb{N} \ni m - 1 < -x < m$. Then $-m < x < -m + 1$, so we may take $n = -m + 1$. Uniqueness follows by the argument given in Thm.1.5.3.

6. Let F be an Archimedean ordered field with at least one irrational element, z. Then $|z|$ is irrational. Let $a < b$ in F. Then $\dfrac{a}{|z|} < \dfrac{b}{|z|}$. Since the rationals are dense in F by (a), $\exists$ rational $r \ni \dfrac{a}{|z|} < r < \dfrac{b}{|z|}$. We can choose r to be nonzero, since if $r = 0$, we simply choose rational $r' \ni r < r' < \dfrac{b}{|z|}$. Then, by Ex.1.4.6, $r|z|$ is irrational, and $a < r|z| < b$.

7. Let $x < y$ in an ordered F. Then $x < \frac{x+y}{2} < y$ (Thm.1.2.10 d).

8. Suppose S is dense in F, and let $a < b$ in F. Then $\exists s_1 \in S \ni a < s_1 < b$, and $\exists s_2 \in S \ni s_1 < s_2 < b$, and so on by mathematical induction: if we have $s_1, s_2, \cdots, s_k$ in $S \ni a < s_1 < s_2 < \cdots < s_k < b$, we can choose $s_{k+1} \in S \ni s_k < s_{k+1} < b$. Then $\{s_k : k \in \mathbb{N}\}$ is a set of infinitely many different elements of S lying between a and b.

9. Suppose that $\forall \varepsilon > 0$, $x \leq a + \varepsilon$. Then $\forall \varepsilon > 0$, $x - a \leq \varepsilon$, so by (a), $x - a \leq 0$; i.e., $x \leq a$.

10. Suppose that $\forall \varepsilon > 0$, $|x| \leq \varepsilon$. By (a), $|x| \leq 0$. But $|x| \geq 0$, so $|x| = 0$, which implies $x = 0$.

11. Suppose that $\forall \varepsilon > 0$, $|a - b| \leq \varepsilon$. By (a), $|a - b| \leq 0$. But $|a - b| \geq 0$, so $|a - b| = 0$; i.e., $a = b$.

12. (a) Suppose that $\forall \varepsilon > 0$, $x \geq -\varepsilon$. Then $\forall \varepsilon > 0$, $-x \leq \varepsilon$, so by the forcing principle, $-x \leq 0$. $\therefore x \geq 0$.
(b) Suppose that $\forall \varepsilon > 0$, $x \geq a - \varepsilon$ $(x - a \geq -\varepsilon)$. Then by (a), $x - a \geq 0$. $\therefore x \geq a$.

1.6 The Completeness Property

EXERCISE SET 1.6-A

1. (a) Yes; $3, 4, 86$; 3 (b) Yes; $3, 4, 86$; 3 (c) Yes; $4, 4.01, 86$; 3 (d) Yes; $5, 6, 29$; 5 (e) Yes; $0, 0.2, 86$; 0 (f) No (g) Yes; $-100, 0, 25$; none (h) Yes; $1, 2, 24$; 1 (i) No (j) Yes; $1, 1.8, 50$; 1 (k) Yes; $2, 3, 86$; 2 (l) Yes; $3/2, 2, 3$; $3/2$ (m) Yes; $2, 3, 86$; 1.5 [Draw graph of $f(x) = 1 + \frac{1}{x}$ for $x > 2$.] (n) Yes; $1, 2, 10$; 1

2. (a) Yes; $-1, -2, -100$; -1 (b) Yes; $-1, -2, -50$; -1 (c) Yes; $1, 0, -20$; 1 (d) Yes; $5, 0, -5$; 5 (e) No (f) Yes; $0, -1, -50$; 0 (g) Yes; $-100, 0, 25$; none (h) Yes; $0, -1, -10$; 0 (i) Yes; $0, -1, -100$; 0 [Draw graph of $f(x) = \frac{1}{x}$.] (j) Yes; $1/2, 0, -1$; $1/2$ (k) Yes; $1, 0, -100$; 1 [Draw graph.] (l) Yes; $1, 0, -1$; 1 (m) Yes; $2, 1, -100$; 1 [Draw graph.] (n) Yes; $-1, -2, -20$; -1

3. Examples given in Ex.1 and 2.

4. Suppose $u = \min A$ and $v = \min A$. Then $u \in A$ and $\forall x \in A$, $u \leq x$. Also, $v \in A$ and $\forall x \in A$, $v \leq x$. Thus, $u \leq v$ and $v \leq u$. $\therefore u = v$.

5. Alter the proof already given that shows S has a maximum element.

6. (i) By defn. of (a, b), b is an upper bound for (a, b).
(ii) Suppose v is an upper bound for (a, b). We want to prove $b \leq v$. For contradiction, suppose $b > v$. Let $c = \frac{v+b}{2}$. Then $a \leq v < c < b$. Then $c \in (a, b)$ and $c > v$, contradicting the fact that v is an upper bound for (a, b). $\therefore b \leq v$.
By (i) and (ii) together, $b = \sup(a, b)$.

7. Alter the proof of part (a) given.

8. Suppose $m = \min A$. Then $m \in A$ and $\forall a \in A$, $m \leq a$. Thus, m is a lower bound for A. If v is any lower bound for A, then $v \leq m$, since $m \in A$. $\therefore m = \inf A$. The argument for max. is similar.

9. If $u = \inf A \in A$, then $u \in A$ and $\forall a \in A$, $a \geq u$, so by defn., $u = \min A$.

10. Let $n \in \mathbb{N}$. Then $n + 1 \in \mathbb{N}$. Therefore, n cannot be the maximum element of $\mathbb{N}$. If the field F is Archimedean, then $\mathbb{N}$ has no upper bound in F, so it cannot have a supremum. However, $\mathbb{N}$ does have a minimum element, 1. $\therefore 1 = \inf \mathbb{N}$.

11. Let F be Archimedean, $A \subseteq F$, and $u \in F$.
($\Rightarrow$) Suppose $u = \inf A$. Let $\varepsilon > 0$. Then $\forall x \in A$, $x \geq u > u - \varepsilon$. Also, $u + \varepsilon > \inf A$, so $u + \varepsilon$ is not a lower bound for A, so $\exists x \in A \ni x < u + \varepsilon$.
($\Leftarrow$) Suppose (a) and (b) hold. Then,
(1) $\forall x \in A$, $x > u - \varepsilon$. By Ex.1.5.12, $x \geq u$.
(2) Suppose v is a lower bound for A. For contradiction, suppose $v > u$. Let $\varepsilon = v - u$. By (b), $\exists x \in A \ni x < u + \varepsilon = v$. Contradiction. Therefore, all lower bounds of A are $\leq u$.
By (1) and (2) together, $u = \inf A$.

12. u is a lower bound for $A \;\Leftrightarrow\; \forall a \in A, u \leq a \;\Leftrightarrow\; \forall a \in A, -a \leq -u \;\Leftrightarrow\; -u$ is an upper bound for -A.

13. The ($\Leftarrow$) direction is trivial; we prove ($\Rightarrow$). Suppose $u = \sup A$ and $u \notin A$. Let $\varepsilon > 0$. Then
(a) $\forall a \in A$, $x < u < u + \varepsilon$.
(b) $u - \varepsilon$ is not an upper bound for A, so $\exists\, a_1 \in A \ni u - \varepsilon < a_1 < u$. Now a_1 is not an upper bound for A, so $\exists\, a_2 \in A \ni u - \varepsilon < a_1 < a_2 < u$. Apply mathematical induction. If $a_1, a_2, \cdots, a_k \in A \ni u - \varepsilon < a_1 < a_2 < \cdots < a_k < u$, then a_k is not an upper bound for A, so $\exists\, a_{k+1} \in A \ni a_k < a_{k+1} < u$. By math induction, $\{a_k : k \in \mathbb{N}\}$ consists of infinitely many members of A greater than $u - \varepsilon$.

14. For $v \notin A$, $v = \inf A \Leftrightarrow \forall\varepsilon > 0$, (a) $\forall x \in A$, $x > v - \varepsilon$, and (b) $\exists$ infinitely many $x \in A \ni x < v + \varepsilon$. The proof is a straightforward modification of that of Ex.13.

EXERCISE SET 1.6-B

1. Suppose A is a nonempty set with a lower bound in a complete ordered F. By Ex.1.6-A.12, the set $-A = \{-a : a \in A\}$ is bounded above. By completeness, $\exists u = \sup(-A)$. Then
(a) $\forall a \in A$, $-a \leq u$, so $a \geq -u$.
(b) If v is any lower bound for A, then by Ex.1.6-A.12, $-v$ is an upper bound for $-A$, so $-v \geq u$. That is, $v \leq -u$.

By (a) and (b) together, $-u = \inf A$.

2. Suppose $A \subseteq B$ and $A \neq \varnothing$.
(a) If B is bounded below, then $\forall b \in B$, $b \geq \inf B$. Then $\forall a \in A$, $a \in B$, so $a \geq \inf B$. Then $\inf B$ is a lower bound for A, so $\inf B \leq \inf A$.
(b) If B is bounded above, then $\forall b \in B$, $b \leq \sup B$. Then $\forall a \in A$, $a \in B$, so $a \leq \sup B$. Then $\sup B$ is an upper bound for A, so $\sup B \geq \sup A$.

3. Let A be a nonempty subset of ordered F with an upper bound in F, and B be the set of all upper bounds of A in F. By completeness, $\exists u = \sup A$. then $u \in B$ and $\forall b \in B$, $u \leq b$. $\therefore u = \min B$.

4. Suppose $A \neq \varnothing$ and is bounded below in F. Let $B = \{\text{all lower bounds of } A \text{ in } F\}$. By completeness, $\exists u = \inf A \in B$ and u is the largest member of B. That is, $\inf A = \max B$.

5. Let $a = \sup A$, $b = \sup B$ and $c = \max\{a, b\}$. Then
(a) Let $x \in A \cup B$. Then either $x \in A$ so $x \leq a \leq c$, or $x \in B$ so $x \leq b \leq c$. Thus $x \leq c$.
(b) If d is any upper bound for $A \cup B$ then d is any upper bound for A and d is any upper bound for B, so $d \geq a$ and $d \geq b$; thus $d \geq c$.
By (a) and (b) together, $c = \sup A \cup B$.

6. Let $a > 0$ in a complete F, and let $A = \{x > 0 : x^2 < a\}$ and $B = \{x > 0 : x^2 > a\}$.

Case 1 ($a > 1$): Then $1 \in A$, so A is a nonempty set bounded above by a (show). By completeness, $\exists u = \sup A$. Then $u \geq 1$ since $1 \in A$. From here, follow the proof of Thm.1.6.10 verbatim, replacing a in (1) by x, and 2 throughout by a.

Case 2 ($0 < a < 1$): Then $\frac{1}{a} > 1$ and by Case 1, $\exists u \in F \ni u^2 = \frac{1}{a}$. Then $\left(\frac{1}{u}\right)^2 = a$.

Case 3 ($a = 1$): trivial.

7. (a) $\forall x \in X$, $f(x)+g(x) \leq \sup\{f(x) : x \in X\}+\sup\{g(x) : x \in X\}$. Thus, $\sup\{f(x) : x \in X\}+\sup\{g(x) : x \in X\}$ is an upper bound for $\{f(x) + g(x) : x \in X\}$, so it is $\geq$ the least upper bound.

(b) $\forall x \in X$, $f(x) + g(x) \geq \inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\}$. Thus, $\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\}$ is a lower bound for $\{f(x) + g(x) : x \in X\}$, so it is $\leq$ the greatest lower bound.