

Mathematical Preliminaries

Note: An asterisk (*) before an exercise indicates that there is a solution in the Student Study Guide.

Exercise Set 1.1, page 14

- *1. For each part, $f \in C[a, b]$ on the given interval. Since $f(a)$ and $f(b)$ are of opposite sign, the Intermediate Value Theorem implies that a number c exists with $f(c) = 0$.
2. (a) $[0, 1]$
(b) $[0, 1], [4, 5], [-1, 0]$
*(c) $[-2, -2/3], [0, 1], [2, 4]$
(d) $[-3, -2], [-1, -0.5],$ and $[-0.5, 0]$
3. For each part, $f \in C[a, b]$, f' exists on (a, b) and $f(a) = f(b) = 0$. Rolle's Theorem implies that a number c exists in (a, b) with $f'(c) = 0$. For part (d), we can use $[a, b] = [-1, 0]$ or $[a, b] = [0, 2]$.
4. The maximum value for $|f(x)|$ is given below.
*(a) $(2 \ln 2)/3 \approx 0.4620981$
(b) 0.8
(c) 5.164000
(d) 1.582572
- *5. For $x < 0$, $f(x) < 2x + k < 0$, provided that $x < -\frac{1}{2}k$. Similarly, for $x > 0$, $f(x) > 2x + k > 0$, provided that $x > -\frac{1}{2}k$. By Theorem 1.11, there exists a number c with $f(c) = 0$. If $f(c) = 0$ and $f(c') = 0$ for some $c' \neq c$, then by Theorem 1.7, there exists a number p between c and c' with $f'(p) = 0$. However, $f'(x) = 3x^2 + 2 > 0$ for all x .
6. Suppose p and q are in $[a, b]$ with $p \neq q$ and $f(p) = f(q) = 0$. By the Mean Value Theorem, there exists $\xi \in (a, b)$ with
$$f(p) - f(q) = f'(\xi)(p - q).$$
But, $f(p) - f(q) = 0$ and $p \neq q$. So $f'(\xi) = 0$, contradicting the hypothesis.
7. (a) $P_2(x) = 0$
(b) $R_2(0.5) = 0.125$; actual error = 0.125

(c) $P_2(x) = 1 + 3(x - 1) + 3(x - 1)^2$

(d) $R_2(0.5) = -0.125$; actual error = -0.125

8. $P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

x	0.5	0.75	1.25	1.5
$P_3(x)$	1.2265625	1.3310547	1.5517578	1.6796875
$\frac{P_3(x)}{\sqrt{x+1}}$	1.2247449	1.3228757	1.5	1.5811388
$ \sqrt{x+1} - P_3(x) $	0.0018176	0.0081790	0.0517578	0.0985487

*9. Since

$$P_2(x) = 1 + x \quad \text{and} \quad R_2(x) = \frac{-2e^\xi(\sin \xi + \cos \xi)}{6}x^3$$

for some ξ between x and 0, we have the following:

(a) $P_2(0.5) = 1.5$ and $|f(0.5) - P_2(0.5)| \leq 0.0932$;

(b) $|f(x) - P_2(x)| \leq 1.252$;

(c) $\int_0^1 f(x) dx \approx 1.5$;

(d) $|\int_0^1 f(x) dx - \int_0^1 P_2(x) dx| \leq \int_0^1 |R_2(x)| dx \leq 0.313$, and the actual error is 0.122.

10. $P_2(x) = 1.461930 + 0.617884(x - \frac{\pi}{6}) - 0.844046(x - \frac{\pi}{6})^2$ and $R_2(x) = -\frac{1}{3}e^\xi(\sin \xi + \cos \xi)(x - \frac{\pi}{6})^3$ for some ξ between x and $\frac{\pi}{6}$.

(a) $P_2(0.5) = 1.446879$ and $f(0.5) = 1.446889$. An error bound is 1.01×10^{-5} , and the actual error is 1.0×10^{-5} .

(b) $|f(x) - P_2(x)| \leq 0.135372$ on $[0, 1]$

(c) $\int_0^1 P_2(x) dx = 1.376542$ and $\int_0^1 f(x) dx = 1.378025$

(d) An error bound is 7.403×10^{-3} , and the actual error is 1.483×10^{-3} .

11. $P_3(x) = (x - 1)^2 - \frac{1}{2}(x - 1)^3$

(a) $P_3(0.5) = 0.312500$, $f(0.5) = 0.346574$. An error bound is $0.291\bar{6}$, and the actual error is 0.034074.

(b) $|f(x) - P_3(x)| \leq 0.291\bar{6}$ on $[0.5, 1.5]$

(c) $\int_{0.5}^{1.5} P_3(x) dx = 0.08\bar{3}$, $\int_{0.5}^{1.5} (x - 1) \ln x dx = 0.088020$

(d) An error bound is $0.058\bar{3}$, and the actual error is 4.687×10^{-3} .

12. (a) $P_3(x) = -4 + 6x - x^2 - 4x^3$; $P_3(0.4) = -2.016$

(b) $|R_3(0.4)| \leq 0.05849$; $|f(0.4) - P_3(0.4)| = 0.013365367$

(c) $P_4(x) = -4 + 6x - x^2 - 4x^3$; $P_4(0.4) = -2.016$

(d) $|R_4(0.4)| \leq 0.01366$; $|f(0.4) - P_4(0.4)| = 0.013365367$

13. $P_4(x) = x + x^3$

(a) $|f(x) - P_4(x)| \leq 0.012405$

(b) $\int_0^{0.4} P_4(x) dx = 0.0864$, $\int_0^{0.4} xe^{x^2} dx = 0.086755$

(c) 8.27×10^{-4}

(d) $P_4'(0.2) = 1.12$, $f'(0.2) = 1.124076$. The actual error is 4.076×10^{-3} .

*14. First we need to convert the degree measure for the sine function to radians. We have $180^\circ = \pi$ radians, so $1^\circ = \frac{\pi}{180}$ radians. Since,

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad \text{and} \quad f'''(x) = -\cos x,$$

we have $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$.

The approximation $\sin x \approx x$ is given by

$$f(x) \approx P_2(x) = x, \quad \text{and} \quad R_2(x) = -\frac{\cos \xi}{3!}x^3.$$

If we use the bound $|\cos \xi| \leq 1$, then

$$\left| \sin \frac{\pi}{180} - \frac{\pi}{180} \right| = \left| R_2 \left(\frac{\pi}{180} \right) \right| = \left| \frac{-\cos \xi}{3!} \left(\frac{\pi}{180} \right)^3 \right| \leq 8.86 \times 10^{-7}.$$

15. Since $42^\circ = 7\pi/30$ radians, use $x_0 = \pi/4$. Then

$$\left| R_n \left(\frac{7\pi}{30} \right) \right| \leq \frac{\left(\frac{\pi}{4} - \frac{7\pi}{30} \right)^{n+1}}{(n+1)!} < \frac{(0.053)^{n+1}}{(n+1)!}.$$

For $|R_n(\frac{7\pi}{30})| < 10^{-6}$, it suffices to take $n = 3$. To 7 digits,

$$\cos 42^\circ = 0.7431448 \quad \text{and} \quad P_3(42^\circ) = P_3\left(\frac{7\pi}{30}\right) = 0.7431446,$$

so the actual error is 2×10^{-7} .

*16. (a) $P_3(x) = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{23}{648}x^3$

(b) We have

$$f^{(4)}(x) = \frac{-119}{1296}e^{x/2} \sin \frac{x}{3} + \frac{5}{54}e^{x/2} \cos \frac{x}{3},$$

so

$$\left| f^{(4)}(x) \right| \leq \left| f^{(4)}(0.60473891) \right| \leq 0.09787176, \quad \text{for } 0 \leq x \leq 1,$$

and

$$|f(x) - P_3(x)| \leq \frac{|f^{(4)}(\xi)|}{4!}|x|^4 \leq \frac{0.09787176}{24}(1)^4 = 0.004077990.$$

17. (a) $P_3(x) = \ln(3) + \frac{2}{3}(x-1) + \frac{1}{9}(x-1)^2 - \frac{10}{81}(x-1)^3$
 (b) $\max_{0 \leq x \leq 1} |f(x) - P_3(x)| = |f(0) - P_3(0)| = 0.02663366$
 (c) $\tilde{P}_3(x) = \ln(2) + \frac{1}{2}x^2$
 (d) $\max_{0 \leq x \leq 1} |f(x) - \tilde{P}_3(x)| = |f(1) - \tilde{P}_3(1)| = 0.09453489$
 (e) $P_3(0)$ approximates $f(0)$ better than $\tilde{P}_3(1)$ approximates $f(1)$.
18. $P_n(x) = \sum_{k=0}^n x^k, n \geq 19$
19. $P_n(x) = \sum_{k=0}^n \frac{1}{k!} x^k, n \geq 7$
20. For n odd, $P_n(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + \frac{1}{n}(-1)^{(n-1)/2}x^n$. For n even, $P_n(x) = P_{n-1}(x)$.
21. A bound for the maximum error is 0.0026.
22. (a) $P_n^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, 1, \dots, n$. The shapes of P_n and f are the same at x_0 .
 (b) $P_2(x) = 3 + 4(x-1) + 3(x-1)^2$.
23. (a) The assumption is that $f(x_i) = 0$ for each $i = 0, 1, \dots, n$. Applying Rolle's Theorem on each on the intervals $[x_i, x_{i+1}]$ implies that for each $i = 0, 1, \dots, n-1$ there exists a number z_i with $f'(z_i) = 0$. In addition, we have

$$a \leq x_0 < z_0 < x_1 < z_1 < \dots < z_{n-1} < x_n \leq b.$$

- (b) Apply the logic in part (a) to the function $g(x) = f'(x)$ with the number of zeros of g in $[a, b]$ reduced by 1. This implies that numbers w_i , for $i = 0, 1, \dots, n-2$ exist with
- $$g'(w_i) = f''(w_i) = 0, \quad \text{and} \quad a < z_0 < w_0 < z_1 < w_1 < \dots < w_{n-2} < z_{n-1} < b.$$
- (c) Continuing by induction following the logic in parts (a) and (b) provides $n+1-j$ distinct zeros of $f^{(j)}$ in $[a, b]$.
- (d) The conclusion of the theorem follows from part (c) when $j = n$, for in this case there will be (at least) $(n+1) - n = 1$ zero in $[a, b]$.
- *24. First observe that for $f(x) = x - \sin x$ we have $f'(x) = 1 - \cos x \geq 0$, because $-1 \leq \cos x \leq 1$ for all values of x . Also, the statement clearly holds when $|x| \geq \pi$, because $|\sin x| \leq 1$.
- (a) The observation implies that $f(x)$ is non-decreasing for all values of x , and in particular that $f(x) > f(0) = 0$ when $x > 0$. Hence for $x \geq 0$, we have $x \geq \sin x$, and when $0 \leq x \leq \pi$, $|\sin x| = \sin x \leq x = |x|$.
- (b) When $-\pi < x < 0$, we have $\pi \geq -x > 0$. Since $\sin x$ is an odd function, the fact (from part (a)) that $\sin(-x) \leq (-x)$ implies that $|\sin x| = -\sin x \leq -x = |x|$.
 As a consequence, for all real numbers x we have $|\sin x| \leq |x|$.
25. Since $R_2(1) = \frac{1}{6}e^\xi$, for some ξ in $(0, 1)$, we have $|E - R_2(1)| = \frac{1}{6}|1 - e^\xi| \leq \frac{1}{6}(e - 1)$.
26. (a) Use the series

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} \quad \text{to integrate} \quad \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

and obtain the result.

(b) We have

$$\begin{aligned} \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdots (2k+1)} &= \frac{2}{\sqrt{\pi}} \left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots \right] \\ &\quad \cdot \left[x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + \frac{8}{105}x^7 + \frac{16}{945}x^9 + \cdots \right] \\ &= \frac{2}{\sqrt{\pi}} \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 + \cdots \right] = \operatorname{erf}(x) \end{aligned}$$

(c) 0.8427008

(d) 0.8427069

(e) The series in part (a) is alternating, so for any positive integer n and positive x we have the bound

$$\left| \operatorname{erf}(x) - \frac{2}{\sqrt{\pi}} \sum_{k=0}^n \frac{(-1)^k x^{2k+1}}{(2k+1)k!} \right| < \frac{x^{2n+3}}{(2n+3)(n+1)!} .$$

We have no such bound for the positive term series in part (b).

27. (a) Let x_0 be any number in $[a, b]$. Given $\epsilon > 0$, let $\delta = \epsilon/L$. If $|x - x_0| < \delta$ and $a \leq x \leq b$, then $|f(x) - f(x_0)| \leq L|x - x_0| < \epsilon$.

(b) Using the Mean Value Theorem, we have

$$|f(x_2) - f(x_1)| = |f'(\xi)||x_2 - x_1|,$$

for some ξ between x_1 and x_2 , so

$$|f(x_2) - f(x_1)| \leq L|x_2 - x_1|.$$

(c) One example is $f(x) = x^{1/3}$ on $[0, 1]$.

*28. (a) The number $\frac{1}{2}(f(x_1) + f(x_2))$ is the average of $f(x_1)$ and $f(x_2)$, so it lies between these two values of f . By the Intermediate Value Theorem 1.11 there exist a number ξ between x_1 and x_2 with

$$f(\xi) = \frac{1}{2}(f(x_1) + f(x_2)) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

(b) Let $m = \min\{f(x_1), f(x_2)\}$ and $M = \max\{f(x_1), f(x_2)\}$. Then $m \leq f(x_1) \leq M$ and $m \leq f(x_2) \leq M$, so

$$c_1 m \leq c_1 f(x_1) \leq c_1 M \quad \text{and} \quad c_2 m \leq c_2 f(x_2) \leq c_2 M.$$

Thus

$$(c_1 + c_2)m \leq c_1 f(x_1) + c_2 f(x_2) \leq (c_1 + c_2)M$$

and

$$m \leq \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} \leq M.$$

By the Intermediate Value Theorem 1.11 applied to the interval with endpoints x_1 and x_2 , there exists a number ξ between x_1 and x_2 for which

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

(c) Let $f(x) = x^2 + 1$, $x_1 = 0$, $x_2 = 1$, $c_1 = 2$, and $c_2 = -1$. Then for all values of x ,

$$f(x) > 0 \quad \text{but} \quad \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} = \frac{2(1) - 1(2)}{2 - 1} = 0.$$

29. (a) Since f is continuous at p and $f(p) \neq 0$, there exists a $\delta > 0$ with

$$|f(x) - f(p)| < \frac{|f(p)|}{2},$$

for $|x - p| < \delta$ and $a < x < b$. We restrict δ so that $[p - \delta, p + \delta]$ is a subset of $[a, b]$. Thus, for $x \in [p - \delta, p + \delta]$, we have $x \in [a, b]$. So

$$-\frac{|f(p)|}{2} < f(x) - f(p) < \frac{|f(p)|}{2} \quad \text{and} \quad f(p) - \frac{|f(p)|}{2} < f(x) < f(p) + \frac{|f(p)|}{2}.$$

If $f(p) > 0$, then

$$f(p) - \frac{|f(p)|}{2} = \frac{f(p)}{2} > 0, \quad \text{so} \quad f(x) > f(p) - \frac{|f(p)|}{2} > 0.$$

If $f(p) < 0$, then $|f(p)| = -f(p)$, and

$$f(x) < f(p) + \frac{|f(p)|}{2} = f(p) - \frac{f(p)}{2} = \frac{f(p)}{2} < 0.$$

In either case, $f(x) \neq 0$, for $x \in [p - \delta, p + \delta]$.

(b) Since f is continuous at p and $f(p) = 0$, there exists a $\delta > 0$ with

$$|f(x) - f(p)| < k, \quad \text{for} \quad |x - p| < \delta \quad \text{and} \quad a < x < b.$$

We restrict δ so that $[p - \delta, p + \delta]$ is a subset of $[a, b]$. Thus, for $x \in [p - \delta, p + \delta]$, we have

$$|f(x)| = |f(x) - f(p)| < k.$$

Exercise Set 1.2, page 28

1. We have

	Absolute error	Relative error
(a)	0.001264	4.025×10^{-4}
(b)	7.346×10^{-6}	2.338×10^{-6}
(c)	2.818×10^{-4}	1.037×10^{-4}
(d)	2.136×10^{-4}	1.510×10^{-4}
(e)	2.647×10^1	1.202×10^{-3}
(f)	1.454×10^1	1.050×10^{-2}
(g)	420	1.042×10^{-2}
(h)	3.343×10^3	9.213×10^{-3}

2. The largest intervals are:

- (a) (3.1412784, 3.1419068)
- (b) (2.7180100, 2.7185536)
- * (c) (1.4140721, 1.4143549)
- (d) (1.9127398, 1.9131224)

3. The largest intervals are

- (a) (149.85, 150.15)
- (b) (899.1, 900.9)
- (c) (1498.5, 1501.5)
- (d) (89.91, 90.09)

4. The calculations and their errors are:

- (a) (i) 17/15 (ii) 1.13 (iii) 1.13 (iv) both 3×10^{-3}
- (b) (i) 4/15 (ii) 0.266 (iii) 0.266 (iv) both 2.5×10^{-3}
- (c) (i) 139/660 (ii) 0.211 (iii) 0.210 (iv) 2×10^{-3} , 3×10^{-3}
- (d) (i) 301/660 (ii) 0.455 (iii) 0.456 (iv) 2×10^{-3} , 1×10^{-4}

5. We have

	Approximation	Absolute error	Relative error
(a)	134	0.079	5.90×10^{-4}
(b)	133	0.499	3.77×10^{-3}
(c)	2.00	0.327	0.195
(d)	1.67	0.003	1.79×10^{-3}
* (e)	1.80	0.154	0.0786
(f)	-15.1	0.0546	3.60×10^{-3}
(g)	0.286	2.86×10^{-4}	10^{-3}
(h)	0.00	0.0215	1.00

6. We have

	Approximation	Absolute error	Relative error
(a)	133.9	0.021	1.568×10^{-4}
(b)	132.5	0.001	7.55×10^{-6}
(c)	1.700	0.027	0.01614
(d)	1.673	0	0
(e)	1.986	0.03246	0.01662
(f)	-15.16	0.005377	3.548×10^{-4}
(g)	0.2857	1.429×10^{-5}	5×10^{-5}
(h)	-0.01700	0.0045	0.2092

7. We have

	Approximation	Absolute error	Relative error
(a)	133	0.921	6.88×10^{-3}
(b)	132	0.501	3.78×10^{-3}
(c)	1.00	0.673	0.402
(d)	1.67	0.003	1.79×10^{-3}
*(e)	3.55	1.60	0.817
(f)	-15.2	0.0454	0.00299
(g)	0.284	0.00171	0.00600
(h)	0	0.02150	1

8. We have

	Approximation	Absolute error	Relative error
(a)	133.9	0.021	1.568×10^{-4}
(b)	132.5	0.001	7.55×10^{-6}
(c)	1.600	0.073	0.04363
(d)	1.673	0	0
(e)	1.983	0.02945	0.01508
(f)	-15.15	0.004622	3.050×10^{-4}
(g)	0.2855	2.143×10^{-4}	7.5×10^{-4}
(h)	-0.01700	0.0045	0.2092

9. We have

	Approximation	Absolute error	Relative error
*(a)	3.14557613	3.983×10^{-3}	1.268×10^{-3}
(b)	3.14162103	2.838×10^{-5}	9.032×10^{-6}

10. We have

	Approximation	Absolute error	Relative error
(a)	2.7166667	0.0016152	5.9418×10^{-4}
(b)	2.718281801	2.73×10^{-8}	1.00×10^{-8}

11. (a) We have

$$\lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x - \sin x} = \lim_{x \rightarrow 0} \frac{-x \sin x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{-\sin x - x \cos x}{\sin x} = \lim_{x \rightarrow 0} \frac{-2 \cos x + x \sin x}{\cos x} = -2$$

(b) $f(0.1) \approx -1.941$

(c) $\frac{x(1 - \frac{1}{2}x^2) - (x - \frac{1}{6}x^3)}{x - (x - \frac{1}{6}x^3)} = -2$

(d) The relative error in part (b) is 0.029. The relative error in part (c) is 0.00050.

12. (a) $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{1} = 2$

(b) $f(0.1) \approx 2.05$

(c) $\frac{1}{x} \left(\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \right) - \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \right) \right) = \frac{1}{x} \left(2x + \frac{1}{3}x^3 \right) = 2 + \frac{1}{3}x^2;$
 using three-digit rounding arithmetic and $x = 0.1$, we obtain 2.00.

(d) The relative error in part (b) is = 0.0233. The relative error in part (c) is = 0.00166.

13. We have

	x_1	Absolute error	Relative error	x_2	Absolute error	Relative error
(a)	92.26	0.01542	1.672×10^{-4}	0.005419	6.273×10^{-7}	1.157×10^{-4}
(b)	0.005421	1.264×10^{-6}	2.333×10^{-4}	-92.26	4.580×10^{-3}	4.965×10^{-5}
(c)	10.98	6.875×10^{-3}	6.257×10^{-4}	0.001149	7.566×10^{-8}	6.584×10^{-5}
(d)	-0.001149	7.566×10^{-8}	6.584×10^{-5}	-10.98	6.875×10^{-3}	6.257×10^{-4}

14. We have

	Approximation for x_1	Absolute error	Relative error
(a)	92.24	0.004580	4.965×10^{-5}
(b)	0.005417	2.736×10^{-6}	5.048×10^{-4}
(c)	10.98	6.875×10^{-3}	6.257×10^{-4}
(d)	-0.001149	7.566×10^{-8}	6.584×10^{-5}

	Approximation for x_2	Absolute error	Relative error
(a)	0.005418	2.373×10^{-6}	4.377×10^{-4}
(b)	-92.25	5.420×10^{-3}	5.875×10^{-5}
(c)	0.001149	7.566×10^{-8}	6.584×10^{-5}
(d)	-10.98	6.875×10^{-3}	6.257×10^{-4}

15. The machine numbers are equivalent to
- (a) 3224
 - (b) -3224
 - * (c) 1.32421875
 - (d) 1.3242187500000002220446049250313080847263336181640625
16. (a) Next Largest: 3224.00000000000045474735088646411895751953125;
Next Smallest: 3223.9999999999954525264911353588104248046875
- (b) Next Largest: $-3224.00000000000045474735088646411895751953125$;
Next Smallest: $-3223.9999999999954525264911353588104248046875$
- * (c) Next Largest: 1.3242187500000002220446049250313080847263336181640625;
Next Smallest: 1.3242187499999997779553950749686919152736663818359375
- (d) Next Largest: 1.324218750000000444089209850062616169452667236328125;
Next Smallest: 1.32421875
17. (b) The first formula gives -0.00658 , and the second formula gives -0.0100 . The true three-digit value is -0.0116 .
18. (a) -1.82
- (b) 7.09×10^{-3}
- (c) The formula in (b) is more accurate since subtraction is not involved.
19. The approximate solutions to the systems are
- (a) $x = 2.451, y = -1.635$
 - (b) $x = 507.7, y = 82.00$
20. (a) $x = 2.460 \quad y = -1.634$
- (b) $x = 477.0 \quad y = 76.93$
- *21. (a) In nested form, we have $f(x) = (((1.01e^x - 4.62)e^x - 3.11)e^x + 12.2)e^x - 1.99$.
- (b) -6.79
 - (c) -7.07
 - (d) The absolute errors are

$$|-7.61 - (-6.71)| = 0.82 \quad \text{and} \quad |-7.61 - (-7.07)| = 0.54.$$

Nesting is significantly better since the relative errors are

$$\left| \frac{0.82}{-7.61} \right| = 0.108 \quad \text{and} \quad \left| \frac{0.54}{-7.61} \right| = 0.071,$$

22. We have $39.375 \leq \text{Volume} \leq 86.625$ and $71.5 \leq \text{Surface Area} \leq 119.5$.

23. (a) $n = 77$

(b) $n = 35$

*24. When $d_{k+1} < 5$,

$$\left| \frac{y - fl(y)}{y} \right| = \frac{0.d_{k+1} \dots \times 10^{n-k}}{0.d_1 \dots \times 10^n} \leq \frac{0.5 \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

When $d_{k+1} > 5$,

$$\left| \frac{y - fl(y)}{y} \right| = \frac{(1 - 0.d_{k+1} \dots) \times 10^{n-k}}{0.d_1 \dots \times 10^n} < \frac{(1 - 0.5) \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

25. (a) $m = 17$

(b) We have

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1) \dots (m-k-1)(m-k)!}{k!(m-k)!} = \binom{m}{k} \binom{m-1}{k-1} \dots \binom{m-k-1}{1}$$

(c) $m = 181707$

(d) 2,597,000; actual error 1960; relative error 7.541×10^{-4}

26. (a) The actual error is $|f'(\xi)\epsilon|$, and the relative error is $|f'(\xi)\epsilon| \cdot |f(x_0)|^{-1}$, where the number ξ is between x_0 and $x_0 + \epsilon$.

(b) (i) 1.4×10^{-5} ; 5.1×10^{-6} (ii) 2.7×10^{-6} ; 3.2×10^{-6}

(c) (i) 1.2; 5.1×10^{-5} (ii) 4.2×10^{-5} ; 7.8×10^{-5}

27. (a) 124.03

(b) 124.03

(c) -124.03

(d) -124.03

(e) 0.0065

(f) 0.0065

(g) -0.0065

(h) -0.0065

*28. Since $0.995 \leq P \leq 1.005$, $0.0995 \leq V \leq 0.1005$, $0.082055 \leq R \leq 0.082065$, and $0.004195 \leq N \leq 0.004205$, we have $287.61^\circ \leq T \leq 293.42^\circ$. Note that $15^\circ\text{C} = 288.16\text{K}$.

When P is doubled and V is halved, $1.99 \leq P \leq 2.01$ and $0.0497 \leq V \leq 0.0503$ so that $286.61^\circ \leq T \leq 293.72^\circ$. Note that $19^\circ\text{C} = 292.16\text{K}$. The laboratory figures are within an acceptable range.

Exercise Set 1.3, page 39

1. (a) $\frac{1}{1} + \frac{1}{4} \dots + \frac{1}{100} = 1.53$; $\frac{1}{100} + \frac{1}{81} + \dots + \frac{1}{1} = 1.54$.

The actual value is 1.549. Significant round-off error occurs much earlier in the first method.

(b) The following algorithm will sum the series $\sum_{i=1}^N x_i$ in the reverse order.

INPUT $N; x_1, x_2, \dots, x_N$

OUTPUT SUM

STEP 1 Set $SUM = 0$

STEP 2 For $j = 1, \dots, N$ set $i = N - j + 1$
 $SUM = SUM + x_i$

STEP 3 OUTPUT(SUM);
 STOP.

2. We have

	Approximation	Absolute Error	Relative Error
(a)	2.715	3.282×10^{-3}	1.207×10^{-3}
(b)	2.716	2.282×10^{-3}	8.394×10^{-4}
(c)	2.716	2.282×10^{-3}	8.394×10^{-4}
(d)	2.718	2.818×10^{-4}	1.037×10^{-4}

*3. (a) 2000 terms

(b) 20,000,000,000 terms

4. 4 terms

*5. 3 terms

6. (a) $O\left(\frac{1}{n}\right)$

(b) $O\left(\frac{1}{n^2}\right)$

(c) $O\left(\frac{1}{n^2}\right)$

(d) $O\left(\frac{1}{n}\right)$

7. The rates of convergence are:

(a) $O(h^2)$

(b) $O(h)$

(c) $O(h^2)$

(d) $O(h)$

*8. (a) $n(n+1)/2$ multiplications; $(n+2)(n-1)/2$ additions.

(b) $\sum_{i=1}^n a_i \left(\sum_{j=1}^i b_j \right)$ requires n multiplications; $(n+2)(n-1)/2$ additions.

9. The following algorithm computes $P(x_0)$ using nested arithmetic.

INPUT $n, a_0, a_1, \dots, a_n, x_0$
 OUTPUT $y = P(x_0)$

STEP 1 Set $y = a_n$.

STEP 2 For $i = n-1, n-2, \dots, 0$ set $y = x_0 y + a_i$.

STEP 3 OUTPUT (y);
 STOP.

*10. The following algorithm uses the most effective formula for computing the roots of a quadratic equation.

INPUT A, B, C .
 OUTPUT x_1, x_2 .

STEP 1 If $A = 0$ then

if $B = 0$ then OUTPUT ('NO SOLUTIONS');
 STOP.

else set $x_1 = -C/B$;
 OUTPUT ('ONE SOLUTION', x_1);
 STOP.

STEP 2 Set $D = B^2 - 4AC$.

STEP 3 If $D = 0$ then set $x_1 = -B/(2A)$;
 OUTPUT ('MULTIPLE ROOTS', x_1);
 STOP.

STEP 4 If $D < 0$ then set

$b = \sqrt{-D}/(2A)$;
 $a = -B/(2A)$;
 OUTPUT ('COMPLEX CONJUGATE ROOTS');
 $x_1 = a + bi$;
 $x_2 = a - bi$;
 OUTPUT (x_1, x_2);
 STOP.

STEP 5 If $B \geq 0$ then set

$d = B + \sqrt{D}$;
 $x_1 = -2C/d$;
 $x_2 = -d/(2A)$

else set

$d = -B + \sqrt{D}$;
 $x_1 = d/(2A)$;
 $x_2 = 2C/d$.

STEP 6 OUTPUT (x_1, x_2);
 STOP.

11. The following algorithm produces the product $P = (x - x_0), \dots, (x - x_n)$.

INPUT $n, x_0, x_1, \dots, x_n, x$
 OUTPUT P .

STEP 1 Set $P = x - x_0$;
 $i = 1$.

STEP 2 While $P \neq 0$ and $i \leq n$ set
 $P = P \cdot (x - x_i)$;
 $i = i + 1$

STEP 3 OUTPUT (P);
 STOP.

12. The following algorithm determines the number of terms needed to satisfy a given tolerance.

INPUT number x , tolerance TOL , maximum number of iterations M .
 OUTPUT number N of terms or a message of failure.

STEP 1 Set $SUM = (1 - 2x)/(1 - x + x^2)$;
 $S = (1 + 2x)/(1 + x + x^2)$;
 $N = 2$.

STEP 2 While $N \leq M$ do Steps 3–5.

STEP 3 Set $j = 2^{N-1}$;
 $y = x^j$
 $t_1 = \frac{jy}{x}(1 - 2y)$;
 $t_2 = y(y - 1) + 1$;
 $SUM = SUM + \frac{t_1}{t_2}$.

STEP 4 If $|SUM - S| < TOL$ then
 OUTPUT (N);
 STOP.

STEP 5 Set $N = N + 1$.

STEP 6 OUTPUT('Method failed');
 STOP.

When $TOL = 10^{-6}$, we need to have $N \geq 4$.

13. (a) If $|\alpha_n - \alpha|/(1/n^p) \leq K$, then

$$|\alpha_n - \alpha| \leq K(1/n^p) \leq K(1/n^q) \quad \text{since } 0 < q < p.$$

Thus

$$|\alpha_n - \alpha|/(1/n^p) \leq K \quad \text{and} \quad \{\alpha_n\}_{n=1}^{\infty} \rightarrow \alpha$$

with rate of convergence $O(1/n^p)$.

(b)

n	$1/n$	$1/n^2$	$1/n^3$	$1/n^5$
5	0.2	0.04	0.008	0.0016
10	0.1	0.01	0.001	0.0001
50	0.02	0.0004	8×10^{-6}	1.6×10^{-7}
100	0.01	10^{-4}	10^{-6}	10^{-8}

The most rapid convergence rate is $O(1/n^4)$.

14. (a) If $F(h) = L + O(h^p)$, there is a constant $k > 0$ such that

$$|F(h) - L| \leq kh^p,$$

for sufficiently small $h > 0$. If $0 < q < p$ and $0 < h < 1$, then $h^q > h^p$. Thus, $kh^p < kh^q$, so

$$|F(h) - L| \leq kh^q \quad \text{and} \quad F(h) = L + O(h^q).$$

(b) For various powers of h we have the entries in the following table.

h	h^2	h^3	h^4
0.5	0.25	0.125	0.0625
0.1	0.01	0.001	0.0001
0.01	0.0001	0.00001	10^{-8}
0.001	10^{-6}	10^{-9}	10^{-12}

The most rapid convergence rate is $O(h^4)$.

- *15. Suppose that for sufficiently small $|x|$ we have positive constants k_1 and k_2 independent of x , for which

$$|F_1(x) - L_1| \leq K_1|x|^\alpha \quad \text{and} \quad |F_2(x) - L_2| \leq K_2|x|^\beta.$$

Let $c = \max(|c_1|, |c_2|, 1)$, $K = \max(K_1, K_2)$, and $\delta = \max(\alpha, \beta)$.

(a) We have

$$\begin{aligned} |F(x) - c_1L_1 - c_2L_2| &= |c_1(F_1(x) - L_1) + c_2(F_2(x) - L_2)| \\ &\leq |c_1|K_1|x|^\alpha + |c_2|K_2|x|^\beta \leq cK[|x|^\alpha + |x|^\beta] \\ &\leq cK|x|^\gamma[1 + |x|^{\delta-\gamma}] \leq \tilde{K}|x|^\gamma, \end{aligned}$$

for sufficiently small $|x|$ and some constant \tilde{K} . Thus, $F(x) = c_1L_1 + c_2L_2 + O(x^\gamma)$.

(b) We have

$$\begin{aligned} |G(x) - L_1 - L_2| &= |F_1(c_1x) + F_2(c_2x) - L_1 - L_2| \\ &\leq K_1|c_1x|^\alpha + K_2|c_2x|^\beta \leq Kc^\delta[|x|^\alpha + |x|^\beta] \\ &\leq Kc^\delta|x|^\gamma[1 + |x|^{\delta-\gamma}] \leq \tilde{K}|x|^\gamma, \end{aligned}$$

for sufficiently small $|x|$ and some constant \tilde{K} . Thus, $G(x) = L_1 + L_2 + O(x^\gamma)$.

*16. Since

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+1} = x \quad \text{and} \quad x_{n+1} = 1 + \frac{1}{x_n},$$

we have

$$x = 1 + \frac{1}{x}, \quad \text{so} \quad x^2 - x - 1 = 0.$$

The quadratic formula implies that

$$x = \frac{1}{2} \left(1 + \sqrt{5} \right).$$

This number is called the *golden ratio*. It appears frequently in mathematics and the sciences.

*17. (a) To save space we will show the Maple output for each step in one line. Maple would produce this output on separate lines.

```
n := 98; f := 1; s := 1
```

```
n := 98  f := 1  s := 1
```

```
for i from 1 to n do
```

```
  l := f + s; f := s; s := l; od :
```

```
l :=2  f := 1  s := 2
```

```
l :=3  f := 2  s := 3
```

```
⋮
```

```
l :=218922995834555169026  f := 135301852344706746049  s := 218922995834555169026
```

```
l :=354224848179261915075
```

$$(b) \quad F_{100} := \frac{1}{\text{sqrt}(5)} \left(\left(\frac{(1 + \text{sqrt}(5))}{2} \right)^{100} - \left(\frac{(1 - \text{sqrt}(5))}{2} \right)^{100} \right)$$

$$F_{100} := \frac{1}{\sqrt{5}} \left(\left(\frac{1}{2} + \frac{1}{2}\sqrt{5} \right)^{100} - \left(\frac{1}{2} - \frac{1}{2}\sqrt{5} \right)^{100} \right)$$

```
evalf(F100)
```

```
0.3542248538 × 1021
```