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2 Coordinating Coordinates

2.1 Starting with $\vec{r} = \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath}$:

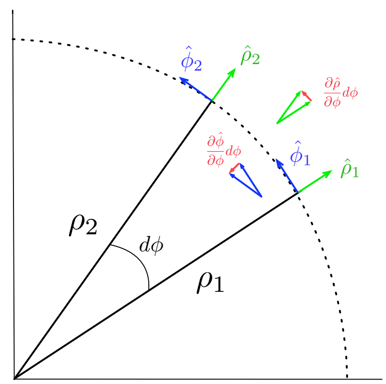
$$\hat{\rho} = \frac{\partial\vec{r}/\partial\rho}{|\partial\vec{r}/\partial\rho|} = \frac{\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\hat{\imath} + \sin\phi\hat{\jmath} \;\checkmark$$

and

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = \frac{-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath} \;\checkmark$$

Similar manipulations in spherical coordinates verify Eqn. (2.16).

2.2



2.3 Cylindrical: $d\vec{r} = \frac{\partial\vec{r}}{\partial\rho}d\rho + \frac{\partial\vec{r}}{\partial\phi}d\phi + \frac{\partial\vec{r}}{\partial z}dz$:

$$\begin{aligned}
\frac{\partial\vec{r}}{\partial\rho} &= \hat{\rho}\left|\frac{\partial\vec{r}}{\partial\rho}\right| = \hat{\rho}\,|\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}| = \hat{\rho} \\
\frac{\partial\vec{r}}{\partial\phi} &= \hat{\phi}\left|\frac{\partial\vec{r}}{\partial\phi}\right| = \hat{\phi}\,|-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}| = \hat{\phi}\rho \\
\frac{\partial\vec{r}}{\partial z} &= \hat{k}\left|\frac{\partial\vec{r}}{\partial z}\right| = \hat{k}
\end{aligned}$$

Spherical: $d\vec{r} = \frac{\partial\vec{r}}{\partial r}dr + \frac{\partial\vec{r}}{\partial\theta}d\theta + \frac{\partial\vec{r}}{\partial\phi}d\phi$:

$$\begin{aligned}
\frac{\partial\vec{r}}{\partial r} &= \hat{r}\left|\frac{\partial\vec{r}}{\partial r}\right| = \hat{r}\left|\sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k}\right| = \hat{r} \\
\frac{\partial\vec{r}}{\partial\theta} &= \hat{\theta}\left|\frac{\partial\vec{r}}{\partial\theta}\right| = \hat{\theta}\left|r\left(\cos\theta\cos\phi\hat{\imath} + \cos\theta\sin\phi\hat{\jmath} - \sin\theta\hat{k}\right)\right| = \hat{\theta}\,r \\
\frac{\partial\vec{r}}{\partial\phi} &= \hat{\phi}\left|\frac{\partial\vec{r}}{\partial\phi}\right| = \hat{\phi}\,|r\,(-\sin\theta\sin\phi\hat{\imath} + \sin\theta\cos\phi\hat{\jmath})| = \hat{\phi}\,r\sin\theta
\end{aligned}$$

2.4 From Eqn. (2.18), the matrix mapping $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ to $\{\hat{\rho}, \hat{\phi}, \hat{k}\}$ is $M = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Similarly, Eqn. (2.19) gives the matrix N mapping $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ to $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$. So the matrix mapping spherical coordinates into cylindrical coordinates is MN^{-1}. Since these are rotations, we can save a lot of work invoking $N^{-1} = N^T$. Then that the mapping from $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$ to $\{\hat{\rho}, \hat{\phi}, \hat{k}\}$ multiplies out to be

$$MN^T = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} .$$

The inverse transformation is NM^T — which is just the transpose of MN^T.

2.5 Writing out the matrix equation

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{pmatrix} ,$$

it's straightforward to verify that $\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\phi} = 1$ and that $\hat{r}\times\hat{\theta} = \hat{\phi}$, etc.

2.6 (a) cartesian: $x^2 + y^2 + z^2 = 1$
cylindrical: $\rho^2 + z^2 = 1$
spherical: $r = 1$

(b) cartesian: $x^2 + y^2 = 1$
cylindrical: $\rho = 1$
spherical: $r\sin\theta = 1$

2.7 The direction cosines are the cartesian components of a unit vector from the origin making angles α, β, γ with the axes. Thus, given two different unit vectors $\vec{A}$ and $\vec{B}$, we see $\vec{A}\cdot\vec{B} = \cos\theta$ gives the identity in (b); part (a) is just a special case of this result.

2.8 Decomposing the vectors into cartesian components, but using spherical coordinates,

$$\vec{r} = r\sin\theta\cos\phi\hat{\imath} + r\sin\theta\sin\phi\hat{\jmath} + r\cos\theta\hat{k}$$
$$\vec{r}\,' = r'\sin\theta'\cos\phi'\hat{\imath} + r'\sin\theta'\sin\phi'\hat{\jmath} + r'\cos\theta'\hat{k}$$

Then

$$\begin{aligned} \vec{r}\cdot\vec{r}\,' &\equiv rr'\cos\gamma \\ &= rr'\sin\theta\sin\theta'\cos\phi\cos\phi' + rr'\sin\theta\sin\theta'\sin\phi\sin\phi' + rr'\cos\theta\cos\theta' . \end{aligned}$$

Solving:

$$\begin{aligned} \cos\gamma = &= \sin\theta\sin\theta'\left(\cos\phi\cos\phi' + \sin\phi\sin\phi'\right) + \cos\theta\cos\theta' \\ &= \sin\theta\sin\theta'\cos(\phi - \phi') + \cos\theta\cos\theta' . \end{aligned}$$

2.9 Executing the steps outlined in Example 2.2:

$$\begin{aligned}\vec{a} &= \frac{d}{dt}\left(\dot\rho\hat\rho + \rho\dot\phi\hat\phi\right)\\ &= \ddot\rho\hat\rho + \dot\rho\dot{\hat\rho} + \dot\rho\dot\phi\hat\phi + \rho\ddot\phi\hat\phi + \rho\dot\phi\dot{\hat\phi}\\ &= \ddot\rho\hat\rho + \dot\rho\dot\phi\hat\phi + \dot\rho\dot\phi\hat\phi + \rho\ddot\phi\hat\phi - \rho\dot\phi\dot\phi\hat\rho\\ &= \ddot\rho\hat\rho + 2\dot\rho\dot\phi\hat\phi + \rho\ddot\phi\hat\phi - \rho\dot\phi\dot\phi\hat\rho\\ &= \left(\ddot\rho - \rho\dot\phi^2\right)\hat\rho + \left(\rho\ddot\phi + 2\dot\rho\dot\phi\right)\hat\phi\\ &= \left(\ddot\rho - \rho\omega^2\right)\hat\rho + \left(\rho\alpha + 2\dot\rho\omega\right)\hat\phi\ .\end{aligned}$$

2.10 First, $|J|^2 = |J|\,|J| = |J^T|\,|J| = |J^T J|$. Then

$$\begin{aligned}|J|^2 &= \left|\begin{pmatrix}\partial x/\partial u & \partial y/\partial u\\ \partial x/\partial v & \partial y/\partial v\end{pmatrix}\begin{pmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{pmatrix}\right|\\ &= \begin{vmatrix}(\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2 & \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\\ \frac{\partial x}{\partial v}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v}\frac{\partial y}{\partial u} & (\frac{\partial x}{\partial v})^2 + (\frac{\partial y}{\partial v})^2\end{vmatrix}\end{aligned}$$

Before trying to calculate this horrific determinant, note that since $\hat u \sim \partial\vec r/\partial u$ and $\hat v \sim \partial\vec r/\partial v$, the off-diagonal terms are just $\hat u\cdot\hat v$ — which vanishes for an orthogonal system. Moreover, the diagonal terms are just the scale factors h_u^2 and h_v^2. Thus

$$|J|^2 = \begin{vmatrix}h_u^2 & 0\\ 0 & h_v^2\end{vmatrix} = h_u^2 h_v^2\ . \quad \checkmark$$

2.11 Since $\hat\theta$ and $\hat\phi$ span the tangent plane to the sphere, then using little more than $\hat\imath\times\hat\jmath = \hat k$ does the trick:

$$\begin{aligned}\hat n \equiv \hat\theta\times\hat\phi &= \left(\hat\imath\cos\theta\cos\phi + \hat\jmath\cos\theta\sin\phi - \hat k\sin\theta\right)\times\left(-\hat\imath\sin\phi + \hat\jmath\cos\phi\right)\\ &= \hat k\cos\theta\cos^2\phi + \hat k\cos\theta\sin^2\phi + \hat\jmath\sin\theta\sin\phi + \hat\imath\sin\theta\cos\phi\\ &= \hat\imath\sin\theta\cos\phi + \hat\jmath\sin\theta\sin\phi + \hat k\cos\theta \equiv \hat r \quad \checkmark\end{aligned}$$

2.12 Area elements

(a) In cylindrical coordinates, the scale factors are $h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$. Then

i. on surface of constant ρ, $d\vec a = \hat\rho\,\rho\,d\phi\,dz$
ii. on surface of constant ϕ, $d\vec a = \hat\phi\,d\rho\,dz$
iii. on surface of constant z, $d\vec a = \hat k\,\rho\,d\rho\,d\phi$

(b) In spherical coordinates, the scale factors are $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$. Then

i. on surface of constant r, $d\vec a = \hat r\,r^2\sin\theta\,d\theta\,d\phi = \hat r\,r^2 d\Omega$
ii. on surface of constant θ, $d\vec a = \hat\theta\,r\sin\theta\,dr\,d\phi$
iii. on surface of constant ϕ, $d\vec a = \hat\phi\,r\,dr\,d\theta$

2.13 (a) Directly leveraging Eqn. (2.12) immediately yields

radial equation: $m\ddot r - mr\dot\phi^2 = -\frac{k}{r^2}$
angular equation: $r\ddot\phi + 2\dot r\dot\phi = 0$

(b) Simplifying the angular equation as $\frac{1}{r}\frac{d}{dt}\left(r^2\dot\phi\right) = \frac{1}{mr}\frac{d}{dt}\left(mr^2\dot\phi\right) = 0$ reveals angular momentum conservation: $\ell \equiv mr^2\dot\phi = \text{constant}$

(c) Using Eqn. (2.9) to express the kinetic energy in polar coordinates gives

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)\cdot\left(\frac{d\vec{r}}{dt}\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \ .$$

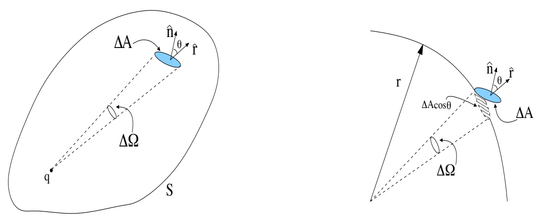
Using angular momentum conservation in part (b), the kinetic energy is

$$KE = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} \ .$$

For a circular orbit $\dot{r} = 0$, so the entire kinetic energy is due to angular motion.

2.14 (a) A point charge q has electric field $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$, so that

$$d\Phi = \vec{E}\cdot\hat{n}\,dA = \frac{q}{4\pi\epsilon_0 r^2}\ \hat{r}\cdot\hat{n}\ dA = \frac{q}{4\pi\epsilon_0 r^2}\ \cos\theta\ dA$$



(b)

(c) The component of the vector-valued area $d\vec{A} \equiv dA\ \hat{n}$ perpendicular to the radial line is $dA\cos\theta$, and subtends the same solid angle as dA; using the definition of solid angle (which requires r perpendicular to the subtended area),

$$d\Omega = \frac{dA\cos\theta}{r^2}$$

we have

$$d\Phi = \frac{q}{4\pi\epsilon_0}\ d\Omega \ .$$

Since the total solid angle is 4π, the net flux is

$$\Phi = \oint_S \vec{E}\cdot\hat{n}\ dA = \frac{q}{4\pi\epsilon_0}\oint_S d\Omega = \frac{q}{\epsilon_0},$$

where no explicit integration was necessary!

2.15 Starting with

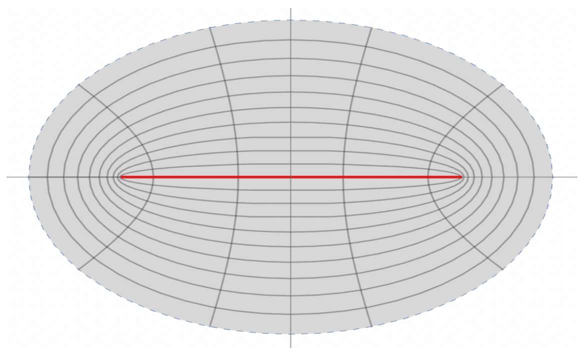
$$x = r\sin\theta\cos\phi \ , \qquad y = r\sin\theta\sin\phi \ , \qquad z = r\cos\theta$$

we get

$$\begin{aligned} |J| &\equiv \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix} \\ &= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = \ldots = r^2\sin\theta \ . \end{aligned}$$

2.16 (a) $\frac{x^2}{a^2\cosh^2 u}+\frac{y^2}{a^2\sinh^2 u} = \cos^2\phi+\sin^2\phi = 1$. For constant u, this describes an ellipse with semi-major and minor axes $a\cosh u$, $a\sinh u$ and foci at $c = \pm\sqrt{a^2\cosh^2 u - a^2\sinh^2 u} = \pm a$.

```
elliptic[u_, φ_] := {a Cosh[u]Cos[φ], a Sinh[u]Sin[φ]};
a = 1;
uPlots = ParametricPlot[elliptic[u, φ], {u, 0, 1}, {φ, 0, 2π},
        BoundaryStyle → Dashed, Mesh → 9, Frame → False,
        PlotStyle → Gray, Ticks → None, PlotRange → All];
zeroPlot = ParametricPlot[elliptic[0, φ], {φ, 0, 2π},
        PlotStyle → {Red, Thick}, Ticks → None];
Show[uPlots,zeroPlot]
```



(b) The easiest approach is via Eqn. (2.27b):

$$\begin{aligned}
h_u^2 &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 \\
&= (a\sinh u\cos\phi)^2 + (a\cosh u\sin\phi)^2 \\
&= a^2\left[\sinh^2 u\left(1-\sin^2\phi\right) + \left(1+\sinh^2 u\right)\sin^2\phi\right] = a^2\left(\sinh^2 u + \sin^2\phi\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
h_\phi^2 &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 \\
&= (-a\cosh u\sin\phi)^2 + (a\sinh u\cos\phi)^2 \\
&= a^2\left[\left(1+\sinh^2 u\right)\sin^2\phi + \sinh^2 u\left(1-\sin^2\phi\right)\right] = a^2\left(\sinh^2 u + \sin^2\phi\right)
\end{aligned}$$

2.17 (a) – $\frac{x^2+y^2}{a^2\cosh^2 u} + \frac{z^2}{a^2\sinh^2 u} = \sin^2\theta + \cos^2\theta = 1$. For constant u, this is the equation of an ellipsoidal surface. Note that the surface intersects the xy-plane in circles of radius $a\cosh u$, whereas ellipses in the xz- and yz-planes have semi-major axis $a\cosh u$, and semi-minor axis $a\sinh u$.

– $\frac{x^2+y^2}{a^2\sin^2\theta} - \frac{z^2}{a^2\cos^2\theta} = \cosh^2 u - \sinh^2 u = 1$. For constant θ, this is the equation of a hyperbolic surface. Once again, the curves in the xy-plane are circles (this time of radius $a\sin\theta$), but those in the xz- and yz-planes are hyperbolas.

(b) Starting with $x = a\cosh u\sin\theta\cos\phi$, $y = a\cosh u\sin\theta\sin\phi$, $z = a\sinh u\cos\theta$, the easiest approach is via Eqn. (2.27b):

$$\begin{aligned}
h_u^2 &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\
&= (a \sinh u \sin\theta \cos\phi)^2 + (a \sinh u \sin\theta \sin\phi)^2 + (a \cosh u \cos\theta)^2 \\
&= a^2 \left(\sinh^2 u \sin^2\theta + \cosh^2 u \cos^2\theta\right) \\
&= a^2 \left[\sinh^2 u \left(1 - \cos^2\theta\right) + \left(1 + \sinh^2 u\right)\cos^2\theta\right] = a^2 \left(\sinh^2 u + \cos^2\theta\right) \quad \checkmark
\end{aligned}$$

It's easy to see that h_θ gives the same. As for h_ϕ:

$$h_\phi^2 = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = (-a\cosh u \sin\theta \sin\phi)^2 + (a \cosh u \sin\theta\cos\phi)^2 = a^2\cosh^2 u \sin^2\theta \quad \checkmark$$

(c) Easy: $|J| = h_u h_\theta h_\phi = a^3 \cosh u \sin\theta \left(\sinh^2 u + \cos^2\theta\right)$. But if you prefer the long way:

$$\begin{aligned}
|J| &\equiv \begin{vmatrix} \partial x/\partial u & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial u & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial u & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix} \\
&= a^3 \begin{vmatrix} \sinh u \sin\theta\cos\phi & \cosh u \cos\theta\cos\phi & -\cosh u \sin\theta \sin\phi \\ \sinh u \sin\theta \sin\phi & \cosh u\cos\theta\sin\phi & \cosh u \sin\theta\cos\phi \\ \cosh u \cos\theta & -\sinh u \sin\theta & 0 \end{vmatrix} \\
&= \ldots = a^3 \cosh u \sin\theta \left(\sinh^2 u + \cos^2\theta\right) \ .
\end{aligned}$$

(d) Surfaces of constant u have

$$da = h_\theta h_\phi \, d\theta \, d\phi = a^2 \cosh u \sin\theta \sqrt{\sinh^2 u + \cos^2\theta}\, d\theta\, d\phi \ .$$

Since $h_u = h_\theta$, surfaces of constant θ have the almost-identical

$$da = h_u h_\phi \, du \, d\phi = a^2 \cosh u \sin\theta \sqrt{\sinh^2 u + \cos^2\theta}\, du\, d\phi \ .$$

2.18 Using the h_i for Problem 2.17 in Eqns. (2.27), and with

$$\vec{r} = a\cosh u \sin\theta\cos\phi\,\hat\imath + a\cosh u\sin\theta\sin\phi\,\hat\jmath + a\sinh u\cos\theta\,\hat{k} \ ,$$

we find

$$\hat{e}_u = \frac{1}{h_u}\frac{\partial\vec{r}}{\partial u} = \frac{1}{\sqrt{\sinh^2 u + \cos^2\theta}}\left(\sinh u\sin\theta\cos\phi\,\hat\imath + \sinh u \sin\theta\sin\phi\,\hat\jmath + a\cosh u\cos\theta\,\hat{k}\right)$$

$$\hat{e}_\theta = \frac{1}{h_\theta}\frac{\partial\vec{r}}{\partial \theta} = \frac{1}{\sqrt{\sinh^2 u + \cos^2\theta}}\left(\cosh u\cos\theta\cos\phi\,\hat\imath + \cosh u \cos\theta\sin\phi\,\hat\jmath - \sinh u\sin\theta\,\hat{k}\right)$$

and

$$\hat{e}_\phi = \frac{1}{h_\phi}\frac{\partial\vec{r}}{\partial \phi} = \frac{1}{\cosh u \sin\theta}\left(-\cosh u\sin\theta\sin\phi\,\hat\imath + \cosh u \sin\theta\cos\phi\,\hat\jmath\right) \ .$$

Multiple use of the identities $\cos^2 + \sin^2 = 1$ and $\cosh^2 - \sinh^2 = 1$ readily demonstrates that these vectors form an orthonormal set.

2.19 From Eqns. (2.27), we have $h_i\hat{e}_i = \partial\vec{r}/du_i$:

$$\begin{aligned}
h_r\hat{r} &= (\sin\psi\sin\theta\cos\phi, \sin\psi\sin\theta\sin\phi, \sin\psi\cos\theta, \cos\psi) \\
h_\psi\hat{\psi} &= r\,(\cos\psi\sin\theta\cos\phi, \cos\psi\sin\theta\sin\phi, \cos\psi\cos\theta, -\sin\psi) \\
h_\theta\hat{\theta} &= r\,(\sin\psi\cos\theta\cos\phi, \sin\psi\cos\theta\sin\phi, -\sin\psi\sin\theta, 0) \\
h_\phi\hat{\phi} &= r\,(-\sin\psi\sin\theta\sin\phi, \sin\psi\sin\theta\cos\phi, 0, 0)
\end{aligned}$$

Each is indeed orthogonal to the other three, and the magnitude of each gives the scale factor:

$$h_r = 1 \quad h_\psi = r \quad h_\theta = r\sin\psi \quad h_\phi = r\sin\psi\sin\theta \ ,$$

which together produce the required line element in $\mathbb{R}^4$. Finally, the product of the four gives the Jacobian, $|J| = r^3\sin^2\psi\sin\theta$.

2.20 For hyperspherical coordinates

$$\begin{aligned} x_4 &= r\cos\psi \\ x_3 &= r\sin\psi\cos\theta \\ x_2 &= r\sin\psi\sin\theta\cos\phi \\ x_1 &= r\sin\psi\sin\theta\sin\phi \ , \end{aligned}$$

the Jacobian matrix is (in reverse order, as suggested)

$$\begin{aligned} |J| &= \left|\frac{\partial(x_4,x_3,x_2,x_1)}{\partial(r,\psi,\theta,\phi)}\right| \\ &= \begin{vmatrix} \cos\psi & -r\sin\psi & 0 & 0 \\ \sin\psi\cos\theta & r\cos\psi\cos\theta & -r\sin\psi\sin\theta & 0 \\ \sin\psi\sin\theta\cos\phi & r\cos\psi\sin\theta\cos\phi & r\sin\psi\cos\theta\cos\phi & -r\sin\psi\sin\theta\sin\phi \\ \sin\psi\sin\theta\sin\phi & r\cos\psi\sin\theta\sin\phi & r\sin\psi\cos\theta\sin\phi & r\sin\psi\sin\theta\cos\phi \end{vmatrix} \\ &= \cos\psi \begin{vmatrix} r\cos\psi\cos\theta & -r\sin\psi\sin\theta & 0 \\ r\cos\psi\sin\theta\cos\phi & r\sin\psi\cos\theta\cos\phi & -r\sin\psi\sin\theta\sin\phi \\ r\cos\psi\sin\theta\sin\phi & r\sin\psi\cos\theta\sin\phi & r\sin\psi\sin\theta\cos\phi \end{vmatrix} \\ &\quad + r\sin\psi \begin{vmatrix} \sin\psi\cos\theta & -r\sin\psi\sin\theta & 0 \\ \sin\psi\sin\theta\cos\phi & r\sin\psi\cos\theta\cos\phi & -r\sin\psi\sin\theta\sin\phi \\ \sin\psi\sin\theta\sin\phi & r\sin\psi\cos\theta\sin\phi & r\sin\psi\sin\theta\cos\phi \end{vmatrix} \ . \end{aligned}$$

Now each additive element of a determinant has exactly one contribution from each row and column. So a common multiplicative factor of a column or row can be moved outside the determinant. We can do just this with all the factors of $\sin\psi$ and $\cos\psi$, giving

$$|J| = \left(r\cos^2\psi\sin^2\psi + r\sin^4\psi\right) \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} \ .$$

Of course, we could also have taken out two more factors of r, but leaving them helps us recognize the remaining 3×3 determinant as the Jacobian $r^2\sin\theta$ of standard spherical coordinates in $\mathbb{R}^3$. Thus

$$|J| = (r\cos^2\psi\sin^2\psi + r\sin^4\psi)\, r^2\sin\theta = r^3\sin^2\psi\sin\theta \ .$$

3 Complex Numbers

3.1 (a) $|e^{-i\pi}|^2 = e^{i\pi}e^{-i\pi} = 1$ ✓
(b) $\pi^{ie} = e^{ie\ln\pi}$ ✓
(c) $(-\pi)^{ie} = (e^{i\pi}e^{\ln\pi})^{ie} = e^{-e\pi}e^{ie\ln\pi}$ ×
(d) $i^{e\pi} = \left(e^{i\pi/2}\right)^{e\pi} = e^{ie\pi^2/2}$ ✓
(e) $(-i)^{e\pi^i} = (e^{i3\pi/2})^{e^{(1+i\ln\pi)}} = e^{i3e\pi/2}e^{-3\pi\ln(\pi)/2}$ ×

3.2 In the cartesian representation $z = a + ib$,

$$|z|^2 = zz^* = (a+ib)(a-ib) = a^2+b^2$$
$$|z^2| = |(a+ib)^2| = |(a^2-b^2)+i(2ab)| = \sqrt{(a^2-b^2)^2+(2ab)^2} = \sqrt{(a^2+b^2)^2} = a^2+b^2$$

In the polar representation $z = re^{i\varphi}$ it's even easier:

$$|z|^2 = zz^* = re^{i\varphi}\,re^{-i\varphi} = r^2$$
$$|z^2| = |(re^{i\varphi})^2| = |r^2e^{2i\varphi}| = r^2$$

In general, $|z^n| = |z|^n = r^n$.

3.3 Starting with $z = \cos\theta + i\sin\theta$, we have

$$dz = (-\sin\theta + i\cos\theta)d\theta$$
$$= i(\cos\theta + i\sin\theta)d\theta = izd\theta\ .$$

Thus

$$\int\frac{dz}{z} = i\int d\theta \quad\Longrightarrow\quad z = e^{i\theta}\ ,$$

where the constant of integration is determined from $z(\theta=0)=1$.

3.4 Compute the square in two equivalent ways:

1) $|(a+ib)(c+id)|^2 = |(ac-bd)+i(bc+ad)|^2 = (ac-bd)^2+(bc+ad)^2 \equiv p^2+q^2$
2) $|(a+ib)(c+id)|^2 = (a^2+b^2)(c^2+d^2) \equiv MN$.

Thus $MN = p^2+q^2$. For example,

$$\left.\begin{array}{r}M = 13 = 2^2+3^2\\ N = 25 = 3^2+4^2\end{array}\right\}MN = 325 = (2\cdot3-3\cdot4)^2+(3\cdot3-2\cdot4)^2 = 6^2+17^2$$

3.5 (a) $z_1 = i+\sqrt{3} : |z_1| = \sqrt{1+3} = 2, \quad \arg(z_1) = \arctan\left(1/\sqrt{3}\right) = \frac{\pi}{6} \quad\Longrightarrow\quad z_1 = 2e^{i\pi/6}$
$z_2 = i-\sqrt{3} : |z_2| = 2, \quad \arg(z_1) = \arctan\left(-1/\sqrt{3}\right) = \frac{5\pi}{6} \quad\Longrightarrow\quad z_2 = 2e^{i5\pi/6} = 2e^{-i\pi/6}$
(b) $2i = 2e^{i\pi/2} \rightsquigarrow \sqrt{2i} = \pm\sqrt{2}e^{i\pi/4} = \pm\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \pm(1+i)$
$2+2i\sqrt{3} = 4e^{i\pi/3} \rightsquigarrow \sqrt{2+2i\sqrt{3}} = \pm2e^{i\pi/6} = \pm2\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) = \pm(\sqrt{3}+i)$