CHAPTER 2

2.1 $u_1^{(1)} = U_1 \quad u_1^{(2)} = u_2^{(1)} = U_2 \quad u_2^{(2)} = u_1^{(3)} = U_3 \quad u_2^{(3)} = U_4$

U_i = GLOBAL DISPLACEMENT i, $i = 1, 4$

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix}$$

2.2

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+2k_3 & -2k_3 \\ 0 & 0 & -2k_3 & 2k_3 \end{bmatrix}$$

2.3

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & \cdots & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 & \cdots & 0 \\ 0 & -k_2 & k_1+k_3 & -k_3 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & k_{N-2}+k_{N-1} & -k_{N-1} \\ 0 & 0 & \cdots & & -k_{N-1} & k_{N-1} \end{bmatrix}$$

2.4

$$[K] = \begin{bmatrix} 50 & -50 & 0 \\ -50 & 75 & -25 \\ 0 & -25 & 25 \end{bmatrix} \text{ LB/IN.}$$

$U_2 = \delta = 0.75$ IN.

$U_1 = 0 \longrightarrow$ DISPLACEMENT CONSTRAINT

$$\begin{bmatrix} 75 & -25 \\ -25 & 25 \end{bmatrix} \begin{Bmatrix} 0.75 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_3 \end{Bmatrix}$$

FIRST EQUATION: $75(0.75) - 25U_3 = 0 \Rightarrow U_3 = 2.25$ IN.

SECOND EQUATION: $-25(0.75) + 25U_3 = F_3 \Rightarrow F_3 = 37.5$ LB.

2.5

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} = \begin{bmatrix} 30 & -30 & 0 & 0 \\ -30 & 70 & -40 & 0 \\ 0 & -40 & 70 & -30 \\ 0 & 0 & -30 & 30 \end{bmatrix} \text{ LB/IN.}$$

$U_2 = 0 \quad U_4 = 1$ IN. $\quad U_1 = 0$ (BOUNDARY CONDITION)

$$\begin{bmatrix} 70 & -40 & 0 \\ -40 & 70 & -30 \\ 0 & -30 & 30 \end{bmatrix} \begin{Bmatrix} 0 \\ U_3 \\ 1 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ 0 \\ F_4 \end{Bmatrix}$$

$$\left.\begin{aligned} -40U_3 &= F_2 \\ 70U_3 - 30 &= 0 \\ -30U_3 + 30 &= F_4 \end{aligned}\right\} \quad \begin{aligned} U_3 &= 3/7 \text{ IN.} \\ F_2 &= 17.14 \text{ LB.} \\ F_4 &= 17.14 \text{ LB.} \end{aligned}$$

2.6 (a)

$$[K] = \begin{bmatrix} k & -k & 0 & 0 \\ -k & k+3k & -3k & 0 \\ 0 & -3k & 3k+2k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix} = \begin{bmatrix} k & -k & 0 & 0 \\ -k & 4k & -3k & 0 \\ 0 & -3k & 5k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix}$$

(b) $U_e = \frac{1}{2}k(U_2-U_1)^2 + \frac{1}{2}(3k)(U_3-U_2)^2 + \frac{1}{2}(2k)(U_4-U_3)^2$

$$\frac{\partial U_e}{\partial U_1} = F_1 = -k(U_2-U_1)$$

$$\frac{\partial U_e}{\partial U_2} = k(U_2-U_1) - (3k)(U_3-U_2) = F_2$$

$$\frac{\partial U_e}{\partial U_3} = (3k)(U_3-U_2) - (2k)(U_4-U_3) = F_3$$

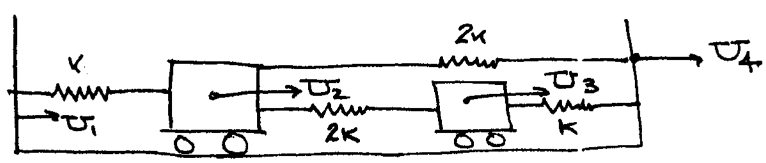
$$\frac{\partial U_e}{\partial U_4} = (2k)(U_4-U_3)$$

IN MATRIX FORM:

$$\begin{bmatrix} k & -k & 0 & 0 \\ -k & 4k & -3k & 0 \\ 0 & -3k & 5k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

RESULTS ARE IDENTICAL AS EXPECTED.

2.7



$$[K] = \begin{bmatrix} k & -k & 0 & 0 \\ -k & 5k & -2k & -2k \\ 0 & -2k & 3k & -k \\ 0 & -2k & -k & 3k \end{bmatrix} \qquad [K]\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ -F_1 \\ F_2 \\ R_4 \end{Bmatrix}$$

R_1 AND R_4 ARE REACTIONS

(b) APPLYING THE CONSTRAINTS $U_1 = U_4 = 0$ AND SUBSTITUTING NUMERICAL VALUES

$$\begin{bmatrix} 250 & -100 \\ -100 & 150 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 15 \end{Bmatrix}$$

GIVES

$$U_2 = -0.055 \text{ IN.} \qquad U_3 = 0.064 \text{ IN.}$$

2.8 $$U_e = \tfrac{1}{2} K_1 (U_2 - U_1)^2 + \tfrac{1}{2} K_2 (U_3 - U_2)^2 + \tfrac{1}{2} K_3 (U_4 - U_3)^2 + \tfrac{1}{2} K_4 (U_5 - U_4)^2$$

$$\frac{\partial U_e}{\partial U_1} = -K_1 (U_2 - U_1) = F_1$$

$$\frac{\partial U_e}{\partial U_2} = K_1 (U_2 - U_1) - K_2 (U_3 - U_2) = F_2$$

$$\frac{\partial U_e}{\partial U_3} = K_2 (U_3 - U_2) - K_3 (U_4 - U_3) = F_3$$

$$\frac{\partial U_e}{\partial U_4} = K_3 (U_4 - U_3) - K_4 (U_5 - U_4) = F_4$$

$$\frac{\partial U_e}{\partial U_5} = -K_4 (U_5 - U_4) = F_1$$

$\underline{2.9}$ $K_1 = K_2 = K_3 = K_4 = 10 \text{ N/mm}$ $F_2 = 20 N$ $F_3 = 25 N$

$F_4 = 40 N$ $U_1 = U_5 = 0$

$$\begin{bmatrix} 10 & -10 & 0 & 0 & 0 \\ -10 & 20 & -10 & 0 & 0 \\ 0 & -10 & 20 & -10 & 0 \\ 0 & 0 & -10 & 20 & -10 \\ 0 & 0 & 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \\ U_4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ -20 \\ 25 \\ 40 \\ R_5 \end{Bmatrix}$$

$$\begin{bmatrix} 20 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 20 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 25 \\ 40 \end{Bmatrix} \Rightarrow \begin{array}{l} U_2 = 0.75 \text{ mm} \\ U_3 = 3.5 \text{ mm} \\ U_4 = 3.75 \text{ mm} \end{array}$$

$R_1 = -10 U_2 = -7.5 \text{ N}$

$R_5 = -10 U_4 = 37.5 \text{ N}$

$f^{(1)} = R_1 = 7.5 N$

$f^{(2)} = 10 (U_3 - U_2) = 27.5 N$

$f^{(3)} = 10 (U_4 - U_3) = 2.5 N$

$f^{(4)} = R_4 = 37.5 N$

$\underline{2.10}$ $$K_e = \frac{AE}{L_e} = \frac{500(207)(10^3)}{500} = 207(10^3) \text{ N/mm}$$

$$[K^{(1)}] = 207(10^3) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = [K^{(2)}]$$

$$[K] = 207(10^3)\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$207(10^3)\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 12(10^3) \end{Bmatrix}$$

$$U_3 = \frac{2(12)}{207} = 0.116 \text{ MM} \qquad U_2 = \frac{12}{207} = 0.058 \text{ MM}$$

$$\sigma^{(1)} = 207(10)^3 \frac{0.058-0}{500} = 24 \text{ MPa}$$

$$\sigma^{(2)} = 207(10^3)\frac{0.116-0.058}{500} = 24 \text{ MPa}$$

BUCKLING SHOULD BE CONSIDERED.

2.11

$$[K^{(1)}] = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 3(10^6)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ LB/IN}$$

$$[K^{(2)}] = \frac{A_2 E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1.125(10^6)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ LB/IN}$$

$$[K] = 10^6\begin{bmatrix} 3 & -3 & 0 \\ -3 & 4.125 & -1.125 \\ 0 & -1.125 & 1.125 \end{bmatrix}$$

$$10^6\begin{bmatrix} 3 & -3 & 0 \\ -3 & 4.125 & -1.125 \\ 0 & -1.125 & 1.125 \end{bmatrix}\begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 20(10^3) \end{Bmatrix}$$

$$U_2 = \frac{20(10^3)}{3(10^6)} \cong 6.7(10^{-3})\text{ IN.}$$

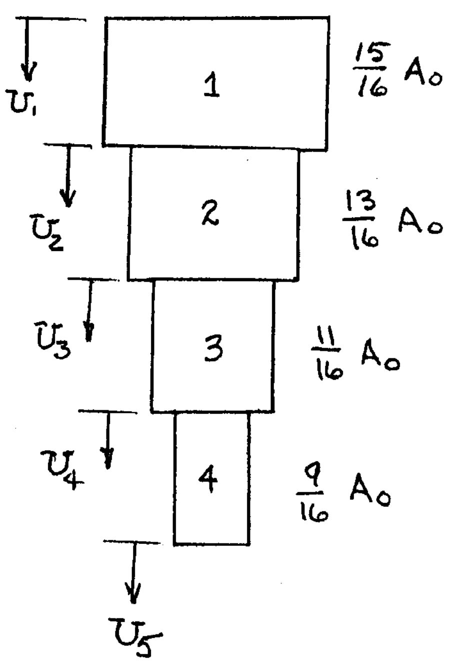
$$U_3 = \frac{1}{1.125(10^6)}\left(20(10^3) + 1.125(10^6)U_2\right) \cong 24.5(10^3)\text{ IN.}$$

$$R_1 = -3(10^6)U_2 = -20(10^3)\text{ LB.}$$

$$\sigma^{(1)} = 15(10^6)\frac{U_2 - U_1}{20} \cong 5025\text{ PSI} \quad \text{(TENSILE)}$$

$$\sigma^{(2)} = 10(10^6)\frac{U_3 - U_2}{20} \cong 8900\text{ PSI} \quad \text{(TENSILE)}$$

2.12



$E = 10(10^6)\text{ LB/IN}^2 \quad A_0 = 4\text{ IN}^2$

$L_e = 5\text{ IN} \quad P = 4000\text{ LB.}$

$$\frac{A_0 E}{L_e} = \frac{4(10)(10^6)}{5} = 8(10^6)\text{ LB/IN}$$

$$[K^{(1)}] = \frac{8(10^6)}{16}\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$[K^{(2)}] = \frac{8(10^6)}{16}\begin{bmatrix} 13 & -13 \\ -13 & 13 \end{bmatrix}$$

$$[K^{(3)}] = \frac{8(10^6)}{16}\begin{bmatrix} 11 & -11 \\ -11 & 11 \end{bmatrix} \qquad [K^{(4)}] = \frac{8(10^6)}{16}\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

ASSEMBLING THE SYSTEM EQUATIONS GIVES

$$\frac{8(10^6)}{16}\begin{bmatrix} 15 & -15 & 0 & 0 & 0 \\ -15 & 28 & -13 & 0 & 0 \\ 0 & -13 & 24 & -11 & 0 \\ 0 & 0 & -11 & 20 & -9 \\ 0 & 0 & 0 & -9 & 9 \end{bmatrix}\begin{Bmatrix} 0 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ 0 \\ 4000 \end{Bmatrix}$$

ELIMINATING THE REACTION EQUATION AND SOLVING THE REMAINING 4×4 SYSTEM GIVES

$U_2 = 5.33(10^{-4})$ IN. $\quad U_3 = 1.149(10^{-3})$ IN.

$U_4 = 1.876(10^{-3})$ IN. $\quad U_5 = 2.765(10^{-3})$ IN.

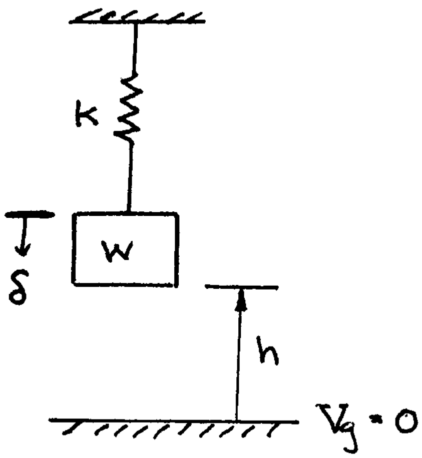
$$\sigma^{(1)} = 10(10^6)\frac{U_2 - U_1}{5} = 1066 \text{ psi}$$

$$\sigma^{(2)} = 10(10^6)\frac{U_3 - U_2}{5} = 1232 \text{ psi}$$

$$\sigma^{(3)} = 10(10^6)\frac{U_4 - U_3}{5} = 1454 \text{ psi}$$

$$\sigma^{(4)} = 10(10^6)\frac{U_5 - U_4}{5} = 1778 \text{ psi}$$

2.13



WHEN WEIGHT IS AT HEIGHT h ABOVE AN ARBITRARY DATUM FOR GRAVITATIONAL POTENTIAL ENERGY, THE SPRING IS UNDEFORMED. FOR ANY OTHER POSITION δ, POTENTIAL ENERGY IS

$$V = \frac{1}{2} K \delta^2 + W(h - \delta)$$

FOR MINIMUM POTENTIAL ENERGY

$$\frac{dV}{d\delta} = 0 = K\delta - W$$

$\therefore \quad K\delta = W$ AND THIS IS THE EQUILIBRIUM FORCE EQUATION.

2.14

CHECK THE REQUIRED NODAL CONDITIONS:

$$\left.\begin{aligned} N_1(0) &= \cos\frac{\pi 0}{2L} = 1 \\ N_1(L) &= \cos\frac{\pi}{2} = 0 \\ N_2(0) &= \sin\frac{\pi 0}{2L} = 0 \\ N_2(1) &= \sin\frac{\pi}{2} = 1 \end{aligned}\right\} \quad \text{AS REQUIRED}$$

$$\epsilon = \frac{du}{dx} = \frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2 = \frac{\pi}{2L}\left(-u_1 \sin\frac{\pi x}{2L} + u_2 \cos\frac{\pi x}{2L}\right)$$

$$\epsilon(x=0) = \frac{\pi}{2L} u_2 \qquad \epsilon(x=L) = -\frac{\pi}{2L} u_1$$

THE STRAIN VARIATION IS CLEARLY NOT ACCEPTABLE

PHYSICALLY. THIS IS A CASE IN WHICH A SET OF "ADMISSABLE FUNCTIONS" SATISFY THE BOUNDARY CONDITIONS BUT ARE NOT IN ACCORD WITH THE PHYSICS OF THE PROBLEM (CHAPTER 5).

2.15 FROM MECHANICS OF MATERIALS WE HAVE

$$\theta = \theta_2 - \theta_1 = \frac{TL}{JG} \quad (1)$$

AND

$$\tau = \frac{Tr}{J} \quad (2)$$

THE SHEAR STRAIN IS $\gamma = \frac{\tau}{G} = \frac{Tr}{JG}$ (3)

TOTAL STRAIN ENERGY IS

$$U_e = \frac{1}{2} \iiint_V \tau \gamma \, dV = \frac{1}{2} \iiint_V \frac{T^2 r^2}{J^2 G} \, dV$$

USING (1) TO EXPRESS TORQUE IN TERMS OF ANGULAR DISPLACEMENTS WE OBTAIN

$$U_e = \frac{G}{2L^2} \int_0^L (\theta_2 - \theta_1)^2 \iint_A r^2 \, dA \, dx$$

RECOGNIZING

$$\iint_A r^2 \, dA = J = \text{POLAR MOMENT OF INERTIA}$$

$$U_e = \frac{JG}{2L^2}\int_0^L (\theta_2-\theta_1)^2 dx = \frac{JG}{2L}(\theta_2-\theta_1)^2$$

THEN

$$\Pi_p = U_e - W = \frac{JG}{2L}(\theta_2-\theta_1)^2 - T_1\theta_1 - T_2\theta_2$$

$$\frac{\partial \Pi_p}{\partial \theta_1} = 0 = \frac{JG}{L}(\theta_2-\theta_1)(-1) - T_1$$

$$\frac{\partial \Pi_p}{\partial \theta_2} = 0 = \frac{JG}{L}(\theta_2-\theta_1) - T_2$$

IN MATRIX FORM

$$\frac{JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$