

## Answers to selected exercises for chapter 1

1.1 Apply  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ , then

$$\begin{aligned} f_1(t) + f_2(t) &= A_1 \cos \omega t \cos \phi_1 - A_1 \sin \omega t \sin \phi_1 + A_2 \cos \omega t \cos \phi_2 - A_2 \sin \omega t \sin \phi_2 \\ &= (A_1 \cos \phi_1 + A_2 \cos \phi_2) \cos \omega t - (A_1 \sin \phi_1 + A_2 \sin \phi_2) \sin \omega t \\ &= C_1 \cos \omega t - C_2 \sin \omega t, \end{aligned}$$

where  $C_1 = A_1 \cos \phi_1 + A_2 \cos \phi_2$  and  $C_2 = A_1 \sin \phi_1 + A_2 \sin \phi_2$ . Put  $A = \sqrt{C_1^2 + C_2^2}$  and take  $\phi$  such that  $\cos \phi = C_1/A$  and  $\sin \phi = C_2/A$  (this is possible since  $(C_1/A)^2 + (C_2/A)^2 = 1$ ). Now  $f_1(t) + f_2(t) = A(\cos \omega t \cos \phi - \sin \omega t \sin \phi) = A \cos(\omega t + \phi)$ .

1.2 Put  $c_1 = A_1 e^{i\phi_1}$  and  $c_2 = A_2 e^{i\phi_2}$ , then  $f_1(t) + f_2(t) = (c_1 + c_2)e^{i\omega t}$ . Let  $c = c_1 + c_2$ , then  $f_1(t) + f_2(t) = ce^{i\omega t}$ . The signal  $f_1(t) + f_2(t)$  is again a time-harmonic signal with amplitude  $|c|$  and initial phase  $\arg c$ .

1.5 The power  $P$  is given by

$$\begin{aligned} P &= \frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} A^2 \cos^2(\omega t + \phi_0) dt = \frac{A^2 \omega}{4\pi} \int_{-\pi/\omega}^{\pi/\omega} (1 + \cos(2\omega t + 2\phi_0)) dt \\ &= \frac{A^2}{2}. \end{aligned}$$

1.6 The energy-content is  $E = \int_0^\infty e^{-2t} dt = \frac{1}{2}$ .

1.7 The power  $P$  is given by

$$P = \frac{1}{4} \sum_{n=0}^3 |\cos(n\pi/2)|^2 = \frac{1}{2}.$$

1.8 The energy-content is  $E = \sum_{n=0}^\infty e^{-2n}$ , which is a geometric series with sum  $1/(1 - e^{-2})$ .

1.9 **a** If  $u(t)$  is real, then the integral, and so  $y(t)$ , is also real.

**b** Since

$$\left| \int u(\tau) d\tau \right| \leq \int |u(\tau)| d\tau,$$

it follows from the boundedness of  $u(t)$ , so  $|u(\tau)| \leq K$  for some constant  $K$ , that  $y(t)$  is also bounded.

**c** The linearity follows immediately from the linearity of integration. The time-invariance follows from the substitution  $\xi = \tau - t_0$  in the integral  $\int_{t_0}^t u(\tau - t_0) d\tau$  representing the response to  $u(t - t_0)$ .

**d** Calculating  $\int_{t_0}^t \cos(\omega\tau) d\tau$  gives the following response:  $(\sin(\omega t) - \sin(\omega t_0))/\omega = 2 \sin(\omega/2) \cos(\omega t - \omega/2)/\omega$ .

**e** Calculating  $\int_{t_0}^t \sin(\omega\tau) d\tau$  gives the following response:  $(-\cos(\omega t) + \cos(\omega t_0))/\omega = 2 \sin(\omega/2) \sin(\omega t - \omega/2)/\omega$ .

**f** From the response to  $\cos(\omega t)$  in d it follows that the amplitude response is  $|2 \sin(\omega/2)/\omega|$ .

**g** From the response to  $\cos(\omega t)$  in d it follows that the phase response is  $-\omega/2$  if  $2 \sin(\omega/2)/\omega \geq 0$  and  $-\omega/2 + \pi$  if  $2 \sin(\omega/2)/\omega < 0$ . From

phase and amplitude response the frequency response follows:  $H(\omega) = 2 \sin(\omega/2)e^{-i\omega/2}/\omega$ .

- 1.11 **a** The frequency response of the cascade system is  $H_1(\omega)H_2(\omega)$ , since the response to  $e^{i\omega t}$  is first  $H_1(\omega)e^{i\omega t}$  and then  $H_1(\omega)H_2(\omega)e^{i\omega t}$ .  
**b** The amplitude response is  $|H_1(\omega)H_2(\omega)| = A_1(\omega)A_2(\omega)$ .  
**c** The phase response is  $\arg(H_1(\omega)H_2(\omega)) = \Phi_1(\omega) + \Phi_2(\omega)$ .

- 1.12 **a** The amplitude response is  $|1 + i| |e^{-2i\omega}| = \sqrt{2}$ .  
**b** The input  $u[n] = 1$  has frequency  $\omega = 0$ , initial phase 0 and amplitude 1. Since  $e^{i\omega n} \mapsto H(e^{i\omega})e^{i\omega n}$ , the response is  $H(e^0)1 = 1 + i$  for all  $n$ .  
**c** Since  $u[n] = (e^{i\omega n} + e^{-i\omega n})/2$  we can use  $e^{i\omega n} \mapsto H(e^{i\omega})e^{i\omega n}$  to obtain that  $y[n] = (H(e^{i\omega})e^{i\omega n} + H(e^{-i\omega})e^{-i\omega n})/2$ , so  $y[n] = (1 + i) \cos(\omega(n - 2))$ .  
**d** Since  $u[n] = (1 + \cos 4\omega n)/2$ , we can use the same method as in b and c to obtain  $y[n] = (1 + i)(1 + \cos(4\omega(n - 2)))/2$ .

- 1.13 **a** The power is the integral of  $f^2(t)$  over  $[-\pi/|\omega|, \pi/|\omega|]$ , times  $|\omega|/2\pi$ . Now  $\cos^2(\omega t + \phi_0)$  integrated over  $[-\pi/|\omega|, \pi/|\omega|]$  equals  $\pi/|\omega|$  and  $\cos(\omega t) \cos(\omega t + \phi_0)$  integrated over  $[-\pi/|\omega|, \pi/|\omega|]$  is  $(\pi/|\omega|) \cos \phi_0$ . Hence, the power equals  $(A^2 + 2AB \cos(\phi_0) + B^2)/2$ .  
**b** The energy-content is  $\int_0^1 \sin^2(\pi t) dt = 1/2$ .

- 1.14 The power is the integral of  $|f(t)|^2$  over  $[-\pi/|\omega|, \pi/|\omega|]$ , times  $|\omega|/2\pi$ , which in this case equals  $|c|^2$ .

- 1.16 **a** The amplitude response is  $|H(\omega)| = 1/(1 + \omega^2)$ . The phase response is  $\arg H(\omega) = \omega$ .  
**b** The input has frequency  $\omega = 1$ , so it follows from  $e^{i\omega t} \mapsto H(\omega)e^{i\omega t}$  that the response is  $H(1)ie^{it} = ie^{i(t+1)}/2$ .

- 1.17 **a** The signal is not periodic since  $\sin(2N) \neq 0$  for all integer  $N$ .  
**b** The frequency response  $H(e^{i\omega})$  equals  $A(e^{i\omega})e^{i\Phi e^{i\omega}}$ , hence, we obtain that  $H(e^{i\omega}) = e^{i\omega}/(1 + \omega^2)$ . The response to  $u[n] = (e^{2in} - e^{-2in})/2i$  is then  $y[n] = (e^{2i(n+1)} - e^{-2i(n+1)})/(10i)$ , so  $y[n] = (\sin(2n + 2))/5$ . The amplitude is thus  $1/5$  and the initial phase  $2 - \pi/2$ .

- 1.18 **a** If  $u(t) = 0$  for  $t < 0$ , then the integral occurring in  $y(t)$  is equal to 0 for  $t < 0$ . For  $t_0 \geq 0$  the expression  $u(t - t_0)$  is also causal. Hence, the system is causal for  $t_0 \geq 0$ .

**b** It follows from the boundedness of  $u(t)$ , so  $|u(\tau)| \leq K$  for some constant  $K$ , that  $y(t)$  is also bounded (use the triangle inequality and the inequality from exercise 1.9b). Hence, the system is stable.

**c** If  $u(t)$  is real, then the integral is real and so  $y(t)$  is real. Hence, the system is real.

**d** The response is

$$y(t) = \sin(\pi(t - t_0)) + \int_{t-1}^t \sin(\pi\tau) d\tau = \sin(\pi(t - t_0)) - 2(\cos \pi t)/\pi.$$

- 1.19 **a** If  $u[n] = 0$  for  $n < 0$ , then  $y[n]$  is also equal to 0 for  $n < 0$  whenever  $n_0 \geq 0$ . Hence, the system is causal for  $n_0 \geq 0$ .

**b** It follows from the boundedness of  $u[n]$ , so  $|u[n]| \leq K$  for some constant  $K$  and all  $n$ , that  $y[n]$  is also bounded (use the triangle inequality):

$$|y[n]| \leq |u[n - n_0]| + \left| \sum_{l=n-2}^n u[l] \right| \leq K + \sum_{l=n-2}^n |u[l]| \leq K + \sum_{l=n-2}^n K,$$

which equals  $4K$ . Hence, the system is stable.

**c** If  $u[n]$  is real, then  $u[n - n_0]$  is real and also the sum in the expression for  $y[n]$  is real, hence,  $y[n]$  is real. This means that the system is real.

**d** The response to  $u[n] = \cos \pi n = (-1)^n$  is

$$\begin{aligned} y[n] &= (-1)^{n-n_0} + \sum_{l=n-2}^n (-1)^l = (-1)^{n-n_0} + (-1)^n(1 - 1 + 1) \\ &= (-1)^n(1 + (-1)^{n_0}). \end{aligned}$$