Chapter 1

MATRICES

1.1 **a.** $\begin{bmatrix} -5 & 9 & -26 \\ -14 & 12 & -7 \\ 11 & -11 & -13 \end{bmatrix}$

b. $\begin{bmatrix} 8 & 64 & -8 \\ 0 & 48 & -56 \\ 16 & 32 & 96 \end{bmatrix}$

c. $\begin{bmatrix} 47 & 33 & 170 \\ 98 & -12 & -35 \\ -53 & 125 & 235 \end{bmatrix}$

1.2 $\begin{bmatrix} 127 & -64 & 0 \\ 147 & -141 & -175 \\ -40 & 154 & 350 \end{bmatrix}$

1.3 $A+B$, AB, DC are undefined.

$$A+C=\begin{bmatrix} 0 & -1 \\ 1 & 3 \\ 5 & 4 \end{bmatrix}$$

$$BA=\begin{bmatrix} 0 & 12 \\ -4 & 2 \\ -10 & 5 \end{bmatrix}$$

$$CD=\begin{bmatrix} -14 & 3 \\ 10 & -2 \\ 22 & -4 \end{bmatrix}$$

$$D^2=\begin{bmatrix} 14 & -4 \\ 8 & 2 \end{bmatrix}$$

1.4 Let $AB=\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}=\begin{bmatrix} -a+e & -b+f \\ c+e & d+f \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Equate corresponding entries

$$\begin{aligned} -a+e&=1 & -b+f&=0 \\ c+e&=0 & d+f&=1 \\ e&=a+1 & f&=b \\ c=-e&=-a-1 & d&=1-f=1-b \end{aligned}$$

$$B=\begin{bmatrix} a & b \\ -1-a & 1-b \\ 1+a & b \end{bmatrix}$$

Using the associative law, $(BA)^2 B = (BA)(BA)B = B(AB)(AB) = B(I)(I) = B$.

1.5 $A^2 - (a+d)A + (ad-bc)I = A(A-(a+d)I) + (ad-bc)I =$

$A\begin{bmatrix} -d & b \\ c & -a \end{bmatrix} + (ad-bc)I =$

$\begin{bmatrix} -ad+bc & 0 \\ 0 & bc-ad \end{bmatrix} + (ad-bc)I = 0$

1.6

$$\begin{bmatrix} d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1} & 0 \\ 0 & 0 & \dots & 0 & d_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} & a_{nn} \end{bmatrix} =$$

$$\begin{bmatrix} d_1a_{11} & d_1a_{12} & \dots & d_1a_{1,n-1} & d_1a_{1n} \\ d_2a_{21} & d_2a_{22} & \dots & d_2a_{2,n-1} & d_2a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{n-1}a_{n-1,1} & d_{n-1}a_{n-1,2} & \dots & d_{n-1}a_{n-1,n-1} & d_{n-1}a_{n-1,n} \\ d_na_{n1} & d_na_{n2} & \dots & d_na_{n,n-1} & d_na_{nn} \end{bmatrix}$$

1.7 $\begin{bmatrix} 5 & 6 & -1 & 2 \\ -1 & 2 & 1 & -9 \\ 2 & 0 & -1 & 0 \\ 0 & 3 & 28 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -3 \\ 0 \end{bmatrix}$

1.8

$$\begin{aligned} x_1 + 9x_3 &= 1 \\ -8x_1 + 3x_2 + 45x_3 &= 0 \\ 12x_1 - 6x_2 + 55x_3 &= 1 \end{aligned}$$

1.9 **a.** $\begin{bmatrix} 7 & -3 & 8 & -16 & 7 \end{bmatrix}^{\mathrm{T}}$

b. $\begin{bmatrix} 6 & 32 & 4 & 16 & 3 \end{bmatrix}^{\mathrm{T}}$

c.

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22} + a_{23}x_3 \\ a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \\ a_{43}x_3 + a_{44}x_4 \end{bmatrix}$$

d. $y_i = a_{i,i-1}x_{i-1} + a_{ii}x_i + a_{i,i+1}x_{i+1}$, $1 \le i \le n$, where $a_{10} = a_{n,n+1} = 0$.

1.10 The rotation matrix is $R = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix}$. The MATLAB statements graph the line $y = -x+3$ and its rotation 30° counterclockwise (Figure 1.1).

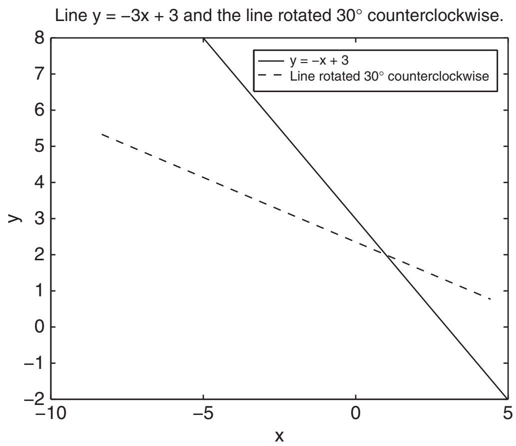

FIGURE 1.1 Problem 1.10 graph

program1_10.m

```
theta = pi/6;
R = [cos(theta) -sin(theta);sin(theta) cos(theta)];
x = [-5 5]';
z = R*[x -x+3]';
plot(x, -x+3, 'k', z(:,1), z(:,2), 'k--');
title('Line y = -3x + 3 and the line rotated 30\circ counterclockwise.');
xlabel('x');
ylabel('y');
legend('y = -x + 3','Line rotated 30\circ counterclockwise','Location','NorthEast');
```

1.11 First translate the line to (0,0). $0 = 4 + tx$, $0 = -1 + ty$, and $tx = -4$, $ty = 1$. Form the translation matrix, T_1, the rotation matrix R, and the translation matrix T_2.

$$T_1 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} cos\frac{\pi}{3} & -sin\frac{\pi}{3} & 0 \\ sin\frac{\pi}{3} & cos\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ Rotate 60°counterclockwise.

$$T_2 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$ Translate back to (4, −1). The required linear transformation is $F = T_2RT_1$, and the following MATLAB statements graph the original and the rotated line.

program1_11.m

```
T1 = [1 0 -4;0 1 1;0 0 1];
T2 = [1 0 4;0 1 -1;0 0 1];
theta = pi/3;
R = [cos(theta) -sin(theta) 0;sin(theta) cos(theta) 0;0 0 1];
F = T2*R*T1;
x = [-5 5]';
y = -x+3;
a = F*[x(1) y(1) 1]';
b = F*[x(2),y(2) 1]';
m = [a(1) b(1)]';
```

```
n = [a(2) b(2)]';
plot(x,y,'k',m,n,'k--',4,-1,'ko');
title(['Line y = -x + 3 and the line rotated 60 degrees'...
      'counterclockwise about (4,-1).']);
xlabel('x');
ylabel('y');
legend('y = -x + 3','Rotated line','(4,-1)','Location','NorthEast');
```

1.12 Let $A = \begin{bmatrix} 1 & 4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{13} & -\frac{4}{13} \\ \frac{3}{13} & \frac{1}{13} \end{bmatrix} = \begin{bmatrix} \frac{1}{13}+\frac{12}{13} & -\frac{4}{13}+\frac{4}{13} \\ -\frac{3}{13}+\frac{3}{13} & \frac{12}{13}+\frac{1}{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

1.13 **a.** $A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$

b. $x = A^{-1}b = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}\begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{11}{6} \end{bmatrix}$

1.14 **a.** $A^2 - 2A + 13I = \begin{bmatrix} -11 & 8 \\ -6 & -11 \end{bmatrix} - 2\begin{bmatrix} 1 & 4 \\ -3 & 1 \end{bmatrix} + 13\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -11-2+13 & 8-8+0 \\ -6+6+0 & -11-2+13 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

b. $A^2 - 2A + 13I = A(A-2I) + 13I = 0$, so $A^{-1}A(A-2I) + 13A^{-1} = 0$,
and $A^{-1} = -\frac{1}{13}(A-2I)$

1.15 $A^2 = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 2 \\ 6 & 4 & 3 \end{bmatrix}$, $A^3 = AA^2 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 2 \\ 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -3 & -3 \\ 6 & 4 & 3 \\ 12 & 9 & 4 \end{bmatrix}$

$$3A^2 - 3A + I = \begin{bmatrix} -3 & 0 & -6 \\ 6 & 3 & 6 \\ 18 & 12 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 3 & -3 \\ 0 & 0 & 3 \\ 6 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -3 & -3 \\ 6 & 4 & 3 \\ 12 & 9 & 4 \end{bmatrix} = A^3$$

1.16 **a.** Assume A is nonsingular, so A^{-1} exists. Since $A^2 = 0$, $A^{-1}(AA) = A^{-1}0$, and $A = 0$.
The zero matrix is not invertible. The assumption that A is nonsingular leads to a contradiction, so A is singular.

b. Assume $A^2 = A$, $A \neq I$, and A is nonsingular. Then, $A^{-1}(AA) = A^{-1}A = I$, and $A = I$, a contradiction.

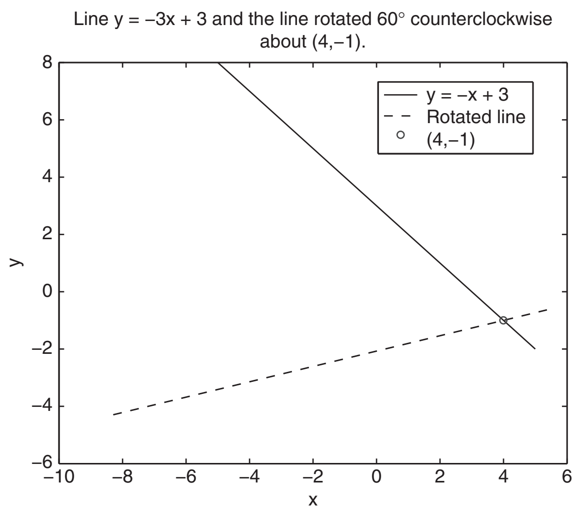

FIGURE 1.2 Problem 1.11 graph

1.17 $XX^{\mathrm{T}} = \begin{bmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \\ 17 & 39 & 61 \end{bmatrix}, X^{\mathrm{T}}X = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix}$

$$YY^{\mathrm{T}} = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 9 & 12 \\ -4 & 12 & 16 \end{bmatrix}, Y^{\mathrm{T}}Y = 26$$

1.18 $AB = \begin{bmatrix} 5 & -27 & 16 \\ 7 & -48 & 23 \\ 8 & -45 & 31 \end{bmatrix}, (AB)^{\mathrm{T}} = \begin{bmatrix} 5 & 7 & 8 \\ -27 & -48 & -45 \\ 16 & 23 & 31 \end{bmatrix}$

$$B^{\mathrm{T}}A^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -7 & 1 \\ 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 4 & 7 & 7 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 8 \\ -27 & -48 & -45 \\ 16 & 23 & 31 \end{bmatrix}$$

1.19 Noting that $A^{\mathrm{T}} = A$, $\left(B^{\mathrm{T}}AB\right)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}\left(B^{\mathrm{T}}\right)^{\mathrm{T}} = B^{\mathrm{T}}AB$, and the matrix is symmetric.

1.20 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

1.21 $AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & 0 & \ldots & 0 \\ 0 & a_{22}b_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \ldots & a_{nn}b_{nn} \end{bmatrix}$

$$BA = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} & 0 & \ldots & 0 \\ 0 & b_{22}ba_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \ldots & b_{nn}a_{nn} \end{bmatrix} = AB$$

1.22 **a.** $trace\,(A+B) = trace\left(\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \ldots & a_{1,n-1}+b_{1,n-1} & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \ldots & a_{2,n-1}+b_{2,n-1} & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1}+b_{n-1,1} & a_{n-1,2}+b_{n-1,2} & \ldots & a_{n-1,n-1}+b_{n-1,n-1} & a_{n-1,n}+b_{n-1,n} \\ a_{n1}+n_{n1} & a_{n2}+b_{n2} & \ldots & a_{n,n-1}+b_{n,n-1} & a_{nn}+b_{nn} \end{bmatrix}\right) =$

$\sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = trace\,(A) + trace\,(B)$

b. $trace\,(cA) = trace\left(\begin{bmatrix} ca_{11} & ca_{12} & \ldots & ca_{1,n-1} & ca_{1n} \\ ca_{21} & ca_{22} & \ldots & ca_{2,n-1} & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ca_{n-1,1} & ca_{n-1,2} & \ldots & ca_{n-1,n-1} & ca_{n-1,n} \\ ca_{n1} & ca_{n2} & \ldots & ca_{n,n-1} & ca_{nn} \end{bmatrix}\right) =$

$\sum_{i=1}^{n} ca_{ii} = c\sum_{i=1}^{n} a_{ii} = c\ trace\,(A)$

1.23 x^{T} has dimension $1 \times n$, A has dimension $n \times n$, and x has dimension $n \times 1$, so $x^{\mathrm{T}}Ax$ has dimension $(1 \times n) \times (n \times n) \times (n \times 1)$, or 1×1.

$$\begin{bmatrix} 1 & 3 & 9 \end{bmatrix}^{\mathrm{T}} A \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = 718$$

1.24 **a.** Exchanging rows with columns and then repeating the action returns the original matrix.

b. Let $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$, $1 \le i \le m$, $1 \le j \le n$. Then, $A^{\mathrm{T}} = \left(a_{ji}\right)$ and $B^{\mathrm{T}} = \left(b_{ji}\right)$, $1 \le j \le n$, $1 \le i \le m$, and $A^{\mathrm{T}} \pm B^{\mathrm{T}} = \left(a_{ji}\right) \pm \left(b_{ji}\right) = \left(a_{ji} \pm b_{ji}\right) = (A \pm B)^{\mathrm{T}}$.

c. $sA = \left(sa_{ij}\right)$, $1 \le i \le m$, $1 \le j \le n$, so $(sA)^{\mathrm{T}} = \left(sa_{ji}\right) = s\left(a_{ji}\right) = sA^{\mathrm{T}}$, $1 \le j \le n$, $1 \le i \le m$.

1.25 We will prove this by using the definition of matrix multiplication. The entry in row i, column j of AB is

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

The entry in row i, column j of $(AB)^{\mathrm{T}}$ is determined by interchanging subscripts, so

$$\left((AB)^{\mathrm{T}}\right)_{ij} = \sum_{k=1}^{n} a_{jk} b_{ki}.$$

Let $\hat{a_{ij}}$ and $\hat{b_{ij}}$ be the entries in row i, column j of A^{T} and B^{T}, respectively, so

$$\left(B^{\mathrm{T}} A^{\mathrm{T}}\right)_{ij} = \sum_{k=1}^{n} \hat{b_{ik}} \hat{a_{kj}}.$$

We have $\hat{a_{ij}} = a_{ji}$ and $\hat{b_{ij}} = b_{ji}$. Thus,

$$\left(B^{\mathrm{T}} A^{\mathrm{T}}\right)_{ij} = \sum_{k=1}^{n} b_{ki} a_{jk} = \sum_{k=1}^{n} a_{jk} b_{ki} = \left((AB)^{\mathrm{T}}\right)_{ij},$$

and $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$.

1.0.1 MATLAB PROBLEMS

1.26 **program1_26.m** and a run.

```
format rational;

A = [1 4 1;1 3 2;-1 2 7];
B = [1 0 1;2 5 12;-9 1 1];

Ainv = inv(A);
Binv = inv(B);
inv(A*B)
Binv*Ainv

ans =
     153/560   -11/28     31/560
     201/56    -75/14     55/56
    -119/80      9/4     -33/80

ans =
     153/560   -11/28     31/560
     201/56    -75/14     55/56
    -119/80      9/4     -33/80
```

1.27

program1_27.m and a run.

```
A = [1 3 -1 -9;0 3 0 1;12 8 -11 0;2 1 5 3];
inv(A)

ans =

    0.0200   -0.2229    0.0593    0.1344
    0.0342    0.3068   -0.0029    0.0004
    0.0467   -0.0200   -0.0284    0.1469
   -0.1027    0.0797    0.0088   -0.0013
```

1.28

a. program1_28a.m and a run.

```
H = hilb(6);
format shortg
inv(H)

ans =

  36         -630         3360         -7560         7560        -2772
 -630         14700       -88200  2.1168e+005  -2.205e+005        83160
  3360       -88200  5.6448e+005 -1.4112e+006   1.512e+006 -5.8212e+005
 -7560  2.1168e+005 -1.4112e+006  3.6288e+006  -3.969e+006  1.5523e+006
  7560  -2.205e+005   1.512e+006  -3.969e+006    4.41e+006 -1.7464e+006
 -2772        83160 -5.8212e+005  1.5523e+006 -1.7464e+006  6.9854e+005
```

It is seems likely that H is ill-conditioned since the entries of H^{-1} are very large.

b. program1_28b.m and a run.

```
syms H
H = sym(hilb(6));
inv(H)

ans =

[    36,    -630,     3360,    -7560,     7560,    -2772]
[  -630,   14700,   -88200,   211680,  -220500,    83160]
[  3360,  -88200,   564480, -1411200,  1512000,  -582120]
[ -7560,  211680, -1411200,  3628800, -3969000,  1552320]
[  7560, -220500,  1512000, -3969000,  4410000, -1746360]
[ -2772,   83160,  -582120,  1552320, -1746360,   698544]
```

1.29 a. tr.m

```
function t = tr(A)
%TR compute the trace of an n x n matrix
%
% Input: the matrix A.
% Output: the trace of A

[m n] = size(A);
if m ~= n
   error('The matrix is not square.');
end

t = sum(diag(A));
```

b. program1_29b.m and a run.

```
tr(A)
trace(A)
H = hilb(15);
tr(H)
trace(H)

ans =
    12

ans =
    12
```

```
ans =
    2.3359

ans =
    2.3359
```

1.30 a. triprod.m

```
function y = triprod(A,x)
%  TRIPROD Multiplication of a tridiagonal matrix with a vector.
%  y = triprod(A,x) returns the product Ax.
%
%  Input: n x n tridiagonal matrix A and an n x 1 vector x.
%  Output: A*x

[m n] = size(A);
if m ~= n
        error('The matrix is not square.');
end

% next error check requires knowledge of the matrix 2-norm.
TMP = diag(diag(A)) + diag(diag(A,-1),-1)+diag(diag(A,1),1);
if norm(A - TMP) >= eps
        error('The matrix does not appear to be tridiagonal.');
end

y = zeros(n,1);
y(1) = A(1,1)*x(1) + A(1,2)*x(2);
for i = 2:n-1
        y(i) = A(i,i-1)*x(i-1) + A(i,i)*x(i) + A(i,i+1)*x(i+1);
end
y(n) = A(n,n-1)*x(n-1) + A(n,n)*x(n);
```

b. The vectors x1 and x2 are those given in Problem 1.9, parts (a) and (b).
Run of **program1_30.m**

```
triprod(A,x1)

ans =
     7
    -3
     8
   -16
     7

A*x1

ans =
     7
    -3
     8
   -16
     7

triprod(A,x2)

ans =
     6
    32
     4
    16
     3
```

```
A*x2

ans =
     6
    32
     4
    16
     3

A(5,1) = 1.0e-10
triprod(A,x1)
Error using triprod (line 16)
The matrix does not appear to be tridiagonal.

Error in program1_30 (line 23)
triprod(A,x1)
```

Chapter 2

LINEAR EQUATIONS

2.1 **a.** $\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \xrightarrow{R2 = R2 - \frac{1}{2}R1} \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & -\frac{3}{2} & -\frac{1}{2} \end{array}\right]$

$-\frac{3}{2}y = -\frac{1}{2} \Rightarrow y = \frac{1}{3}$
$2x + \frac{1}{3} = 3 \Rightarrow x = \frac{4}{3}$

Solution: $\begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}$

b. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 2 & 1 \end{array}\right] \xrightarrow{R3 = R3 - R1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$x_3 = 0$
$x_2 - x_3 = 0 \Rightarrow x_2 = 0$
$x + 2x_2 + x_3 = 1 \Rightarrow x_1 = 1$

Solution: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

2.2 **a.** $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & -1 & 8 \\ 1 & -1 & -1 & -8 \end{array}\right] \xrightarrow[R3 = R3 - R1]{R2 = R2 - 2R1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & -2 & -2 & -10 \end{array}\right]$

$\xrightarrow{R3 = R3 - (-2)\,R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -8 & -2 \end{array}\right]$

$x_3 = \frac{1}{4}, x_2 = 4 + 3\left(\frac{1}{4}\right) = \frac{19}{4}, x_1 = 2 - \frac{19}{4} - \frac{1}{4} = -3$

Solution: $\begin{bmatrix} -3 \\ \frac{19}{4} \\ \frac{1}{4} \end{bmatrix}$

b. $\left[\begin{array}{cccc|c} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{array}\right] \xrightarrow{R1 \leftrightarrow R3} \left[\begin{array}{cccc|c} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{array}\right]$

$\xrightarrow{R4 = R4 - R1} \left[\begin{array}{cccc|c} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{array}\right] \xrightarrow{R3 \leftrightarrow R2} \left[\begin{array}{cccc|c} 2 & 2 & -5 & 2 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{array}\right]$

$\xrightarrow{R4 = R4 - (-1)\,R2} = \left[\begin{array}{cccc|c} 2 & 2 & -5 & 2 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{array}\right] \xrightarrow{R4 = R4 - R3} \left[\begin{array}{cccc|c} 2 & 2 & -5 & 2 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$