Integer Programming: 2nd Edition

Solutions to Certain Exercises in IP Book

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Solutions to Certain Exercises in Chapter 1

3. Modeling disjunctions.

(i) Extend the formulation of discrete alternatives of Section 1.5 to the union of two bounded polyhedra (*polytopes*) $P_k = \{y \in R^n : A^k y \leq b^k, 0 \leq y \leq u\}$ for $k = 1, 2$ where $\max_k \max_i \{a_i^k y - b_i^k : 0 \leq y \leq u\} \leq M$.

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(ii) Show that an extended formulation for $P_1 \cup P_2$ is the set Q:

$$\begin{aligned} y = w^1 + w^2 \\ A^k w^k \leq b^k x^k \quad \text{for } k = 1, 2 \\ 0 \leq w^k \leq u x^k \quad \text{for } k = 1, 2 \\ x^1 + x^2 = 1 \\ y \in \mathbb{R}^n, w^k \in \mathbb{R}^n, x^k \in \{0, 1\} \quad \text{for } k = 1, 2. \end{aligned}$$

Solution:

(i)
$$\begin{array}{rcll} A^k y & \leq & b_k & +M\mathbf{1}\delta_k \; k \in [1, 2] \\ 0 \leq y & \leq & u & \\ \delta_1 + \delta_2 & = & 1 & \\ \delta_k & \in & \{0, 1\} & k \in [1, 2] \end{array}$$

(ii) First we show that $\text{proj}_y(Q) \subseteq P_1 \cup P_2$.
If $x^1 = 1$, then $w^2 = x^2 = 0$ leaving

$$y = w^1, \quad A^1 w^1 \leq b^1, \quad 0 \leq w^1 \leq u$$

and thus $y \in P_1$. Similarly, if $x^2 = 1$, it follows that $y \in P_2$.
Conversely, if $y \in P_1 \cup P_2$, suppose wlog that $y \in P_1$. Then $(y, w^1, w^2, x^1, x^2) \in Q$ with $w^1 = y, w^2 = 0, x^1 = 1, x^2 = 0$.

6. Prove that the set of feasible solutions to the formulation of the traveling salesman problem in Section 1.3 is precisely the set of incidence vectors of tours.

Solution: The solutions of the set

$$\left\{ x \in \mathbb{Z}_+^{n(n-1)} : \sum_{j:j \neq i} x_{ij} = 1 \; i \in [1, n], \sum_{i:i \neq j} x_{ij} = 1 \; j \in [1, n] \right\}$$

are assignments, namely a set of disjoint cycles, see Figure 1.2. The subtour elimination constraints eliminate any solution consisting of two or more cycles. Thus, the only solutions remaining are the tours.

7. The QED Company must draw up a production program for the next nine weeks. Jobs last several weeks and once started must be carried out without interruption. During each week, a certain number of skilled workers are required to work full-time on the job. Thus, if job i lasts p_i weeks, $l_{i,u}$ workers are required in week u for $u = 1, \ldots, p_i$. The total number of workers available in week t is L_t. Typical job data $(i, p_i, l_{i1}, \cdots, l_{ip_i})$ is shown below.

Job	Length	Week1	Week2	Week3	Week4
1	3	2	3	1	—
2	2	4	5	—	—
3	4	2	4	1	5
4	4	3	4	2	2
5	3	9	2	3	—

(i) Formulate the problem of finding a feasible schedule as an IP.
(ii) Formulate when the objective is to minimize the maximum number of workers used during any of the nine weeks.
(iii) Job 1 must start at least two weeks before job 3. Formulate.
(iv) Job 4 must start not later than one week after job 5. Formulate.
(v) Jobs 1 and 2 both need the same machine, and cannot be carried out simultaneously. Formulate.

Solution:

(i) Let $x_t^i = 1$ if job i starts in period t.
Each job i must start in some period and terminate before the end of the time horizon

$$\sum_{u=1}^{9-p_i+1} x_u^i = 1 \; i \in [1,5].$$

If job i starts in period u, the number of workers required by job i in period t is $\ell_{i,t-u+1}$ for $t \in [u, u+p_i-1]$. So the bound on the number of workers available in period t gives:

$$\sum_{i=1}^{5} \sum_{u=t-p_i+1}^{t} \ell_{i,t-u+1} x_u^i \leq L_t \; t \in [1,9].$$

The remaining constraints are $x_t^i \in \{0,1\}$ and $x_u^i = 0$ if $u > 9 - p_i + 1$.

(ii) Using (i), let $\eta_t = \sum_{i=1}^{5} \sum_{u=t-p_i+1}^{t} \ell_{i,t-u+1} x_u^i$ be the number of workers used in period t. Add the constraints $\eta_t \leq \eta$ and the objective function $\min \eta$.

(iii) If job 1 has not started in the first t periods, job 3 cannot start in the first $t+2$ periods.

$$\sum_{u=1}^{t} x_u^1 \geq \sum_{u=1}^{t+2} x_u^3 \; t \in [1,7].$$