

CHAPTER 2

DTMCs: Transient Behavior

Modeling Exercises

2.1. The state space of  $\{X_n, n \geq 0\}$  is  $S = \{0, 1, 2, 3, \dots\}$ . Suppose  $X_n = i$ . Then the age of the lightbulb in place at time  $n$  is  $i$ . If this light bulb does not fail at time  $n + 1$ , then  $X_{n+1} = i + 1$ . If it fails at time  $n + 1$ , then a new lightbulb is put in at time  $n + 1$  with age 0, making  $X_{n+1} = 0$ . Let  $Z$  be the lifetime of a lightbulb. We have

$$\begin{aligned} P(X_{n+1} = 0 | X_n = i, X_{n-1}, \dots, X_0) &= P(\text{lightbulb of age } i \text{ fails at age } i + 1) \\ &= P(Z = i + 1 | Z > i) \\ &= \frac{p_{i+1}}{\sum_{j=i+1}^{\infty} p_j} \end{aligned}$$

Similarly

$$\begin{aligned} P(X_{n+1} = 0 | X_n = i, X_{n-1}, \dots, X_0) &= P(Z > i + 1 | Z > i) \\ &= \frac{\sum_{j=i+2}^{\infty} p_j}{\sum_{j=i+1}^{\infty} p_j} \end{aligned}$$

It follows that  $\{X_n, n \geq 0\}$  is a success-runs DTMC with

$$p_i = \frac{\sum_{j=i+2}^{\infty} p_j}{\sum_{j=i+1}^{\infty} p_j},$$

and

$$q_i = \frac{p_{i+1}}{\sum_{j=i+1}^{\infty} p_j},$$

for  $i \in S$ .

2.2 The state space of  $\{Y_n, n \geq 0\}$  is  $S = \{1, 2, 3, \dots\}$ . Suppose  $Y_n = i > 1$ , then the remaining life decreases by one at time  $n + 1$ . Thus  $X_{n+1} = i - 1$ . If  $Y_n = 1$ , a new light bulb is put in place at time  $n + 1$ , thus  $Y_{n+1}$  is the lifetime of the new light bulb. Let  $Z$  be the lifetime of a light bulb. We have

$$P(Y_{n+1} = i - 1 | X_n = i, X_{n-1}, \dots, X_0) = 1, \quad i \geq 2,$$

and

$$P(X_{n+1} = k | X_n = 1, X_{n-1}, \dots, X_0) = P(Z = k) = p_k, \quad k \geq 1.$$

2.3. Initially the urn has  $w + b$  balls. At each stage the number of balls in the urn increases by  $k - 1$ . Hence after  $n$  stages, the urn has  $w + b + n(k - 1)$  balls.  $X_n$  of them are black, and the remaining are white. Hence the probability of getting a black ball on the  $n + 1$ st draw is

$$\frac{X_n}{w + b + n(k - 1)}.$$

If the  $n + 1$ st draw is black,  $X_{n+1} = X_n + k - 1$ , and if it is white,  $X_{n+1} = X_n$ . Hence

$$P(X_{n+1} = i | X_n = i) = 1 - \frac{i}{w + b + n(k - 1)},$$

and

$$P(X_{n+1} = i + k - 1 | X_n = i) = \frac{i}{w + b + n(k - 1)}.$$

Thus  $\{X_n, n \geq 0\}$  is a DTMC, but it is not time homogeneous.

2.4.  $\{X_n, n \geq 0\}$  is a DTMC with state space  $\{0 = \text{dead}, 1 = \text{alive}\}$  because the movements of the cat and the mouse are independent of the past while the mouse is alive. Once the mouse is dead, it stays dead. If the mouse is still alive at time  $n$ , he dies at time  $n + 1$  if both the cat and mouse choose the same node to visit at time  $n + 1$ . There are  $N - 2$  ways for this to happen. In total there are  $(N - 1)^2$  possible ways for the cat and the mouse to choose the new nodes. Hence

$$P(X_{n+1} = 0 | X_n = 1) = \frac{N - 2}{(N - 1)^2}.$$

Hence the transition probability matrix is given by

$$P = \begin{bmatrix} 1 & 0 \\ \frac{N-2}{(N-1)^2} & 1 - \frac{N-2}{(N-1)^2} \end{bmatrix}.$$

2.5. Let  $X_n = 1$  if the weather is sunny on day  $n$ , and 2 if it is rainy on day  $n$ . Let  $Y_n = (X_{n-1}, X_n)$ , be the vector of weather on day  $n - 1$  and  $n, n \geq 1$ . Now suppose  $Y_n = (1, 1)$ . This means the weather was sunny on day  $n - 1$  and  $n$ . Then, it will be sunny on day  $n + 1$  with probability .8 and the new weather vector will be  $Y_{n+1} = (1, 1)$ . On the other hand it will rain on day  $n + 1$  with probability .2, and the weather vector will be  $Y_{n+1} = (1, 2)$ . These probabilities do not depend on the weather up to time  $n - 2$ , i.e., they are independent of  $Y_1, Y_2, \dots, Y_{n-2}$ . Similar analysis in other states of  $Y_n$  shows that  $\{Y_n, n \geq 1\}$  is a DTMC on state space  $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$  with the following transition probability matrix:

$$P = \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ .75 & .25 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}.$$

2.6. The state space is  $S = \{0, 1, \dots, K\}$ . Let

$$\alpha_i = \binom{K}{i} p^i (1-p)^{K-i}, \quad 0 \leq i \leq K.$$

Thus, when a functioning system fails,  $i$  components fail simultaneously with probability  $\alpha_i$ ,  $i \geq 1$ . The  $\{X_n, n \geq 0\}$  is a DTMC with transition probabilities:

$$\begin{aligned} p_{0,i} &= \alpha_i, & 0 \leq i \leq K, \\ p_{i,i-1} &= 1, & 1 \leq i \leq K. \end{aligned}$$

2.7. Suppose  $X_n = i$ . Then,  $X_{n+1} = i + 1$  if the first coin shows heads, while the second shows tails, which will happen with probability  $p_1(1-p_2)$ , independent of the past. Similarly,  $X_{n+1} = i - 1$  if the first coin shows tails and the second coin shows heads, which will happen with probability  $p_2(1-p_1)$ , independent of the past. If both coins show heads, or both show tails,  $X_{n+1} = i$ . Hence,  $\{X_n, n \geq 0\}$  is a space homogeneous random walk on  $S = \{\dots, -2, -1, 0, 1, 2, \dots\}$  (see Example 2.5) with

$$p_i = p_1(1-p_2), \quad q_i = p_2(1-p_1), \quad r_i = 1 - p_i - q_i.$$

2.8. We define  $X_n$ , the state of the weather system on the  $n$ th day, as the length of the current sunny or rainy spell. The state is  $k$ , ( $k = 1, 2, 3, \dots$ ), if the weather is sunny and this is the  $k$ th day of the current sunny spell. The state is  $-k$ , ( $k = 1, 2, 3, \dots$ ), if the weather is rainy and this is the  $k$ th day of the current rainy spell. Thus the state space is  $S = \{\pm 1, \pm 2, \pm 3, \dots\}$ .

Now suppose  $X_n = k$ , ( $k = 1, 2, 3, \dots$ ). If the sunny spell continues for one more day, then  $X_{n+1} = k + 1$ , or else a rainy spell starts, and  $X_{n+1} = -(k + 1)$ . Similarly, suppose  $X_n = -k$ . If the rainy spell continues for one more day, then  $X_{n+1} = -(k + 1)$ , or else a sunny spell starts, and  $X_{n+1} = 1$ . The Markov property follows from the fact that the lengths of the sunny and rainy spells are independent. Hence, for  $k = 1, 2, 3, \dots$ ,

$$\begin{aligned} \mathrm{P}(X_{n+1} = k + 1 | X_n = k) &= p_k, \\ \mathrm{P}(X_{n+1} = -1 | X_n = k) &= 1 - p_k, \\ \mathrm{P}(X_{n+1} = -(k + 1) | X_n = -k) &= q_k, \\ \mathrm{P}(X_{n+1} = 1 | X_n = -k) &= 1 - q_k. \end{aligned}$$

2.9.  $Y_n$  is the outcome of the  $n$ th toss of a six sided fair die.  $S_n = Y_1 + \dots + Y_n$ .  $X_n = S_n \pmod{7}$ . Hence we see that

$$X_{n+1} = X_n + Y_{n+1} \pmod{7}.$$

Since  $Y_n$ 's are iid, the above equation implies that  $\{X_n, n \geq 0\}$  is a DTMC with state space  $S = \{0, 1, 2, 3, 4, 5, 6\}$ . Now, for  $i, j \in S$ , we have

$$\begin{aligned} P(X_{n+1} = j | X_n = i) &= P(X_n + Y_{n+1} \pmod{7} = j | X_n = i) \\ &= P(i + Y_{n+1} \pmod{7} = j) \\ &= \begin{cases} 0 & \text{if } i = j \\ \frac{1}{6} & \text{if } i \neq j. \end{cases} \end{aligned}$$

Thus the transition probability matrix is given by

$$P = \begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \end{bmatrix}.$$

2.10. State space of  $\{X_n, n \geq 0\}$  is  $S = \{0, 1, \dots, r-1\}$ . We have

$$X_{n+1} = X_n + Y_{n+1} \pmod{r},$$

which shows that  $\{X_n, n \geq 0\}$  is a DTMC. We have

$$P(X_{n+1} = j | X_n = i) = P(Y_{n+1} = (j - i) \pmod{r}) = \sum_{m=0}^{\infty} \alpha_{j-i+mr}.$$

Here we assume that  $\alpha_k = 0$  for  $k \leq 0$ .

2.11. Let  $B_n$  ( $G_n$ ) be the bar the boy (girl) is in on the  $n$ th night. Then  $\{(B_n, G_n), n \geq 0\}$  is a DTMC on  $S = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$  with the following transition probability matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a(1-d) & ad & (1-a)(1-d) & (1-a)d \\ (1-b)c & (1-b)(1-c) & bc & b(1-c) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The story ends in bar  $k$  if the bivariate DTMC gets absorbed in state  $(k, k)$ , for  $k = 1, 2$ .

2.12. Let  $Q$  be the transition probability matrix of  $\{Y_n, n \geq 0\}$ . Suppose  $Z_m = f(i)$ , that the DTMC  $Y$  is in state  $i$  when the filled gas for the  $m$ th time. Then, the student fills gas next after  $11 - i$  days. The DTMC  $Y$  will be in state  $j$  at that time with probability  $[Q^{11-i}]_{ij}$ . This shows that  $\{Z_m, m \geq 0\}$  is a DTMC with state space  $\{f(0), f(1), \dots, f(10)\}$ , with transition probabilities

$$P(Z_{m+1} = f(j) | Z_m = f(i)) = [Q^{11-i}]_{ij}.$$

2.13. Following the analysis in Example 2.1b, we see that  $\{X_n, n \geq 0\}$  is a DTMC on state space  $S = \{1, 2, 3, \dots, k\}$  with the following transition probabilities:

$$P(X_{n+1} = i | X_n = i) = p_i, \quad 1 \leq i \leq k,$$

$$P(X_{n+1} = i + 1 | X_n = i) = 1 - p_i, \quad 1 \leq i \leq k - 1,$$

$$P(X_{n+1} = 1 | X_n = k) = 1 - p_k.$$

2.14. Let the state space be  $\{0, 1, 2, 12\}$ , where the state is 0 if both components are working, 1 if component 1 alone is down, 2 if component 2 alone is down, and 12 if components 1 and 2 are down. Let  $X_n$  be the state on day  $n$ .  $\{X_n, n \geq 0\}$  is a DTMC on  $\{0, 1, 2, 12\}$  with transition matrix

$$P = \begin{bmatrix} \alpha_0 & \alpha_1\alpha_2 & \alpha_{12} & 0 \\ r_1 & 1 - r_1 & 0 & 0 \\ r_2 & 0 & 1 - r_2 & 0 \\ 0 & 0 & r_1 & 1 - r_1 \end{bmatrix}.$$

Here we have assumed that if both components fail, we repair component 1 first, and then component 2.

2.15. Let  $X_n$  be the pair that played the  $n$ th game. Then  $X_0 = (1, 2)$ . Suppose  $X_n = (1, 2)$ . Then the  $n$ th game is played between player 1 and 2. With probability  $b_{12}$  player 1 wins the game, and the next game is played between players 1 and 3, thus making  $X_{n+1} = (1, 3)$ . On the other hand, player 2 wins the game with probability  $b_{21}$ , and the next game is played between players 2 and 3, thus making  $X_{n+1} = (2, 3)$ . Since the probabilities of winning are independent of the past, it is clear that  $\{X_n, n \geq 0\}$  is a DTMC on state space  $\{(1, 2), (2, 3), (1, 3)\}$ . Using the arguments as above, we see that the transition probabilities are given by

$$P = \begin{bmatrix} 0 & b_{21} & b_{12} \\ b_{23} & 0 & b_{32} \\ b_{13} & b_{31} & 0 \end{bmatrix}.$$

2.16. Let  $X_n$  be the number of beers at home when Mr. Al Anon goes to the store. Then  $\{(X_n, Y_n), n \geq 0\}$  is DTMC on state space

$$S = \{(0, L), (1, L), (2, L), (3, L), (4, L), (0, H), (1, H), (2, H), (3, H), (4, H)\}$$

with the following transition probability matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 \end{bmatrix}.$$

2.17. We see that

$$X_{n+1} = \max\{X_n, Y_{n+1}\}.$$

Since the  $Y_n$ 's are iid,  $\{X_n, n \geq 0\}$  is a DTMC. The state space is  $S = \{0, 1, \dots, M\}$ . Now, for  $0 \leq i < j \leq M$ ,

$$p_{i,j} = \mathbb{P}(\max\{X_n, Y_{n+1}\} = j | X_n = i) = \mathbb{P}(Y_n = j) = \alpha_j.$$

Also,

$$p_{i,i} = \mathbb{P}(\max\{X_n, Y_{n+1}\} = i | X_n = i) = \mathbb{P}(Y_n \leq i) = \sum_{k=0}^i \alpha_k.$$

2.18. Let  $Y_n = u$  if the machine is up at time  $n$  and  $d$  if it is down at time  $n$ . If  $Y_n = u$ , let  $X_n$  be the remaining up time at time  $n$ ; and if  $Y_n = d$ , let  $X_n$  be the remaining down time at time  $n$ . Then  $\{(X_n, Y_n), n \geq 0\}$  is a DTMC with state space

$$S = \{(i, j) : i \geq 1, j = u, d\}$$

and transition probabilities

$$p_{(i,j),(i-1,j)} = 1, \quad i \geq 2, j = u, d,$$

$$p_{(1,u),(i,d)} = d_i, \quad p_{(1,d),(i,u)} = u_i, \quad i \geq 1.$$

2.19. Let  $X_n$  be the number of messages in the inbox at 8:00am on day  $n$ . Ms. Friendly answers  $Z_n = \text{Bin}(X_n, p)$  emails on day  $n$ . Hence  $X_n - Z_n = \text{Bin}(X_n, 1-p)$  emails are left for the next day.  $Y_n$  is the number messages that arrive during 24 hours on day  $n$ . Hence at the beginning of the next day there  $X_{n+1} = Y_n + \text{Bin}(X_n, 1-p)$  in her mail box. Since  $\{Y_n, n \geq 0\}$  is iid,  $\{X_n, n \geq 0\}$  is a DTMC.

2.20. Let  $X_n$  be the number of bytes in this buffer in slot  $n$ , after the input during the slot and the removal (playing) of any bytes. We assume that the input during the slot occurs before the removal. Thus

$$X_{n+1} = \max\{\min\{X_n + A_{n+1}, B\} - 1, 0\}.$$

Thus if  $X_n = 0$  and there is no input,  $X_{n+1} = 0$ . Similarly, if  $X_n = B$ ,  $X_{n+1} = B - 1$ . The process  $\{X_n, n \geq 0\}$  is a random walk on  $\{0, \dots, B - 1\}$  with the following transition probabilities:

$$\begin{aligned} p_{0,0} &= \alpha_0 + \alpha_1, & p_{0,1} &= \alpha_2, \\ p_{i,i-1} &= \alpha_0, & p_{i,i} &= \alpha_1, & p_{i,i+1} &= \alpha_2, & 0 < i < B - 1, \\ p_{B-1,B-1} &= \alpha_1 + \alpha_2; & p_{B-1,B-2} &= \alpha_0. \end{aligned}$$

2.21. Let  $X_n$  be the number of passengers on the bus when it leaves the  $n$ th stop. Let  $D_{n+1}$  be the number of passengers that alight at the  $(n + 1)$ st stop. Since each person on board the bus gets off with probability  $p$  in an independent fashion,  $D_{n+1}$  is  $\text{Bin}(X_n, p)$  random variable. Also,  $X_n - D_{n+1}$  is a  $\text{Bin}(X_n, 1 - p)$  random variable.  $Y_{n+1}$  is the number of people that get on the bus at the  $(n + 1)$ st bus stop. Hence

$$X_{n+1} = \min\{X_n - D_{n+1} + Y_{n+1}, B\}.$$

Since  $\{Y_n, n \geq 0\}$  is a sequence of iid random variables, it follows from the above recursive relationship, that  $\{X_n, n \geq 0\}$  is a DTMC. The state space is  $\{0, 1, \dots, B\}$ . For  $0 \leq i \leq B$ , and  $0 \leq j < B$ , we have

$$\begin{aligned} p_{i,j} &= \mathbb{P}(X_{n+1} = j | X_n = i) \\ &= \mathbb{P}(X_n - D_{n+1} + Y_{n+1} = j | X_n = i) \\ &= \mathbb{P}(Y_{n+1} - \text{Bin}(i, p) = j - i) \\ &= \sum_{k=0}^i \mathbb{P}(Y_{n+1} - \text{Bin}(i, p) = j - i | \text{Bin}(i, p) = k) \mathbb{P}(\text{Bin}(i, p) = k) \\ &= \sum_{k=0}^i \mathbb{P}(Y_{n+1} = k + j - i | \text{Bin}(i, p) = k) \binom{i}{k} p^k (1 - p)^{i-k} \\ &= \sum_{k=0}^i \binom{i}{k} p^k (1 - p)^{i-k} \alpha_{k+j-i}, \end{aligned}$$

where we use the convention that  $\alpha_k = 0$  if  $k < 0$ . Finally,

$$p_{i,B} = 1 - \sum_{j=0}^{B-1} p_{i,j}.$$

2.22. The state space is  $\{-1, 0, 1, 2, \dots, k-1\}$ . The system is in state  $-1$  at time  $n$  if it is in economy mode after the  $n$ -th item is produced (and possibly inspected). It is in state  $i$  ( $1 \leq i \leq k$ ) if it is in 100% inspection mode and  $i$  consecutive non-defective items have been found so far. The transition probabilities are

$$\begin{aligned} p_{-1,0} &= p/r, & p_{-1,-1} &= 1 - p/r, \\ p_{i,i+1} &= 1 - p, & p_{i,0} &= p, & 0 \leq i \leq k - 2 \end{aligned}$$

$$p_{k-1,-1} = 1 - p, p_{k-1,0} = p.$$

2.23.  $X_n$  is the amount on hand at the beginning of the  $n$ th day, and  $D_n$  is the demand during the  $n$ th day. Hence the amount on hand at the end of the  $n$ th day is  $X_n - D_n$ . If this is  $s$  or more, no order is placed, and hence the amount on hand at the beginning of the  $(n+1)$ st day is  $X_n - D_n$ . On the other hand, if  $X_n - D_n < s$ , then the inventory is brought upto  $S$  at the beginning of the next day, thus making  $X_{n+1} = S$ . Thus

$$X_{n+1} = \begin{cases} X_n - D_n & \text{if } X_n - D_n \geq s, \\ S & \text{if } X_n - D_n < s. \end{cases}$$

Since  $\{D_n, n \geq 0\}$  are iid,  $\{X_n, n \geq 0\}$  is a DTMC on state space  $\{s, s+1, \dots, S-1, S\}$ . We compute the transition probabilities next. For  $s \leq j \leq i \leq S, j \neq S$ , we have

$$\begin{aligned} P(X_{n+1} = j | X_n = i) &= P(X_n - D_n = j | X_n = i) \\ &= P(D_n = i - j) = \alpha_{i-j}. \end{aligned}$$

and for  $s \leq i < S, j = S$  we have

$$\begin{aligned} P(X_{n+1} = S | X_n = i) &= P(X_n - D_n < s | X_n = i) \\ &= P(D_n > i - s) = \sum_{k=i-s}^{\infty} \alpha_k. \end{aligned}$$

Finally

$$\begin{aligned} P(X_{n+1} = S | X_n = S) &= P(X_n - D_n < s, \text{ or } X_n - D_n = S | X_n = S) \\ &= P(D_n > S - s) + P(D_n = 0) = \sum_{k=S-s+1}^{\infty} \alpha_k + \alpha_0. \end{aligned}$$

The transition probability matrix is given below:

$$P = \begin{bmatrix} \alpha_0 & 0 & 0 & \dots & 0 & b_0 \\ \alpha_1 & \alpha_0 & 0 & \dots & 0 & b_1 \\ \alpha_2 & \alpha_1 & \alpha_0 & \dots & 0 & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{S-s-1} & \alpha_{S-s-2} & \alpha_{S-s-3} & \dots & \alpha_0 & b_{S-s-1} \\ \alpha_{S-s} & \alpha_{S-s-1} & \alpha_{S-s-2} & \dots & \alpha_1 & \alpha_0 + b_S \end{bmatrix},$$

where

$$b_j = P(D_n > j) = \sum_{k=j+1}^{\infty} \alpha_k.$$

2.24. The state space of  $\{(X_n, Y_n), n \geq 0\}$  is

$$S = \{(i, j) : i \geq 0, j = 1, 2\}.$$

Let

$$\beta_k^i = \sum_{j=k}^{\infty} \alpha_j^i, \quad k \geq 1, i = 1, 2.$$

The transition probabilities are given by (see solution to Modeling Exercise 2.1)

$$\begin{aligned} p_{(i,1),(i+1,1)} &= \beta_{i+2}^1 / \beta_{i+1}^1, \quad i \geq 0, \\ p_{(i,2),(i+1,2)} &= \beta_{i+2}^2 / \beta_{i+1}^2, \quad i \geq 0, \\ p_{(i,1),(0,j)} &= v_j \alpha_{i+1}^1 / \beta_{i+1}^1, \quad i \geq 0, \\ p_{(i,2),(0,j)} &= v_j \alpha_{i+1}^2 / \beta_{i+1}^2, \quad i \geq 0. \end{aligned}$$

2.25.  $X_n$  is the number of bugs in the program just before running it for the  $n$ th time. Suppose  $X_n = k$ . Then no is discovered on the  $n$ th run with probability  $1 - \beta_k$ , and hence  $X_{n+1} = k$ . A bug will be discovered on the  $n$  run with probability  $\beta_k$ , in which case  $Y_n$  additional bugs are introduced, (with  $P(Y_n = i) = \alpha_i$ ,  $i = 0, 1, 2$ ) and  $X_{n+1} = k - 1 + Y_n$ . Hence, given  $X_n = k$ ,

$$X_{n+1} = \begin{cases} k - 1 & \text{with probability } \beta_k \alpha_0 = q_k \\ k & \text{with probability } \beta_k \alpha_1 + 1 - \beta_k = r_k \\ k + 1 & \text{with probability } \beta_k \alpha_2 = p_k \end{cases}$$

Thus  $\{X_n, n \geq 0\}$  is a DTMC with state space  $\{0, 1, 2, \dots\}$  with transition probability matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ q_1 & r_1 & p_1 & 0 & 0 & \dots \\ 0 & q_2 & r_2 & p_2 & 0 & \dots \\ 0 & 0 & q_3 & r_3 & p_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

2.26.  $X_n$  = number of active rumor mongers at time  $n$ .

$Y_n$  = number of individuals who have not heard the rumor up to and including time  $n$ .

$Z_n$  = number of individuals who have heard the rumor up to and including time  $n$ , but have stopped spreading it.

The rumor spreading process is modeled as a three dimensional process  $\{(X_n, Y_n, Z_n), n \geq 0\}$ . We shall show that it is a DTMC.

Since the total number of individuals in the colony is  $N$ , we must have

$$X_n + Y_n + Z_n = N, \quad n \geq 0.$$

Now let  $A_n$  be the number of individuals who hear the rumor for the first time at time  $n + 1$ . Now, an individual who has not heard the rumor by time  $n$  does not hear it by time  $n + 1$  if each the  $X_n$  rumor mongers at time  $n$  fails to contact him at time  $n + 1$ . The probability of that is  $((N - 2)/(N - 1))^{X_n}$ . Hence

$$A_n \sim \text{Bin}(Y_n, 1 - ((N - 2)/(N - 1))^{X_n}).$$

Similarly, let  $B_n$  be the number of active rumor-mongers at time  $n$  that become inactive at time  $n + 1$ . An active rumor monger becomes inactive if he contacts a person who has already heard the rumor. The probability of that is  $(X_n + Y_n - 1)/(N - 1)$ . Hence

$$B_n \sim \text{Bin}(X_n, (X_n + Y_n - 1)/(N - 1)).$$

Now, from the definitions of the various random variables involved,

$$X_{n+1} = X_n - B_n + A_n,$$

$$Y_{n+1} = Y_n - A_n,$$

$$Z_{n+1} = Z_n + B_n.$$

Thus  $\{(X_n, Y_n, Z_n), n \geq 0\}$  is a DTMC.

2.27.  $\{X_n, n \geq 0\}$  is a DTMC with state space  $S = \{rr, dr, dd\}$ , since gene type of the  $n+1$ st generation only depends on that of the parents in the  $n$ th generation. We are given that  $X_0 = rr$ . Hence, the parents of the first generation are  $rr, dd$ . Hence  $X_1$  is  $dr$  with probability 1. If  $X_n$  is  $dr$ , then the parents of the  $(n+1)$ st generation are  $dr$  and  $dd$ . Hence the  $(n+1)$ th generation is  $dr$  or  $dd$  with probability .5 each. Once the  $n$ th generation is  $dd$  it stays  $dd$  from then on. Hence transition probability matrix is given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & .5 & .5 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.28. Using the analysis in 2.27, we see that  $\{X_n, n \geq 0\}$  is a DTMC with state space  $S = \{rr, dr, dd\}$  with the following transition probability matrix:

$$P = \begin{bmatrix} .5 & .5 & 0 \\ .25 & .5 & .25 \\ 0 & .5 & .5 \end{bmatrix}.$$

2.29. Let  $X_n$  be the number of recipients in the  $n$ th generation. There are 20 recipients to begin with. Hence  $X_0 = 20$ . Let  $Y_{i,n}$  be the number of letters sent out by the  $i$ th recipient in the  $n$ th generation. The  $\{Y_{i,n} : n \geq 0, i = 1, 2, \dots, X_n\}$  are iid random variables with common pmf given below:

$$P(Y_{i,n} = 0) = 1 - \alpha; \quad P(Y_{i,n} = 20) = \alpha.$$

The number of recipients in the  $(n+1)$ st generation are given by

$$X_{n+1} = \sum_{i=1}^{X_n} Y_{i,n}.$$

Thus  $\{X_n, n \geq 0\}$  is a branching process, following the terminology of Section 2.2.

Note that we cannot start with  $X_0 = 1$  since we would need to use  $Y_{1,0} = 20$  with probability 1, which is different distribution from the other  $Y_{i,n}$ s. This would invalidate the assumptions of a branching process.

2.30. Let  $X_n$  be the number of backlogged packets at the beginning of the  $n$ th slot. Furthermore, let  $I_n$  be the collision indicator defined as follows:  $I_n = id$  if there are no transmissions in the  $(n - 1)$ st slot (idle slot),  $I_n = s$  if there is exactly 1 transmission in the  $(n - 1)$ st slot (successful slot), and  $I_n = e$  if there are 2 or more transmissions in the  $(n - 1)$ st slot (error or collision in the slot). We shall model the state of the system at the beginning of the  $n$ th slot by  $(X_n, I_n)$ . Now suppose  $X_n = i, I_n = s$ . Then, the backlogged packets retry with probability  $r$ . Hence, we get

$$\begin{aligned} P(X_{n+1} = i - 1, I_{n+1} = s | X_n = i, I_n = s) &= (1 - p)^{N-i} ir(1 - r)^{i-1}, \\ P(X_{n+1} = i, I_{n+1} = s | X_n = i, I_n = s) &= (N - i)p(1 - p)^{N-i-1}(1 - r)^i, \\ P(X_{n+1} = i, I_{n+1} = id | X_n = i, I_n = s) &= (1 - p)^{(N-i)}(1 - r)^i, \\ P(X_{n+1} = i, I_{n+1} = e | X_n = i, I_n = s) &= (1 - p)^{(N-i)}(1 - (1 - r)^i - ir(1 - r)^{i-1}). \\ P(X_{n+1} = i + 1, I_{n+1} = e | X_n = i, I_n = s) &= (N - i)p(1 - p)^{N-i-1}(1 - (1 - r)^i) \\ P(X_{n+1} = i + j, I_{n+1} = e | X_n = i, I_n = s) &= \binom{N-i}{j} p^j (1 - p)^{N-i-j}, \quad 2 \leq j \leq N - i. \end{aligned}$$

Next suppose  $X_n = i, I_n = id$ . Then, the backlogged packets retry with probability  $r'' > r$ . The above equations become:

$$\begin{aligned} P(X_{n+1} = i - 1, I_{n+1} = s | X_n = i, I_n = id) &= (1 - p)^{N-i} ir'(1 - r')^{i-1}, \\ P(X_{n+1} = i, I_{n+1} = s | X_n = i, I_n = id) &= (N - i)p(1 - p)^{N-i-1}(1 - r')^i, \\ P(X_{n+1} = i, I_{n+1} = id | X_n = i, I_n = id) &= (1 - p)^{(N-i)}(1 - r')^i, \\ P(X_{n+1} = i, I_{n+1} = e | X_n = i, I_n = id) &= (1 - p)^{(N-i)}(1 - (1 - r')^i - ir'(1 - r')^{i-1}). \\ P(X_{n+1} = i + 1, I_{n+1} = e | X_n = i, I_n = id) &= (N - i)p(1 - p)^{N-i-1}(1 - (1 - r')^i) \\ P(X_{n+1} = i + j, I_{n+1} = e | X_n = i, I_n = id) &= \binom{N-i}{j} p^j (1 - p)^{N-i-j}, \quad 2 \leq j \leq N - i. \end{aligned}$$

Finally, suppose  $X_n = i, I_n = e$ . Then, the backlogged packets retry with probability  $r'' < r$ . The above equations become:

$$\begin{aligned} P(X_{n+1} = i - 1, I_{n+1} = s | X_n = i, I_n = e) &= (1 - p)^{N-i} ir''(1 - r'')^{i-1}, \\ P(X_{n+1} = i, I_{n+1} = s | X_n = i, I_n = e) &= (N - i)p(1 - p)^{N-i-1}(1 - r'')^i, \\ P(X_{n+1} = i, I_{n+1} = id | X_n = i, I_n = e) &= (1 - p)^{(N-i)}(1 - r'')^i, \\ P(X_{n+1} = i, I_{n+1} = e | X_n = i, I_n = e) &= (1 - p)^{(N-i)}(1 - (1 - r'')^i - ir''(1 - r'')^{i-1}). \\ P(X_{n+1} = i + 1, I_{n+1} = e | X_n = i, I_n = e) &= (N - i)p(1 - p)^{N-i-1}(1 - (1 - r'')^i) \\ P(X_{n+1} = i + j, I_{n+1} = e | X_n = i, I_n = e) &= \binom{N-i}{j} p^j (1 - p)^{N-i-j}, \quad 2 \leq j \leq N - i. \end{aligned}$$

This shows that  $\{(X_n, I_n), n \geq 0\}$  is a DTMC with transition probabilities given

above.

2.31. Let  $X_n$  be the number of packets ready for transmission at time  $n$ . Let  $Y_n$  be the number of packets that arrive during time  $(n, n + 1]$ . If  $X_n = 0$ , no packets are transmitted during the  $n$ th slot and we have

$$X_{n+1} = Y_n.$$

If  $X_n > 0$ , exactly one packet is transmitted during the  $n$ th time slot. Hence,

$$X_{n+1} = X_n - 1 + Y_n.$$

Since  $\{Y_n, n \geq 0\}$  are iid, we see that  $\{X_n, n \geq 0\}$  is identical to the DTMC given in Example 2.16.

2.32. Let  $Y_{i,n}$ ,  $i = 1, 2$ , be the number of non-defective items in the inventory of the  $i$ th machine at time  $n$ , after all production and any assembly at time  $n$  is done. Since the assembly is instantaneous, both  $Y_{1,n}$  and  $Y_{2,n}$  cannot be positive simultaneously. Now define

$$X_n = B_2 + Y_{1,n} - Y_{2,n}.$$

The state space of  $\{X_n, n \geq 0\}$  is  $S = \{0, 1, 2, \dots, B_1 + B_2 - 1, M_1 + M_2\}$ . Now,

$$X_n = k > B_2 \Rightarrow Y_{1,n} = k - B_2, Y_{2,n} = 0,$$

$$X_n = k < B_2 \Rightarrow Y_{1,n} = 0, Y_{2,n} = B_2 - k,$$

$$X_n = k = B_2 \Rightarrow Y_{1,n} = 0, Y_{2,n} = 0.$$

Thus  $X_n$  contains complete information about  $Y_{1,n}$  and  $Y_{2,n}$ .  $\{X_n, n \geq 0\}$  is a random walk on  $S$  as in Example 2.5 with

$$p_{n,n+1} = p_n = \begin{cases} \alpha_1 & \text{if } n = 0, \\ \alpha_1(1 - \alpha_2) & \text{if } 0 < n < B_1 + B_2, \end{cases}$$

$$p_{n,n-1} = q_n = \begin{cases} \alpha_2 & \text{if } n = B_1 + B_2, \\ \alpha_2(1 - \alpha_1) & \text{if } 0 < n < B_1 + B_2, \end{cases}$$

$$p_{n,n} = r_n = \begin{cases} 1 - \alpha_1 & \text{if } n = 0, \\ \alpha_1\alpha_2 + (1 - \alpha_1)(1 - \alpha_2) & \text{if } 0 < n < B_1 + B_2, \\ 1 - \alpha_2 & \text{if } n = B_1 + B_2. \end{cases}$$

2.33. Let  $X_n$  be the age of the light bulb in place at time  $n$ . Using the solution to Modeling Exercise 2.1, we see that  $\{X_n, n \geq 0\}$  is a success-runs DTMC on  $\{0, 1, \dots, K - 1\}$  with

$$q_i = p_{i+1}/b_{i+1}, p_i = 1 - q_i, \quad 0 \leq i \leq K - 2, q_{K-1} = 1,$$

where  $b_i = P(Z_n \geq i) = \sum_{j=i}^{\infty} p_j$ .

2.34. The same three models of reader behavior in Section 2.3.7 work if we consider a citation from paper  $i$  to paper  $j$  as link from webpage  $i$  to web page  $j$ , and action of visiting a page is taken to the same as actually looking up a paper.