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# 1

## Solutions for Chapter 1

### Problem 1.1

In terms of the Cartesian components we have  $\mathbf{r} = (x, y, z)$  and  $\mathbf{p} = (p_x, p_y, p_z)$ . Using  $\mathbf{p} = -i\nabla$  and taking any differentiable function  $\psi(\mathbf{r})$  of position it follows that we have

$$\begin{aligned} [\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}] \psi(\mathbf{r}) &= -i \exp(-i\mathbf{k} \cdot \mathbf{r}) \nabla \psi(\mathbf{r}) + i \nabla \{ \exp(-i\mathbf{k} \cdot \mathbf{r}) \psi(\mathbf{r}) \} \\ &= -i \exp(-i\mathbf{k} \cdot \mathbf{r}) \nabla \psi(\mathbf{r}) + i \exp(-i\mathbf{k} \cdot \mathbf{r}) \nabla \psi(\mathbf{r}) \\ &\quad + i(-i\mathbf{k}) \exp(-i\mathbf{k} \cdot \mathbf{r}) \psi(\mathbf{r}) \\ &= \mathbf{k} \exp(-i\mathbf{k} \cdot \mathbf{r}) \psi(\mathbf{r}). \end{aligned}$$

Since this holds for any  $\psi(\mathbf{r})$  we obtain

$$[\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}] = \mathbf{k} \exp(-i\mathbf{k} \cdot \mathbf{r}).$$

Then we consider  $[\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}^2]$ , which gives

$$\begin{aligned} [\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}^2] &= \mathbf{p} \cdot [\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}] + [\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}] \cdot \mathbf{p} \\ &= \mathbf{k} \cdot \mathbf{p} \exp(-i\mathbf{k} \cdot \mathbf{r}) + \exp(-i\mathbf{k} \cdot \mathbf{r}) \mathbf{k} \cdot \mathbf{p}, \end{aligned}$$

where we made use of the result from the first part of the question in order to obtain the above line. The first term can be rewritten as

$$\mathbf{k} \cdot \mathbf{p} \exp(-i\mathbf{k} \cdot \mathbf{r}) = \exp(-i\mathbf{k} \cdot \mathbf{r}) \mathbf{k} \cdot \mathbf{p} - \exp(-i\mathbf{k} \cdot \mathbf{r}) \mathbf{k} \cdot \mathbf{k},$$

which leads to the final result that

$$[\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}^2] = \exp(-i\mathbf{k} \cdot \mathbf{r}) (2\mathbf{k} \cdot \mathbf{p} - k^2).$$

**Problem 1.2**

We start with the commutator between  $\mathbf{A}_n(\mathbf{r}, t)$  and  $\mathbf{E}_m(\mathbf{r}', t)$  by using the expressions in equations (1.15) and (1.16). The result is

$$\begin{aligned}
[\mathbf{A}_n(\mathbf{r}, t), \mathbf{E}_m(\mathbf{r}', t)] &= \sum_{l, l'} \sum_{\lambda, \lambda'=1,2} \frac{i}{2\varepsilon_0 V} \sqrt{\frac{\omega_{l'}}{\omega_l}} (\hat{e}_{l,\lambda})_n (\hat{e}_{l',\lambda'})_m \\
&\quad \times \left[ \left( a_l e^{i(\mathbf{k}_l \cdot \mathbf{r} - \omega_l t)} + a_l^\dagger e^{-i(\mathbf{k}_l \cdot \mathbf{r} - \omega_l t)} \right), \left( a_{l'} e^{i(\mathbf{k}_{l'} \cdot \mathbf{r}' - \omega_{l'} t)} - a_{l'}^\dagger e^{-i(\mathbf{k}_{l'} \cdot \mathbf{r}' - \omega_{l'} t)} \right) \right] \\
&= \sum_{l, l'} \sum_{\lambda, \lambda'=1,2} \frac{i}{2\varepsilon_0 V} \sqrt{\frac{\omega_{l'}}{\omega_l}} (\hat{e}_{l,\lambda})_n (\hat{e}_{l',\lambda'})_m \left\{ \left[ a_l e^{i(\mathbf{k}_l \cdot \mathbf{r} - \omega_l t)}, -a_{l'}^\dagger e^{-i(\mathbf{k}_{l'} \cdot \mathbf{r}' - \omega_{l'} t)} \right] \right. \\
&\quad \left. + \left[ a_l^\dagger e^{-i(\mathbf{k}_l \cdot \mathbf{r} - \omega_l t)}, a_{l'} e^{i(\mathbf{k}_{l'} \cdot \mathbf{r}' - \omega_{l'} t)} \right] \right\} \\
&= \sum_l \sum_{\lambda, \lambda'=1,2} \frac{-i}{2\varepsilon_0 V} (\hat{e}_{l,\lambda})_n (\hat{e}_{l,\lambda'})_m \left( \exp(i\mathbf{k}_l \cdot \mathbf{R}) + \exp(-i\mathbf{k}_l \cdot \mathbf{R}) \right),
\end{aligned}$$

where we have denoted  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ . Now, since the quantities  $\{\hat{e}_{l,1}, \hat{e}_{l,2}, \hat{\mathbf{k}}_l\}$  form an orthogonal basic set in the 3D space, it follows that we can write  $|\hat{e}_{l,1}\rangle\langle\hat{e}_{l,1}| + |\hat{e}_{l,2}\rangle\langle\hat{e}_{l,2}| + |\hat{\mathbf{k}}_l\rangle\langle\hat{\mathbf{k}}_l| = 1$ , implying

$$\sum_{\lambda=1,2} (\hat{e}_{l,\lambda})_n (\hat{e}_{l,\lambda})_m = \delta_{n,m} - \frac{(\mathbf{k}_l)_n (\mathbf{k}_l)_m}{k_l^2} \equiv \delta_{n,m} - (\hat{\mathbf{k}}_l)_n (\hat{\mathbf{k}}_l)_m.$$

After substituting the above result into the previous commutator expression we have

$$[\mathbf{A}_n(\mathbf{r}, t), \mathbf{E}_m(\mathbf{r}', t)] = \sum_l \frac{-i}{\varepsilon_0 V} \left( \delta_{n,m} - (\hat{\mathbf{k}}_l)_n (\hat{\mathbf{k}}_l)_m \right) \exp(i\mathbf{k}_l \cdot \mathbf{R}).$$

We note that the summation over  $l$  now covers both positive and negative values, since  $\hat{\mathbf{k}}_{-l} = -\hat{\mathbf{k}}_l$ . If we now let  $V$  become macroscopically large ( $V \rightarrow \infty$ ), this summation can be replaced by an integral over the 3D wave vector using

$$\frac{1}{V} \sum_l \cdots \rightarrow \left( \frac{1}{2\pi} \right)^3 \int \cdots d^3k,$$

and we arrive at the final result as

$$[\mathbf{A}_n(\mathbf{r}, t), \mathbf{E}_m(\mathbf{r}', t)] = \frac{-i}{\varepsilon_0} \delta_{n,m}^T(\mathbf{r} - \mathbf{r}').$$

Here we have introduced the transverse delta function  $\delta^T$ , which is defined generally as

$$\delta_{n,m}^T(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3} \int d^3k \left( \delta_{n,m} - (\hat{\mathbf{k}})_n (\hat{\mathbf{k}})_m \right) \exp(i\mathbf{k} \cdot \mathbf{R}).$$

By following the same procedure for the other commutators and using the appropriate terms in equations (1.15) and (1.16), we can find that

$$\begin{aligned} [\mathbf{A}_n(\mathbf{r}, t), \mathbf{A}_m(\mathbf{r}', t)] &= 0, & [\mathbf{H}_n(\mathbf{r}, t), \mathbf{H}_m(\mathbf{r}', t)] &= 0, \\ [\mathbf{E}_n(\mathbf{r}, t), \mathbf{E}_m(\mathbf{r}', t)] &= 0, & [\mathbf{H}_n(\mathbf{r}, t), \mathbf{A}_m(\mathbf{r}', t)] &= 0. \end{aligned}$$

### Problem 1.3

We start by considering the commutation relation  $[a, (a^\dagger)^n]$ , where  $n$  is a positive integer. This leads to

$$\begin{aligned} [a, (a^\dagger)^n] &= [a, a^\dagger (a^\dagger)^{(n-1)}] = [a, a^\dagger] a^{\dagger(n-1)} + a^\dagger [a, a^{\dagger(n-1)}] \\ &= (a^\dagger)^{(n-1)} + a^\dagger [a, a^\dagger (a^\dagger)^{(n-2)}] \\ &= (a^\dagger)^{(n-1)} + (a^\dagger)^{(n-1)} + (a^\dagger)^2 [a, (a^\dagger)^{(n-3)}] \\ &= (a^\dagger)^{(n-1)} + (a^\dagger)^{(n-1)} + \dots + (a^\dagger)^{(n-1)} [a, a^\dagger] \\ &= n(a^\dagger)^{(n-1)} = \frac{\partial (a^\dagger)^n}{\partial a^\dagger}. \end{aligned}$$

In a similar way it is found that

$$[a^\dagger, a^n] = -na^{(n-1)} = -\frac{\partial a^n}{\partial a}.$$

It follows, therefore, that for any function  $f(a, a^\dagger)$  expressible as a Taylor series expansion in positive integer powers of  $a$  and  $a^\dagger$  we obtain

$$\begin{aligned} [a, f(a, a^\dagger)] &= \frac{\partial f(a, a^\dagger)}{\partial a^\dagger}, \\ [a^\dagger, f(a, a^\dagger)] &= -\frac{\partial f(a, a^\dagger)}{\partial a}, \end{aligned}$$

as required.

For the last part of this problem we may start by considering

$$\begin{aligned} &\{\exp(\lambda A) B \exp(-\lambda A)\}^n \\ &= \exp(\lambda A) B \exp(-\lambda A) \exp(\lambda A) B \exp(-\lambda A) \dots \\ &= \exp(\lambda A) B^n \exp(-\lambda A) \end{aligned}$$

for any operators  $A$  and  $B$  and any constant  $\lambda$ . It then follows, for a more general case when we have any function  $f(B)$  which may be expanded as a power series in  $B$ , that we have

$$f\left(\exp(\lambda A) B \exp(-\lambda A)\right) = \exp(\lambda A) f(B) \exp(-\lambda A).$$

Therefore, taking  $\lambda A \equiv -\alpha a^\dagger a$  and  $f(B) \equiv f(a^\dagger, a)$  in the above expression, we may conclude that

$$\begin{aligned} & \exp(-\alpha a^\dagger a) f(a, a^\dagger) \exp(\alpha a^\dagger a) \\ &= f\left(\exp(-\alpha a^\dagger a) a \exp(\alpha a^\dagger a), \exp(-\alpha a^\dagger a) a^\dagger \exp(\alpha a^\dagger a)\right). \end{aligned}$$

The final step is to show that the functional dependences simplify because  $\exp(-\alpha a^\dagger a) a \exp(\alpha a^\dagger a) = a e^\alpha$  and  $\exp(-\alpha a^\dagger a) a^\dagger \exp(\alpha a^\dagger a) = a^\dagger e^{-\alpha}$ . These relationships can easily be proved from first principles by expanding the exponential operator in powers of the exponents and then simplifying using the boson commutation properties. A more elegant alternative is to use the Baker-Campbell-Hausdorff identity quoted in section 8.2 of this book.

Then, using the above results we obtain the required result that

$$\exp(-\alpha a^\dagger a) f(a, a^\dagger) \exp(\alpha a^\dagger a) = f(a e^\alpha, a^\dagger e^{-\alpha}).$$

#### Problem 1.4

(a) We start by defining a quantity  $f(\lambda)$ , where  $\lambda$  is a real variable, by

$$f(\lambda) = \exp(\lambda A) B \exp(-\lambda A).$$

We now differentiate with respect to  $\lambda$ , obtaining

$$\begin{aligned} \frac{d}{d\lambda} f(\lambda) &= \exp(\lambda A) (AB - BA) \exp(-\lambda A) \\ &= \exp(\lambda A) [A, B] \exp(-\lambda A). \end{aligned}$$

Since  $[A, B] = c$  (a scalar constant) we have

$$\frac{d}{d\lambda} f(\lambda) = c,$$

while  $d^2 f/d\lambda^2$  and all higher derivatives are zero. From the Taylor series expansion of  $f(\lambda)$  we have

$$f(\lambda) = f(0) + \lambda \frac{d}{d\lambda} f(0) + \frac{\lambda^2}{2!} \frac{d^2}{d\lambda^2} f(0) + \dots,$$

which simplifies to become

$$\exp(\lambda A) B \exp(-\lambda A) = B + \lambda [A, B].$$

(b) In this case we may define the shorthand

$$g(\lambda) = \exp(\lambda A) \exp(\lambda B).$$

Therefore, on differentiating with respect to  $\lambda$ , we find

$$\begin{aligned}\frac{d}{d\lambda}g(\lambda) &= A \exp(\lambda A) \exp(\lambda B) + \exp(\lambda A) B \exp(\lambda B) \\ &= A \exp(\lambda A) \exp(\lambda B) + (B + \lambda c) \exp(\lambda A) \exp(\lambda B) \\ &= (A + B + \lambda c) g(\lambda) \\ &= (A + B + \lambda [A, B]) g(\lambda),\end{aligned}$$

where in the last step we have used the result from part (a).

(c) Using the above definition of  $g(\lambda)$  we can re-derive the result in part (b) for  $dg(\lambda)/d\lambda$  in a different way as

$$\begin{aligned}\frac{d}{d\lambda}g(\lambda) &= \exp(\lambda A) A \exp(\lambda B) + \exp(\lambda A) \exp(\lambda B) B \\ &= \exp(\lambda A) \exp(\lambda B) [\exp(-\lambda B) A \exp(\lambda B) + B] \\ &= g(\lambda) (A + B + \lambda [A, B]).\end{aligned}$$

By comparison of this result with the result in part (b) we establish that  $g(\lambda)$  commutes with  $A + B + \lambda[A, B]$ , so we can integrate either differential equation to obtain

$$g(\lambda) = \exp\left((A + B)\lambda + \frac{\lambda^2}{2} [A, B]\right) = \exp \lambda(A + B) \exp \frac{\lambda^2}{2} [A, B],$$

which is the required result.

Finally, by substituting  $\lambda = 1$ ,  $A = \alpha a^\dagger$  and  $B = -\alpha^* a$  (which leads to the commutator  $[A, B] = -|\alpha|^2 [a^\dagger, a] = |\alpha|^2$ ) into the result of part (c), we obtain initially

$$e^{\alpha a^\dagger} e^{-\alpha^* a} = \exp(\alpha a^\dagger - \alpha^* a + |\alpha|^2/2).$$

This can be rearranged as

$$D(\alpha) \equiv \exp(\alpha a^\dagger - \alpha^* a) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a},$$

as required.

### Problem 1.5

From equation (1.39) the coherent state  $|\alpha\rangle$  can be written as

$$|\alpha\rangle = e^{-\alpha\alpha^*/2} e^{\alpha a^\dagger} |0\rangle,$$

and therefore we have

$$|\alpha\rangle \langle \alpha| = e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a}.$$

Now we may obtain the partial derivative with respect to  $\alpha$  as

$$\begin{aligned} \frac{\partial}{\partial \alpha} |\alpha\rangle \langle \alpha| &= -\alpha^* e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a} + e^{-\alpha\alpha^*} a^\dagger e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a} \\ &= (-\alpha^* + a^\dagger) e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a} \\ &= (-\alpha^* + a^\dagger) |\alpha\rangle \langle \alpha|. \end{aligned}$$

Combining these last two expressions, it follows that

$$\left( \frac{\partial}{\partial \alpha} + \alpha^* \right) |\alpha\rangle \langle \alpha| = a^\dagger |\alpha\rangle \langle \alpha|.$$

Similarly, on partially differentiating the expression for  $|\alpha\rangle \langle \alpha|$  with respect to  $\alpha^*$ , we obtain

$$\begin{aligned} \frac{\partial}{\partial \alpha^*} |\alpha\rangle \langle \alpha| &= -\alpha e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a} + e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle 0| a e^{\alpha^* a} \\ &= -\alpha e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a} + e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a} a \\ &= |\alpha\rangle \langle \alpha| (-\alpha + a). \end{aligned}$$

In the above we have used the property that the commutator  $[a, e^{\alpha^* a}] = 0$ , and therefore we have

$$\left( \frac{\partial}{\partial \alpha^*} + \alpha \right) |\alpha\rangle \langle \alpha| = |\alpha\rangle \langle \alpha| a.$$

### Problem 1.6

We take the volume to be a cube with sides of length  $L$  (and volume  $V = L^3$ ), where the sides are parallel to the  $x, y, z$  axes. Because of the periodic boundary conditions applied over a length  $L$  we need to have (e.g., in the  $x$  direction)

$$n_x \left( \frac{2\pi}{k_x} \right) = L, \quad \text{so} \quad k_x = \frac{2\pi}{L} n_x,$$

where  $n_x$  is an integer. There are similar results for the wavenumbers in the  $y$  and  $z$  directions. The required integral is

$$\begin{aligned} \frac{1}{V} \int d^3 r e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}} &= \frac{1}{L^3} \int \int \int dx dy dz e^{i(k_x - k'_x)x} e^{i(k_y - k'_y)y} e^{i(k_z - k'_z)z} \\ &= \frac{1}{L} \int_0^L dx e^{i(k_x - k'_x)x} \frac{1}{L} \int_0^L dy e^{i(k_y - k'_y)y} \frac{1}{L} \int_0^L dz e^{i(k_z - k'_z)z}. \end{aligned}$$