Chapter 1: Introduction

1.3.5 Mobile Robot Engineering

1.3.5.1 Mobility

Comment in one or two sentences for each subsystem on how the goal of mobility requires that a mobile robot have such a subsystem.

- position estimation
- perception
- control
- planning
- locomotion
- power/computing

1.3.5.1 Mobility: Solution

a) Closed loop directed motion requires position feedback. Open loop control not usually feasible.
b) Must perceive environment in order to compute terrainability or avoid obstacles, or to satisfy mission objectives (not mobility).
c) Must be able to execute motion commands on real hardware.
d) Must look ahead to predict consequences of decisions now. In this way, avoid getting trapped.
e) Must move.
f) Mobile robots must carry their own power and smarts wherever they go.

Chapter 2: Math Fundamentals

2.2.9 Matrices

2.2.9.1 Matrix Exponential

Show that:

$$\exp\left\{\begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}\right\} = \begin{bmatrix} e^6 & 0 \\ 0 & e^{-2} \end{bmatrix}$$

2.2.9.1 Solution

Using the formula for matrix exponential:

$$\exp\left\{\begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}\right\} = I + \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}^2 + \dots$$

$$\exp\left\{\begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}\right\} = \begin{bmatrix} 1 + 6 + \frac{1}{2}(6)^2 + \dots & 0 \\ 0 & 1 - 2 + \frac{1}{2}(-2)^2 + \dots \end{bmatrix} = \begin{bmatrix} e^6 & 0 \\ 0 & e^{-2} \end{bmatrix}$$

2.2.9.2 Jacobian Determinant

You probably learned in multivariable calculus that the ratio of volumes in a 2D linear mapping is given by the Jacobian determinant. Consider the matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and the vectors $d\underline{x} = \begin{bmatrix} dx & 0 \end{bmatrix}^T$ and $d\underline{y} = \begin{bmatrix} 0 & dy \end{bmatrix}^T$. Using the result for the area of a parallelogram defined by two vectors in Figure 2.8, show that the area formed by the vectors $d\underline{u} = Ad\underline{x}$ and $d\underline{v} = Ad\underline{y}$ is $det(A)dxdy$.

2.2.9.2 Solution

The output vectors are:

$$d\underline{u} = Ad\underline{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} dx \\ 0 \end{bmatrix} = \begin{bmatrix} adx \\ cdx \end{bmatrix} \qquad d\underline{v} = Ad\underline{y} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ dy \end{bmatrix} = \begin{bmatrix} bdy \\ ddy \end{bmatrix}$$

The area of the output region is:

$$Area = \underline{u}_x\underline{v}_y - \underline{u}_y\underline{v}_x = adxddy - cdxbdy = (ad - cb)dxdy$$

2.2.9.3 Fundamental Theorem and Projections

A matrix P is called a *projection matrix* if it is symmetric and $P^2 = P$ which is called the property of *idempotence*.

(i) What happens if you compute $\underline{p}_1 = P\underline{x}$ and then $\underline{p}_2 = P\underline{p}_1$? An important projection matrix can be derived from a general $n \times m$ matrix A (where $m < n$) as follows: $P_A = A(A^T A)^{-1} A^T$.

(ii) Show that P_A satisfies both requirements of a projection matrix.

(iii)Note that $\underline{p}_1 = P_A \underline{x}$ must reside in the column space of A. The orthogonal complement Q_A of P_A is defined as $Q_A = I - P_A$. Note that $Q_A P_A = P_A Q_A = 0$. In what subspace does $\underline{q}_1 = Q_A \underline{x}$ reside?

2.2.9.3 Solution

a) You get $\underline{p}_2 = \underline{p}_1$. The second projection does nothing.

b) The requirements are:

$$P_A^T = A(A^T A)^{-1} A^T = P_A$$

$$P_A P_A = A(A^T A)^{-1} A^T A (A^T A)^{-1} A^T = A(A^T A)^{-1} A^T = P_A$$

c) Q_A extracts the component of $\underline{x}$ which has no projection onto the rows of A. Hence, it is in the nullspace of A.

2.2.9.4 Derivative of the Inverse

Suppose a square matrix $A(t)$ depends on a scalar, say t. Differentiate $A^{-1}A$ and find an expression for $\dot{A}^{-1}$.

2.2.9.4 Solution

The time derivative of the matrix inverse is relatively easy to compute. Because:

$$\frac{d\{A^{-1}A\}}{dt} = \frac{d\{I\}}{dt} = 0$$

We have:

$$\dot{A}^{-1}A + A^{-1}\dot{A} = 0$$

Which gives the time derivative of the matrix inverse as:

$$\dot{A}^{-1} = -A^{-1}\dot{A}A^{-1}$$

2.3.5 Fundamentals of Rigid Transforms

2.3.5.1 Specific Homogeneous Transforms

What do the following transforms do? If it is not obvious, transform the corners of a square to find out.

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.3.5.1 Solution

From left to right: translation, scale, xy shear (preserves z), perspective projection to plane at z = d, orthographic projection to plane at z = 0.

2.3.5.2 Operators and Frames

2D Homogeneous transforms work just like 3D ones except that a rigid body in 2D has only three degrees of freedom – translation along x or y or rotation in the plane. Consider the transform:

$$T = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Recall that the unit vectors and origin of a frame can be represented in its own coordinates as an identity matrix. Let such a matrix represent frame a. Consider the T matrix to be an operator and operate on the unit vectors and origin of frame a expressed in its own coordinates to produce another frame, called b. Write explicit vectors down for the unit vectors and origin of the new frame b. Use a notation that records the coordinate system in which they are expressed.

(ii) Visualization of the New Frame. When a transform is interpreted as an operator, the output vector is expressed in the coordinates of the original frame. Get out some graph paper or draw a grid in your editor with at least 10×10 cells. Draw a set of axes to the bottom left of the paper called frame a. Draw the transformed frame, called b in the right place with respect to frame a based on the above result. Label the axes of both frames with x or y.

(iii) Homogeneous Transforms as Frames. Consider the coordinates of the unit vectors and origin of the transformed frame when expressed with respect to the original frame. Compare these coordinates to the columns of the homogeneous transform. How are they related? Explain why this means homogeneous transforms are also machines to convert coordinates of general points under the same relationship between the two frames involved. HINT: how is a general point related to unit vectors and origin of any frame.

2.3.5.2 Solution

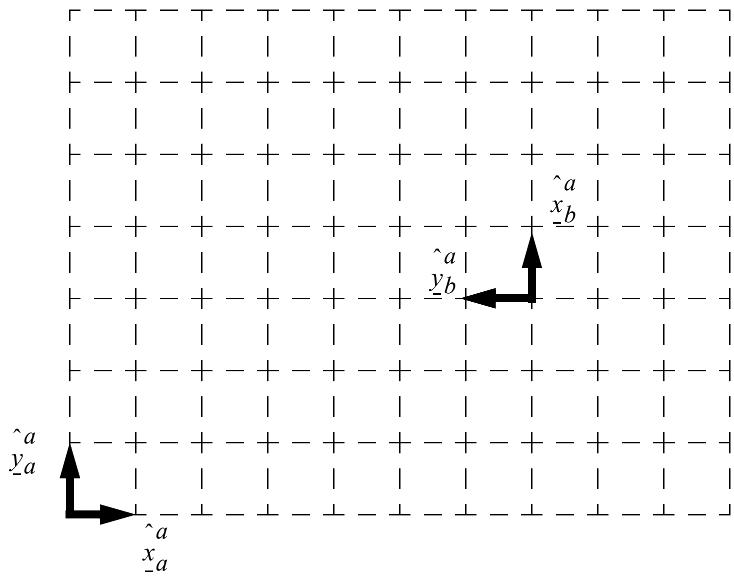
(i) The unit vectors and origin expressed in frame ‘b’ are:

$$\hat{\underline{x}}_b^a = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \hat{\underline{x}}_a^a = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\hat{\underline{y}}_b^a = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \hat{\underline{y}}_a^a = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{\underline{o}}_b^a = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \hat{\underline{o}}_a^a = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

(ii) Frames 'a' and 'b' can be visualized as follows:



(iii) The first column is the transformed x axis, second is the y, third is the origin. Any general point is a linear combination of the basis vectors. Since these basis vectors are transformed correctly, any point will be as well.

2.3.5.3 Pose of a Transform and Operating on a Point

(i) Solving for the Relative Pose. The parameters of the compound homogeneous transform that relates the frames in question 2.3.5.2 can be found using the techniques of inverse kinematics. Write an expression (in the form of a homogeneous transform with three degrees of freedom (or "parameters" in operator form) $\begin{bmatrix} a & b & \psi \end{bmatrix}^T$ for the general relationship between two rigid bodies in 2D, equate it to the above transform.

(ii) Solve the above expression by inspection for the "parameters."

2.3.5.3 Solution

a) $$\begin{bmatrix} c\theta & -s\theta & a \\ s\theta & c\theta & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $$a = 7$$ $$b = 3$$ $$\theta = \pi/2$$

2.3.5.4 Rigid Transforms

Operating on a general point is no different than operating on the origin.

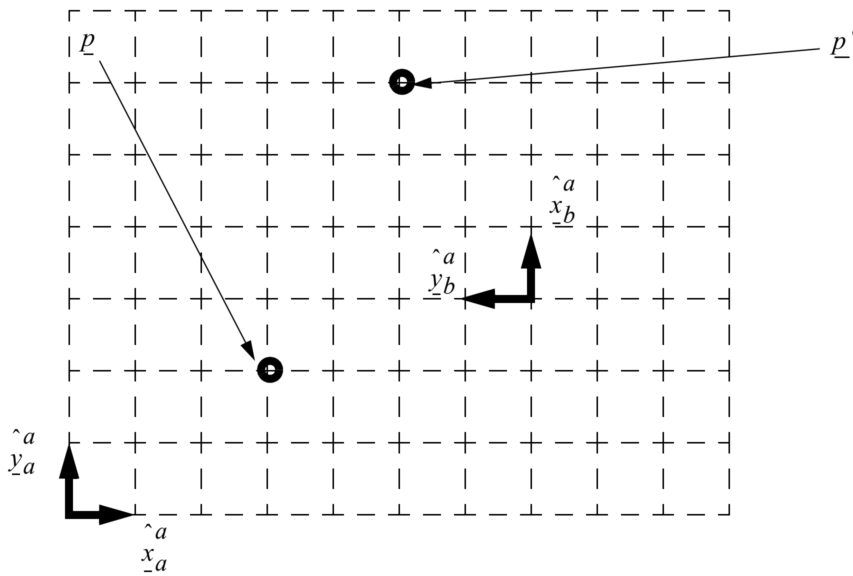
(i) Operate on the point $\underline{p} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^T$ with the transform T from Section 2.3.5.2 and write the coordinates $\underline{p}'$ of the new point. Copy your last figure and draw $\underline{p}$ and $\underline{p}'$ on it. Label each.

(ii) How do the coordinates of $\underline{p}'$ in the new frame (called b) compare to the coordinates of $\underline{p}$ in the old frame (called a)?

(iii) What property of any two points is preserved when they are operated upon by an orthogonal transform and what does this imply about any set of points?

2.3.5.4 Solution

a)
$$\underline{p}' = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \underline{p} = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$$



b) The coordinates of the transformed point in the transformed frame are identical to the coordinates of the original point in the original frame. In other words:

$$\underline{p}^a = \underline{p}'^b$$

c) Lengths are preserved. This shows that the transform moves all of space rigidly.

2.3.5.5 Homogeneous Transforms as Transforms

(i) Transforming a General Point. This exercise is worth extra attention. It illustrates the basic duality of operators and transforms upon which much depends. Copy the last figure including points p and p' on a fresh sheet. Draw the point $\underline{q}^b = \begin{bmatrix} 4 & 1 & 1 \end{bmatrix}^T$. The notation superscript b means the point has been specified with respect to frame b, so make sure to draw it in its correct position with respect to frame b.

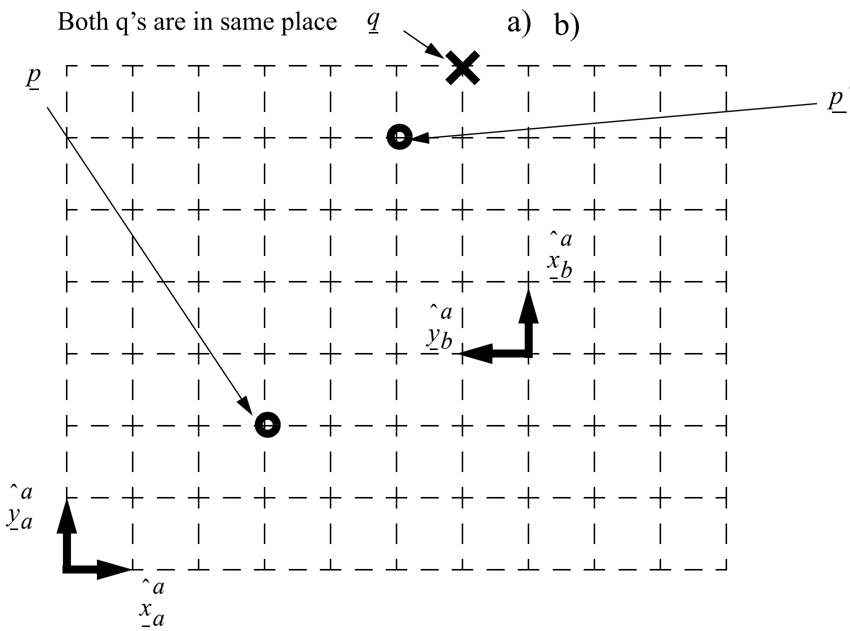
(ii) Write out the multiplication of this point by “the transform” and call the result $\underline{q}^a$. Using a different symbol than the one drawn for $\underline{q}^b$, draw $\underline{q}^a$ in its correct position with respect to frame a.

(iii)Earlier, when p was moved to p', p was expressed in frame a and so was p'. Here you expressed $\underline{q}^b$ in frame b to produce a result $\underline{q}^a$ expressed in frame a. Now the following is the key point. Discuss how interpreting the input differently (i.e in different coordinates) leads to a different interpretation of the ***function of the matrix***.

(iv)How can the function performed on the point $\underline{p}$ be reinterpreted as a different function applied, instead, to p'?

2.3.5.5 Solution

b) $$\underline{q}^a = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \underline{q}^b = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 1 \end{bmatrix}$$



iii) The key difference was that the input was specified wrt frame b, so the matrix is being used as a transform. In all earlier cases, the input was expressed with respect to frame a and the matrix was used as an operator.
iv) The previous operation on point $\underline{p}$ can be interpreted as a conversion of coordinates for p' from frame b to frame a.

2.4.5 Kinematics of Mechanisms

2.4.5.1 Three Link Planar Manipulator

Every roboticist should code manipulator kinematics at least once. Using your favorite programming environment, code the forward and inverse kinematics for the three link manipulator. Pick some random angles and draw a figure. Then compute the end effector pose from the angles and verify that the inverse solution regenerates the angle from the pose. What does the second solution look like? If you are ambitious, try a case near singularity, and experiment with the mechanism Jacobian.

2.4.5.2 Pantograph Leg Mechanism

The pantograph is a four bar linkage that can be used to multiply motion. This one was used on the Dante II robot that ascended and entered an active volcano on Mount Spur in Alaska in 1994. Triangles ABD, ACF, and DEF are all similar. When the actuator pushes