PROBLEM SET 2.1

Problem 1

(a)

$$
\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\
&= 1(-1) - 2(-2) + 3(-1) = 0 \quad \text{Singular} \ \blacktriangleleft
\end{aligned}
$$

(b)

$$
\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 2.11 & -0.80 & 1.72 \\ -1.84 & 3.03 & 1.29 \\ -1.57 & 5.25 & 4.30 \end{vmatrix} \\
&= 2.11\begin{vmatrix} 3.03 & 1.29 \\ 5.25 & 4.30 \end{vmatrix} + 0.80\begin{vmatrix} -1.84 & 1.29 \\ -1.57 & 4.30 \end{vmatrix} + 1.72\begin{vmatrix} -1.84 & 3.03 \\ -1.57 & 5.25 \end{vmatrix} \\
&= 2.11(6.2565) + 0.80(-5.8867) + 1.72(-4.9029) \\
&= 0.058\,867 \quad \text{Ill conditioned} \ \blacktriangleleft
\end{aligned}
$$

(c)

$$
\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} \\
&= 2(3) + 1(-2) = 4 \quad \text{Well-conditioned} \ \blacktriangleleft
\end{aligned}
$$

(d)

$$
\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 4 & 3 & -1 \\ 7 & -2 & 3 \\ 5 & -18 & 13 \end{vmatrix} = 4\begin{vmatrix} -2 & 3 \\ -18 & 13 \end{vmatrix} - 3\begin{vmatrix} 7 & 3 \\ 5 & 13 \end{vmatrix} - 1\begin{vmatrix} 7 & -2 \\ 5 & -18 \end{vmatrix} \\
&= 4(28) - 3(76) - 1(-116) = 0 \quad \text{Singular} \ \blacktriangleleft
\end{aligned}
$$

Problem 2

(a)

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 5/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 21 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 25 \\ 1 & 7 & 39 \end{bmatrix} \quad \blacktriangleleft$$

$$|\mathbf{A}| = |\mathbf{L}|\,|\mathbf{U}| = (1 \times 1 \times 1)(1 \times 3 \times 0) = 0 \quad \blacktriangleleft$$

(b)

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -4 \\ 2 & -4 & 11 \end{bmatrix} \quad \blacktriangleleft$$

$$|\mathbf{A}| = |\mathbf{L}|\,|\mathbf{U}| = (2 \times 1 \times 1)(2 \times 1 \times 1) = 4 \quad \blacktriangleleft$$

Problem 3

First solve $\mathbf{Ly} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 11/13 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} y_1 &= 1 \\ \frac{3}{2}(1) + y_2 &= -1 \qquad y_2 = -\frac{5}{2} \\ \frac{1}{2}(1) + \frac{11}{13}\left(\frac{-5}{2}\right) + y_3 &= 2 \qquad y_3 = \frac{47}{13} \end{aligned}$$

Then solve $\mathbf{Ux} = \mathbf{y}$:

$$\begin{bmatrix} 2 & -3 & -1 \\ 0 & 13/2 & -7/2 \\ 0 & 0 & 32/13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5/2 \\ 47/13 \end{bmatrix}$$

$$\begin{aligned} \frac{32}{13}x_3 &= \frac{47}{13} \qquad x_3 = \frac{47}{32} \quad \blacktriangleleft \\ \frac{13}{2}x_2 - \frac{7}{2}\left(\frac{47}{32}\right) &= -\frac{5}{2} \qquad x_2 = \frac{13}{32} \quad \blacktriangleleft \\ 2x_1 - 3\left(\frac{13}{32}\right) - \frac{47}{32} &= 1 \qquad x_1 = \frac{59}{32} \quad \blacktriangleleft \end{aligned}$$

Problem 4

The augmented coefficient matrix is

$$[\mathbf{A}|\mathbf{b}] = \begin{bmatrix} 2 & -3 & -1 & 3 \\ 3 & 2 & -5 & -9 \\ 2 & 4 & -1 & -5 \end{bmatrix}$$

Elimination phase:

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 - \frac{3}{2} \times \text{row } 1 \\ \text{row } 3 &\leftarrow \text{row } 3 - \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 2 & -3 & -1 & 3 \\ 0 & 13/2 & -7/2 & -27/2 \\ 0 & 7 & 0 & -8 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - \frac{14}{13} \times \text{row } 2$$

$$\begin{bmatrix} 2 & -3 & -1 & 3 \\ 0 & 13/2 & -7/2 & -27/2 \\ 0 & 0 & 49/13 & 85/13 \end{bmatrix}$$

Solution by back substitution:

$$\begin{aligned} \frac{49}{13} x_3 &= \frac{85}{13} \qquad x_3 = \frac{85}{49} = 1.7347 \quad \blacktriangleleft \\ \frac{13}{2} x_2 - \frac{7}{2}\left(\frac{85}{49}\right) &= -\frac{27}{2} \qquad x_2 = -\frac{8}{7} = -1.1429 \quad \blacktriangleleft \\ 2x_1 - 3\left(-\frac{8}{7}\right) - \frac{85}{49} &= 3 \qquad x_1 = \frac{32}{49} = 0.6531 \quad \blacktriangleleft \end{aligned}$$

Problem 5

The augmented coefficient matrix is

$$[\mathbf{A}|\mathbf{B}] = \begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

Before elimination, we exchange rows 2 and 3 in order to reduce the amount of algebra:

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

Elimination phase:

$$\text{row } 2 \leftarrow \text{row } 2 + \frac{1}{2} \times \text{row } 1$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1/2 & 1 & 1/2 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - \frac{1}{2} \times \text{row } 2$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1/2 & 1 & 1/2 & 1 \\ 0 & 0 & 9/4 & -1/2 & -1/4 & -1/2 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

$$\text{row } 4 \leftarrow \text{row } 4 - \frac{4}{9} \times \text{row } 3$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1/2 & 1 & 1/2 & 1 \\ 0 & 0 & 9/4 & -1/2 & -1/4 & -1/2 \\ 0 & 0 & 0 & -16/9 & 1/9 & 2/9 \end{bmatrix}$$

First solution vector by back substitution:

$$\begin{aligned} -\frac{16}{9}x_4 &= \frac{1}{9} & x_4 &= -\frac{1}{16} \\ \frac{9}{4}x_3 - \frac{1}{2}\left(-\frac{1}{16}\right) &= -\frac{1}{4} & x_3 &= -\frac{1}{8} \\ 2x_2 - \frac{1}{2}\left(-\frac{1}{8}\right) + \left(-\frac{1}{16}\right) &= \frac{1}{2} & x_2 &= \frac{1}{4} \\ 2x_1 - \left(-\frac{1}{8}\right) &= 1 & x_1 &= \frac{7}{16} \end{aligned}$$

Second solution vector:

$$\begin{aligned}
-\frac{16}{9}x_4 &= \frac{2}{9} & x_4 = -\frac{1}{8} \\
\frac{9}{4}x_3 - \frac{1}{2}\left(-\frac{1}{8}\right) &= -\frac{1}{2} & x_3 = -\frac{1}{4} \\
2x_2 - \frac{1}{2}\left(-\frac{1}{4}\right) + \left(-\frac{1}{8}\right) &= 1 & x_2 = \frac{1}{2} \\
2x_1 - \left(-\frac{1}{4}\right) &= 0 & x_1 = -\frac{1}{8}
\end{aligned}$$

Therefore,

$$\mathbf{X} = \begin{bmatrix} 7/16 & -1/8 \\ 1/4 & 1/2 \\ -1/8 & -1/4 \\ -1/16 & -1/8 \end{bmatrix} \quad \blacktriangleleft$$

Problem 6

After reordering rows, the augmented coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}$$

Elimination phase:

$$\text{row } 3 \leftarrow \text{row } 3 - \text{row } 2$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\text{row } 4 \leftarrow \text{row } 4 - 2 \times \text{row } 3$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 2 & -3 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\text{row } 5 \leftarrow \text{row } 5 - \text{row } 4$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 2 & -3 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Back substitution:

$$\begin{aligned} -x_5 &= 1 & x_5 = -1 \blacktriangleleft \\ -x_4 + 2(-1) &= -3 & x_4 = 1 \blacktriangleleft \\ x_3 + 1 &= 2 & x_3 = 1 \blacktriangleleft \\ x_2 - 1 + 1 - (-1) &= -1 & x_2 = -2 \blacktriangleleft \\ x_1 + 2(-2) - 2(1) &= -4 & x_1 = 2 \blacktriangleleft \end{aligned}$$

Problem 7

(a)

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Use Gauss elimination storing each multiplier in the location occupied by the element that was eliminated. The multipliers are enclosed in boxes $\boxed{\text{thus}}$.

$$\text{row 2} \leftarrow \text{row 2} - \left(-\frac{1}{4}\right) \times \text{row 2}$$

$$\begin{bmatrix} 4 & -1 & 0 \\ \boxed{-1/4} & 15/4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

$$\text{row 3} \leftarrow \text{row 3} - \left(-\frac{4}{15}\right) \times \text{row 2}$$

$$\begin{bmatrix} 4 & -1 & 0 \\ \boxed{-1/4} & 15/4 & -1 \\ 0 & \boxed{-4/15} & 56/15 \end{bmatrix}$$

Thus

$$\mathbf{U} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 15/4 & -1 \\ 0 & 0 & 56/15 \end{bmatrix} \blacktriangleleft \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 0 & -4/15 & 1 \end{bmatrix} \blacktriangleleft$$

(b)

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T$$

Substituting for $\mathbf{L}\mathbf{L}^T$ from Eq. (2.16), we get

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{11}L_{31} & L_{21}L_{31} + L_{22}L_{32} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{bmatrix}$$

Equating matrices term-by term:

$$\begin{aligned} L_{11}^2 &= 4 \qquad L_{11} = 2 \\ 2L_{21} &= -1 \qquad L_{21} = -\frac{1}{2} \\ 2L_{31} &= 0 \qquad L_{31} = 0 \\ \left(-\frac{1}{2}\right)^2 + L_{22}^2 &= 4 \qquad L_{22} = \frac{\sqrt{15}}{2} \\ -\frac{1}{2}(0) + \frac{\sqrt{15}}{2}L_{32} &= -1 \qquad L_{32} = -\frac{2}{\sqrt{15}} \\ 0^2 + \left(-\frac{2}{\sqrt{15}}\right)^2 + L_{33}^2 &= 4 \qquad L_{32} = 2\sqrt{\frac{14}{15}} \end{aligned}$$

Therefore,

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ -1/2 & \sqrt{15}/2 & 0 \\ 0 & -2/\sqrt{15} & 2\sqrt{14/15} \end{bmatrix} \blacktriangleleft$$

Problem 8

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -4 \\ 9 & -8 & 24 \\ -12 & 24 & -26 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -3 \\ 65 \\ -42 \end{bmatrix}$$

Decomposition of $\mathbf{A}$ (multipliers are enclosed in boxes):

$$\begin{aligned} \text{row 2} &\leftarrow \text{row 2} - (-3) \times \text{row 1} \\ \text{row 3} &\leftarrow \text{row 3} - 4 \times \text{row 1} \end{aligned}$$

$$\begin{bmatrix} -3 & 6 & -4 \\ \boxed{-3} & 10 & 12 \\ \boxed{4} & 0 & -10 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} -3 & 6 & -4 \\ 0 & 10 & 12 \\ 0 & 0 & -10 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= -3 \\ -3(-3) + y_2 &= 65 \qquad y_2 = 56 \\ 4(-3) + y_3 &= -42 \qquad y_3 = -30 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} -10x_3 &= -30 \qquad x_3 = 3 \ \blacktriangleleft \\ 10x_2 + 12(3) &= 56 \qquad x_2 = 2 \ \blacktriangleleft \\ -3x_1 + 6(2) - 4(3) &= -3 \qquad x_1 = 1 \ \blacktriangleleft \end{aligned}$$

Problem 9

$$\mathbf{A} = \begin{bmatrix} 2.34 & -4.10 & 1.78 \\ -1.98 & 3.47 & -2.22 \\ 2.36 & -15.17 & 6.18 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0.02 \\ -0.73 \\ -6.63 \end{bmatrix}$$

Decomposition of $\mathbf{A}$ (multipliers are enclosed in boxes):

$$\begin{aligned} \text{row 2} &\leftarrow \text{row 2} - (-0.846\,154) \times \text{row 1} \\ \text{row 3} &\leftarrow \text{row 3} - 1.008\,547 \times \text{row 1} \end{aligned}$$

$$\begin{bmatrix} 2.34 & -4.10 & 1.78 \\ \boxed{-0.846\,154} & 0.000\ 769 & -0.713\,846 \\ \boxed{1.008\,547} & -11.034\,96 & 4.384\,786 \end{bmatrix}$$

$$\text{row 3} \leftarrow \text{row 3} - (-14349.75) \times \text{row 2}$$

$$\begin{bmatrix} 2.34 & -4.10 & 1.78 \\ \boxed{-0.846\,154} & 0.000\ 769 & -0.713\,846 \\ \boxed{1.008\,547} & \boxed{-14349.75} & -10239.13 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 2.34 & -4.10 & 1.78 \\ 0 & 0.000\ 769 & -0.713\,846 \\ 0 & 0 & -10\,239.1 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -0.846\,154 & 1 & 0 \\ 1.008\,547 & -14349.7 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 0.02 \\ -0.846\,154(0.02) + y_2 &= -0.73 \qquad y_2 = -0.713\,07\,7 \\ 1.008\,547(0.02) - 14349.7(-0.713\,07\,7) + y_3 &= -6.63 \qquad y_3 = -10\,239.1 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} -10\,239.1x_3 &= -10\,239.1 \qquad x_3 = 1.0 \ \blacktriangleleft \\ 0.000\ 769x_2 - 0.713\,846 &= -0.713\,07\,7 \qquad x_2 = 1.0 \ \blacktriangleleft \\ 2.34x_1 - 4.10 + 1.78 &= 0.02 \qquad x_1 = 1.0 \ \blacktriangleleft \end{aligned}$$

Problem 10

$$\mathbf{A} = \begin{bmatrix} 4 & -3 & 6 \\ 8 & -3 & 10 \\ -4 & 12 & -10 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Decomposition of **A** (multipliers are enclosed in boxes):

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 - 2 \times \text{row } 1 \\ \text{row } 3 &\leftarrow \text{row } 3 - (-1) \times \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 4 & -3 & 6 \\ \boxed{2} & 3 & -2 \\ \boxed{-1} & 9 & -4 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - 3 \times \text{row } 2$$

$$\begin{bmatrix} 4 & -3 & 6 \\ \boxed{2} & 3 & -2 \\ \boxed{-1} & \boxed{3} & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -3 & 6 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

<u>First solution vector</u>
Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 1 \\ 2(1) + y_2 &= 0 \qquad y_2 = -2 \\ -1 + 3(-2) + y_3 &= 0 \qquad y_3 = 7 \end{aligned}$$

Solution of $\mathbf{Uy} = \mathbf{x}$:

$$\begin{aligned} 2x_3 &= 7 \qquad x_3 = \frac{7}{2} \\ 3x_2 - 2\left(\frac{7}{2}\right) &= -2 \qquad x_2 = \frac{5}{3} \\ 4x_1 - 3\left(\frac{5}{3}\right) + 6\left(\frac{7}{2}\right) &= 1 \qquad x_1 = -\frac{15}{4} \end{aligned}$$

<u>Second solution vector</u>
Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 0 \\ 2(0) + y_2 &= 1 \qquad y_2 = 1 \\ -1(0) + 3(1) + y_3 &= 0 \qquad y_3 = -3 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} 2x_3 &= -3 & x_3 = -\frac{3}{2} \\ 3x_2 - 2\left(-\frac{3}{2}\right) &= 1 & x_3 = -\frac{2}{3} \\ 4x_1 - 3\left(-\frac{2}{3}\right) + 6\left(-\frac{3}{2}\right) &= 0 & x_1 = \frac{7}{4} \end{aligned}$$

Therefore,

$$\mathbf{X} = \begin{bmatrix} 7/2 & -3/2 \\ 5/3 & -2/3 \\ -15/4 & 7/4 \end{bmatrix} \blacktriangleleft$$

Problem 11

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3/2 \\ 3 \end{bmatrix}$$

Substituting for $\mathbf{LL}^T$ from Eq. (2.16), we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{11}L_{31} & L_{21}L_{31} + L_{22}L_{32} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{bmatrix}$$

Equating matrices term-by term:

$$\begin{aligned} L_{11} &= 1 & L_{21} = 1 \qquad L_{31} = 1 \\ 1^2 + L_{22}^2 &= 2 & L_{22} = 1 \\ (1)(1) + (1)L_{32} &= 2 & L_{32} = 1 \\ 1^2 + 1^2 + L_{33}^2 &= 3 & L_{33} = 1 \end{aligned}$$

Thus

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{L}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 1 \\ 1 + y_2 &= \frac{3}{2} & y_2 = \frac{1}{2} \\ 1 + \frac{1}{2} + y_3 &= 3 & y_3 = \frac{3}{2} \end{aligned}$$

Solution of $\mathbf{L}^T\mathbf{x} = \mathbf{y}$:

$$\begin{aligned} x_3 &= \frac{3}{2} & & \blacktriangleleft \\ x_2 + \frac{3}{2} &= \frac{1}{2} & x_2 &= -1 \quad \blacktriangleleft \\ x_1 - 1 + \frac{3}{2} &= 1 & x_1 &= \frac{1}{2} \quad \blacktriangleleft \end{aligned}$$

Problem 12

$$A = \begin{bmatrix} 4 & -2 & -3 \\ 12 & 4 & -10 \\ -16 & 28 & 18 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1.1 \\ 0 \\ -2.3 \end{bmatrix}$$

Decomposition of $\mathbf{A}$ (multipliers are enclosed in boxes):

$$\begin{aligned} \text{row 2} &\leftarrow \text{row 2} - 3 \times \text{row 1} \\ \text{row 3} &\leftarrow \text{row 3} - (-4) \times \text{row 1} \end{aligned}$$

$$\begin{bmatrix} 4 & -2 & -3 \\ \boxed{3} & 10 & -1 \\ \boxed{-4} & 20 & 6 \end{bmatrix}$$

$$\text{row 3} \leftarrow \text{row 3} - 2 \times \text{row 2}$$

$$\begin{bmatrix} 4 & -2 & -3 \\ \boxed{3} & 10 & -1 \\ \boxed{-4} & \boxed{2} & 8 \end{bmatrix}$$

Therefore

$$\mathbf{U} = \begin{bmatrix} 4 & -2 & -3 \\ 0 & 10 & -1 \\ 0 & 0 & 8 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 1.1 & & \\ 3(1.1) + y_2 &= 0 & y_2 &= -3.3 \\ -4(1.1) + 2(-3.3) + y_3 &= -2.3 & y_3 &= 8.7 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} 8x_3 &= 8.7 & x_3 &= 1.0875 \quad \blacktriangleleft \\ 10x_2 - 1.0875 &= -3.3 & x_2 &= -0.221\,25 \quad \blacktriangleleft \\ 4x_1 - 2(-0.221\,25) - 3(1.0875) &= 1.1 & x_1 &= 0.98 \quad \blacktriangleleft \end{aligned}$$

Problem 13

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & \cdots \\ 0 & 0 & \alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Since the banded structure of a matrix is preserved during decomposition, **L** must be a diagonal matrix. Therefore,

$$\mathbf{L}\mathbf{L}^T = \begin{bmatrix} L_{11}^2 & 0 & 0 & \cdots \\ 0 & L_{22}^2 & 0 & \cdots \\ 0 & 0 & L_{33}^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It follows from $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ that

$$\mathbf{L} = \begin{bmatrix} \sqrt{\alpha_1} & 0 & 0 & \cdots \\ 0 & \sqrt{\alpha_2} & 0 & \cdots \\ 0 & 0 & \sqrt{\alpha_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \blacktriangleleft$$

Problem 14

```
## problem2_1_14
from numpy import dot,array

def gaussElimin(a,b):
    n,m = b.shape     # Replaces n = len(b)
  # Elimination Phase
    for k in range(0,n-1):
        for i in range(k+1,n):
           if a[i,k] != 0.0:
               lam = a [i,k]/a[k,k]
               a[i,k+1:n] = a[i,k+1:n] - lam*a[k,k+1:n]
               b[i] = b[i] - lam*b[k]
  # Back substitution
    for k in range(n-1,-1,-1):
        for i in range(m): # New loop (over all constant vectors)
            b[k,i] = (b[k,i] - dot(a[k,k+1:n],b[k+1:n,i]))/a[k,k]
    return b
```

```
a = array([[2,-1,0],[-1,2,-1],[0,-1,1]])*1.0
b = array([[1,0,0],[0,1,0],[0,0,1]])*1.0
print(''The solution is:\n'',gaussElimin(a,b))
input(''\nPress return to exit'')

The solution is:
[[ 1.  1.  1.]
 [ 1.  2.  2.]
 [ 1.  2.  3.]]
```

Problem 15

This program prompts for the number of equations.

```
## problem2_1_15
from numpy import zeros,array
from LUdecomp import *

n = eval(raw_input(''Number of equations ==> ''))
a = zeros((n,n))
b = zeros(n)
for i in range(n):
   for j in range(n):
      a[i,j] =1.0/(i+j+1)
      b[i] = b[i] + a[i,j]
LUdecomp(a)
LUsolve(a,b)
print(''The solution is:\n'',b)
input(''\nPress return to exit'')
```

The largest n for which 6-figure accuracy is achieved seems to be 8:

```
Number of equations ==> 8
The solution is:
[ 1.          1.          0.99999997  1.00000017  0.99999955  1.00000064
  0.99999955  1.00000013]
```

Problem 16

Forward sustitution The kth equation of $\mathbf{Ly} = \mathbf{b}$ is

$$L_{k1}y_1 + L_{k2}y_2 + \cdots + L_{kk}y_k = b_k$$

Solving for y_k yields

$$\begin{aligned} y_k &= \frac{b_k - (L_{k,1}y_1 + L_{k,2}y_2 + \cdots + L_{k,k-1}y_{k-1})}{L_{kk}} \\ &= b_k - \frac{\left[\begin{array}{cccc} L_{k,1} & L_{k,2} & \cdots & L_{k,k-1} \end{array}\right] \cdot \left[\begin{array}{cccc} y_1 & y_2 & \cdots & y_{k-1} \end{array}\right]}{L_{k,k}} \end{aligned}$$

This expression, evaluated with $k = 1, 2, \ldots, n$ (in that order), constitutes the forward substitution phase In `choleskiSol` the b's are overwritten with y's during the computations.

Back substitution A typical (kth) equation of $\mathbf{L}^T\mathbf{x} = \mathbf{y}$ is

$$L_{k,k}x_k + L_{k+1,k}x_{k+1} + L_{k+2,k}x_{k+2} + \cdots + L_{n,k}x_n = y_k$$

The solution for x_k is

$$\begin{aligned} x_k &= \frac{y_k - (L_{k+1,k}x_{k+1} + L_{k+2,k}x_{k+2} + \ldots L_{n,k}x_n)}{L_{k,k}} \\ &= \frac{y_k - \left[\begin{array}{cccc} L_{k+1,k} & L_{k+2,k} & \cdots & L_{n,k} \end{array}\right] \cdot \left[\begin{array}{cccc} x_{k+1} & x_{k+2} & \cdots & x_n \end{array}\right]}{L_{kk}} \end{aligned}$$

In back substitution we evaluate this expression in the order $k = n, n-1, \ldots, 1$. Note that in `choleskiSol` the vector $\mathbf{x}$ overwrites the vector $\mathbf{y}$.

Problem 17

```
## problem2_1_17
from numpy import zeros,array
from LUdecomp import *

x = array([0,1,3,4],float)
y = array([10,35,31,2],float)
n = len(x)
a = zeros((n,n))
for i in range(n):
   a[0:n,i] = x[0:n]**i
LUdecomp(a)
LUsolve(a,y)
print(''The coefficients are:\n'',y)
input(''\nPress return to exit'')

The coefficients are:
[ 10.  34.  -9.   0.]
```

Problem 18

```
## problem2_1_18
from numpy import zeros,array
from LUdecomp import *

x = array([0,1,3,5,6],float)
y = array([-1,1,3,2,-2],float)
n = len(x)
a = zeros((n,n))
for i in range(n):
   a[0:n,i] = x[0:n]**i
LUdecomp(a)
LUsolve(a,y)
print(''The coefficients are:\n'',y)
input(''\nPress return to exit''

The coefficients are:
 [-1.  2.68333333 -0.875  0.21666667 -0.025]
```

Problem 19

$$\begin{aligned} f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \\ f''(x) &= 2c_2 + 6c_3 x + 12 c_4 x^2 \end{aligned}$$

The specified conditions result in the equations

$$\mathbf{Ac} = \mathbf{y}$$

```
## problem2_1_19
from numpy import zeros,array
from LUdecomp import *
a = zeros((5,5))
a[0] = [1,0,0,0,0]                        # f(0)
a[1] = [1,0.75,0.75**2,0.75**3,0.75**4]  # f(0.75)
a[2] = [1,1,1,1,1]                        # f(1)
a[3] = [0,0,2,0,0]                        # f''(0)
a[4] = [0,0,2,6,12]                       # f''(1)
y = array([1,-0.25,1,0,0])
LUdecomp(a)
LUsolve(a,y)
```