

Chapter 1

Introduction

Exercise 1.1

Let $A \in \mathbb{S}^m$. Show that for arbitrary $M \in \mathbb{R}^{m \times n}$, $A \leq 0$ implies $M^T A M \leq 0$.

Solution I. Since $A \leq 0$, we have

$$y^T A y \leq 0, \forall y \in \mathbb{R}^m.$$

Therefore, for arbitrary $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{m \times n}$, there holds

$$x^T M^T A M x = (Mx)^T A Mx \leq 0.$$

This completes the proof.

Solution II. Since $A \leq 0$, there exists a matrix $T \in \mathbb{R}^{m \times m}$ such that

$$A = -T^T T.$$

Therefore, for arbitrary $M \in \mathbb{R}^{m \times n}$, there holds

$$M^T A M = -(TM)^T (TM) \leq 0.$$

This completes the proof.

Remark. On the other side, if for arbitrary $M \in \mathbb{R}^{m \times n}$, there holds $M^T A M \leq 0$, we can simply choose $M = I$, the identity, and obtain $A \leq 0$. Therefore, we actually have the conclusion that $A \leq 0$ if and only if $M^T A M \leq 0$ for arbitrary $M \in \mathbb{R}^{m \times n}$.

Exercise 1.2 (Duan and Patton (1998), Zhang and Yang (2003), page 175)

Let $A \in \mathbb{C}^{n \times n}$. Show that A is Hurwitz stable if $A + A^H < 0$.

Solution. First, we remark that, like the case for a real matrix, a complex square matrix is called Hurwitz stable if all its eigenvalues have negative real parts. Let λ be an eigenvalue of A , x be a corresponding eigenvector, then we have

$$Ax = \lambda x,$$

and further

$$x^H(A^H + A)x = (\lambda + \bar{\lambda})x^H x.$$

Thus $A^H + A < 0$ implies

$$\operatorname{Re} \lambda(A) = \frac{\lambda + \bar{\lambda}}{2} < 0.$$

This completes the proof.

Exercise 1.3 (Duan and Patton (1998))

Let $A \in \mathbb{R}^{n \times n}$. Show that A is Hurwitz stable if and only if

$$A = PQ, \tag{s1.1}$$

with $P > 0$ and Q being some matrix satisfying $Q + Q^T < 0$.

Solution. Suppose $A = PQ$ holds with $P > 0$ and Q satisfying

$$Q + Q^T < 0. \tag{s1.2}$$

Let

$$\hat{P} = P^{-1} > 0,$$

then it is easy to see

$$A^T \hat{P} + \hat{P} A = Q + Q^T < 0.$$

Therefore, the matrix A is Hurwitz stable.

Conversely, if A is Hurwitz stable, then there exists a matrix $\hat{P} > 0$, such that

$$A^T \hat{P} + \hat{P} A < 0. \tag{s1.3}$$

Let

$$Q = \hat{P} A,$$

we can easily get (s1.1), with $P = \hat{P}^{-1} > 0$, and the matrix Q obviously satisfies (s1.2) because of (s1.3).

Exercise 1.4

Give an example to show that certain set of nonlinear inequalities can be converted into LMIs.

Solution. Let $Q(x) = Q^T(x)$, $R(x) = R^T(x)$ and $S(x)$ depend affinely on x . It is clearly that

$$Q(x) - S(x)R(x)^{-1}S^T(x) > 0, \quad R(x) > 0,$$

is quadratic with respect to $S(x)$. Using Schur completion lemma the above two relations can be equivalently converted into

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0,$$

which is now linear in $S(x)$.

Exercise 1.5

Verify for which integer i the following inequality is true:

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} > \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Solution I. Let

$$\Theta = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & i-1 \\ i-1 & 1 \end{bmatrix}, \quad (\text{s1.4})$$

then we know

$$sI - \Theta = \begin{bmatrix} s-1 & -i+1 \\ -i+1 & s-1 \end{bmatrix},$$

$$\det(sI - \Theta) = (s-1)^2 - (i-1)^2 = (s-i)(s+i-2).$$

Thus

$$\lambda(\Theta) = \{i, 2-i\},$$

which indicates $\Theta > 0$ if and only if $i = 1$. Therefore the conclusion holds if and only if $i = 1$.

Solution II. It follows from (s1.4) and the Schur complement lemma that $\Theta > 0$ if and only if

$$1 - (i-1)^2 > 0,$$

which is equivalent to

$$(i-1)^2 < 1.$$

Obviously, this holds if and only if $i = 1$.

Exercise 1.6

Consider the combined constraints (in the unknown x) of the form

$$\begin{cases} F(x) < 0 \\ Ax = a \end{cases}, \quad (\text{s1.5})$$

where the affine function $F : \mathbb{R}^n \rightarrow \mathbb{S}^m$, matrix $A \in \mathbb{R}^{m \times n}$ and vector $a \in \mathbb{R}^m$ are given, and the equation $Ax = a$ has a solution. Show that (s1.5) can be converted into an LMI.

Solution. Suppose $\text{rank}A = r$, then it is well-known that all the solution vectors of the equation $Ax = a$ constitute a manifold, of dimension r , in \mathbb{R}^n , and a general form of all the solutions can be written as

$$x = x_0 + z_1 e_1 + z_2 e_2 + \cdots + z_r e_r,$$

where x_0 is a particular solution to the matrix equation $Ax = a$, while e_1, e_2, \dots, e_r are a set of linearly independent solutions to the homogeneous equation $Ax = 0$, and $z_i, i = 1, 2, \dots, r$, are a series of arbitrary scalars.

Let

$$\begin{aligned} x &= [x_1 \ x_2 \ \cdots \ x_n]^T, \\ x_0 &= [x_1^0 \ x_2^0 \ \cdots \ x_n^0]^T, \\ e_i &= [e_{1i} \ e_{2i} \ \cdots \ e_{ni}]^T, \ i = 1, 2, \dots, r, \end{aligned}$$

then the components of vector x can be written as

$$x_j = x_j^0 + z_1 e_{j1} + z_2 e_{j2} + \cdots + z_r e_{jr}, \quad j = 1, 2, \dots, n, \quad (\text{s1.6})$$

and the affine function F can be expressed as

$$F(x) = F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n. \quad (\text{s1.7})$$

Substituting (s1.6) into (s1.7), yields,

$$\begin{aligned} F(x) &= F_0 + \left(x_1^0 + z_1 e_{11} + z_2 e_{12} + \cdots + z_r e_{1r}\right) F_1 \\ &\quad + \left(x_2^0 + z_1 e_{21} + z_2 e_{22} + \cdots + z_r e_{2r}\right) F_2 \\ &\quad + \cdots + \left(x_n^0 + z_1 e_{n1} + z_2 e_{n2} + \cdots + z_r e_{nr}\right) F_n \\ &= F_0 + x_1^0 F_1 + \cdots + x_n^0 F_n \\ &\quad + z_1 (e_{11} F_1 + e_{21} F_2 + \cdots + e_{n1} F_n) \\ &\quad + \cdots + z_r (e_{1r} F_1 + e_{2r} F_2 + \cdots + e_{nr} F_n). \end{aligned}$$

Put

$$\begin{aligned} \tilde{F}_0 &= F_0 + x_1^0 F_1 + \cdots + x_n^0 F_n, \\ \tilde{F}_i &= e_{1i} F_1 + e_{2i} F_2 + \cdots + e_{ni} F_n, \quad i = 1, 2, \dots, r, \\ z &= [z_1 \ z_2 \ \cdots \ z_r]^T, \end{aligned}$$

we finally have

$$F(x) = \tilde{F}_0 + z_1 \tilde{F}_1 + \cdots + z_r \tilde{F}_r \triangleq \tilde{F}(z).$$

This implies that $x \in \mathbb{R}^n$ satisfies (1.22) if and only if $\tilde{F}(z) < 0, z \in \mathbb{R}^r$.

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Exercise 1.7

Write the Hermite matrix $A \in \mathbb{C}^{n \times n}$ as $X + iY$ with real X and Y . Show that $A < 0$ only if $X < 0$.

Solution. Considering the conjugate symmetry of matrix A , we know that

$$X^T = X, \quad Y^T = -Y.$$

Therefore, for arbitrary $z \in \mathbb{R}^n$, we have

$$\left(z^T Y z\right)^T = z^T Y^T z = -\left(z^T Y z\right),$$

which results in

$$z^T Y z = 0, \quad \forall z \in \mathbb{R}^n. \quad (\text{s1.8})$$

Using the above relation, we further have

$$\begin{aligned} z^T (X + iY) z &= z^T X z + i z^T Y z \\ &= z^T X z. \end{aligned} \quad (\text{s1.9})$$

When $A < 0$, we have

$$z^T (X + iY) z < 0, \quad \forall z \in \mathbb{R}^n, \quad z \neq 0$$

this, together with (s1.9), implies

$$z^T X z < 0, \quad \forall z \in \mathbb{R}^n, \quad z \neq 0.$$

This gives the negative definiteness of X .

Exercise 1.8

Let A, B be symmetric matrices of the same dimension. Show

1. $A > B$ implies $\lambda_{\max}(A) > \lambda_{\max}(B)$,
2. $\lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B)$.

Solution. Let $M \in \mathbb{S}^n$, $\lambda_{\max}(M)$ be the maximum eigenvalue of matrix M . We can easily show that

$$\lambda_{\max}(M)I \geq M.$$

Then, there holds

$$\lambda_{\max}(M)x^T x \geq x^T M x, \quad \forall x \in \mathbb{R}^n. \quad (\text{s1.10})$$

Proof of conclusion 1

Let x be the eigenvector of matrix B corresponding to the eigenvalue $\lambda_{\max}(B)$, then

$$Bx = \lambda_{\max}(B)x.$$

Considering $A > B$, we have

$$x^T(A - B)x > 0,$$

which means

$$x^T Ax > x^T Bx = \lambda_{\max}(B)x^T x. \quad (\text{s1.11})$$

On the other hand, using (s1.10) and (s1.11), gives

$$\lambda_{\max}(A)x^T x > \lambda_{\max}(B)x^T x,$$

which implies $\lambda_{\max}(A) > \lambda_{\max}(B)$, in view of $x^T x > 0$, $x \neq 0$.

Proof of conclusion 2

Let x be the eigenvector of matrix $A + B$ corresponding to the eigenvalue $\lambda_{\max}(A + B)$, then

$$\lambda_{\max}(A + B)x = (A + B)x,$$

from which we have

$$\lambda_{\max}(A + B)x^T x = x^T Ax + x^T Bx. \quad (\text{s1.12})$$

Using (s1.10) again, we obtain

$$x^T Ax \leq \lambda_{\max}(A)x^T x, \quad x^T Bx \leq \lambda_{\max}(B)x^T x. \quad (\text{s1.13})$$

Combining (s1.12) with (s1.13), yields

$$\lambda_{\max}(A + B)x^T x \leq (\lambda_{\max}(A) + \lambda_{\max}(B))x^T x,$$

which clear implies, in view of $x^T x > 0$, $x \neq 0$, the relation to be proven.