Chapter 1

1.1 *Solution*:
Taking "$\nabla\cdot$" operation on both sides of Eq. (1.1-2) and using $\nabla\cdot(\nabla\times\mathbf{H})=0$, we obtain

$$-\frac{\partial}{\partial t}\nabla\cdot\mathbf{D}=\nabla\cdot\mathbf{J}$$

This proves the conservation of charge after using Eq. (1.1-3).

1.2 *Solution*:
Using $\int\nabla\cdot\mathbf{D}dV=\oint\mathbf{D}\cdot d\mathbf{S}=\oint\mathbf{D}\cdot\mathbf{n}da$, where **n** is an outward normal unit vector. The surface of the pillbox shown in Figure 1.1 can be divided into three parts: a top circle, a bottom circle and a ring.

$$\oint\mathbf{D}\cdot\mathbf{n}da=\int_{S_1}\mathbf{D}\cdot\mathbf{n}da+\int_{S_2}\mathbf{D}\cdot\mathbf{n}da+\int_{\text{ring}}\mathbf{D}\cdot\mathbf{n}da$$

$$=A(\mathbf{D}_1\cdot\mathbf{n}_1+\mathbf{D}_2\cdot\mathbf{n}_2)=A(-\mathbf{D}_1\cdot\mathbf{n}_2+\mathbf{D}_2\cdot\mathbf{n}_2)=A(-\mathbf{D}_1\cdot\mathbf{n}+\mathbf{D}_2\cdot\mathbf{n})=Q$$

where Q is the total charge within the pillbox, A is the area of the circle, $\mathbf{n}_2$ = -$\mathbf{n}_1$ = **n**. The integral over the ring approaches zero. The proof for **B** is similar.

1.3 *Solution*:
Using $\int(\nabla\times\mathbf{H})\cdot d\mathbf{S}=\oint\mathbf{H}\cdot d\vec{\ell}$ and the rectangular path shown in Figure 1.1, we obtain

$\int\mathbf{J}\cdot d\mathbf{S}=\Delta I=\oint\mathbf{H}\cdot d\vec{\ell}=(\mathbf{t}\times\mathbf{n})\cdot(\mathbf{H}_2-\mathbf{H}_1)\Delta L$, where ΔI is the current flowing through the rectangular area, ΔL is the length of the rectangular path, **t** is a unit tangent vector perpendicular to the rectangle and parallel to the interface, **n** is a unit normal to the interface. Since $\mathbf{K}\cdot\mathbf{t}=\Delta I/\Delta L$ and $(\mathbf{t}\times\mathbf{n})\cdot(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{t}\cdot[\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)]$, we obtain (1.1-11). The proof for **E** is similar.

1.4 *Solution*: Direct substitution into Maxwell's equations.

1.5 *Solution*:
Take divergence of the stress tensor

$$\nabla\cdot\mathbf{T}=\nabla\cdot(\varepsilon\mathbf{EE}+\mu\mathbf{HH})-\nabla(\varepsilon E^2+\mu H^2)/2$$
$$=\varepsilon(\nabla\cdot\mathbf{E})\mathbf{E}+\mu(\nabla\cdot\mathbf{H})\mathbf{H}+\varepsilon(\mathbf{E}\cdot\nabla)\mathbf{E}+\mu(\mathbf{H}\cdot\nabla)\mathbf{H}$$
$$-\varepsilon(\mathbf{E}\cdot\nabla)\mathbf{E}-\mu(\mathbf{H}\cdot\nabla)\mathbf{H}-\varepsilon\mathbf{E}\times(\nabla\times\mathbf{E})-\mu\mathbf{H}\times(\nabla\times\mathbf{H})$$
$$=\rho\mathbf{E}-\varepsilon\mathbf{E}\times(\nabla\times\mathbf{E})-\mu\mathbf{H}\times(\nabla\times\mathbf{H})$$

where we assume ε and μ are constants.
Then take the time derivative of the momentum density

$$\frac{\partial\mathbf{P}}{\partial t}=\mu\varepsilon(\frac{\partial}{\partial t}\mathbf{E})\times\mathbf{H}+\mu\varepsilon\mathbf{E}\times\frac{\partial}{\partial t}\mathbf{H}=-\mu\mathbf{H}\times\nabla\times\mathbf{H}-\varepsilon\mathbf{E}\times\nabla\times\mathbf{E}-\mathbf{J}\times\mathbf{B}$$

We note $\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}$ is the Lorentz force.
This proves the equation of motion.

1.6 *Solution*:

(a) Work done = Force x distance. $dW = \mathbf{F}\cdot d\mathbf{x} = q\mathbf{E}\cdot d\mathbf{x} = q\mathbf{E}\cdot \mathbf{v}dt$

(b) $dW = \sum_i \mathbf{F}_i \cdot d\mathbf{x}_i = \sum_i q_i \mathbf{E}\cdot d\mathbf{x}_i$

(c) $dW == \sum_i d\mathbf{x}_i \cdot q_i \mathbf{E} = \mathbf{E}\cdot \sum_i d\mathbf{x}_i q_i = \mathbf{E}\cdot d\mathbf{P}$

1.7 *Solution*:

(a) The flight time is given by

$$\tau = L/\mathrm{v}_g = Ldk/d\omega = L\frac{d}{d\omega}\left(\frac{\omega}{c}n\right) = L\left(\frac{n}{c}+\frac{\omega}{c}\frac{dn}{d\omega}\right).$$

Taking the differential on both sides of $\lambda\omega = 2\pi c$, we obtain $\lambda d\omega + \omega d\lambda = 0$.

Thus, $L\left(\frac{n}{c}+\frac{\omega}{c}\frac{dn}{d\omega}\right) = L\left(\frac{n}{c}-\frac{\lambda}{c}\frac{dn}{d\lambda}\right)$.

(b) Using the result from (a), we can write $D = \frac{d}{d\lambda}\left(\frac{n}{c}-\frac{\lambda}{c}\frac{dn}{d\lambda}\right) = -\frac{\lambda}{c}\frac{d^2n}{d\lambda^2} = -\frac{1}{c\lambda}\lambda^2\frac{d^2n}{d\lambda^2}$

(c) $(\mathrm{v}_2-\mathrm{v}_g)(\mathrm{v}_1-\mathrm{v}_g) = \left(\frac{\omega_2}{k_2}-\frac{\omega_2-\omega_1}{k_2-k_1}\right)\left(\frac{\omega_1}{k_1}-\frac{\omega_2-\omega_1}{k_2-k_1}\right) = \frac{(\omega_1 k_2-\omega_2 k_1)^2}{k_1 k_2 (k_2-k_1)^2} > 0$

1.8 *Solution*:

Develop a simple computer program to plot the dispersion curves.

1.9 *Solution*:

Without loss of generality, we assume

$$E_x = A_x \cos(\omega t - \delta/2)$$
$$E_y = A_y \cos(\omega t + \delta/2)$$

Expand the cosine functions, we obtain

$$E_x/A_x = \cos(\omega t - \delta/2) = \cos(\omega t)\cos(\delta/2) + \sin(\omega t)\sin(\delta/2)$$
$$E_y/A_y = \cos(\omega t + \delta/2) = \cos(\omega t)\cos(\delta/2) - \sin(\omega t)\sin(\delta/2)$$

Addition and subtraction of above equations lead to

$$E_x/A_x + E_y/A_y = 2\cos(\omega t)\cos(\delta/2)$$
$$E_x/A_x - E_y/A_y = 2\sin(\omega t)\sin(\delta/2)$$

and then

$$\sin(\delta/2)[E_x/A_x + E_y/A_y] = 2\cos(\omega t)\sin(\delta/2)\cos(\delta/2) = \cos(\omega t)\sin\delta$$
$$\cos(\delta/2)[E_x/A_x - E_y/A_y] = 2\sin(\omega t)\sin(\delta/2)\cos(\delta/2) = \sin(\omega t)\sin\delta$$

We now add the square of the above equations and obtain

$$\sin^2(\delta/2)[E_x/A_x + E_y/A_y]^2 + \cos^2(\delta/2)[E_x/A_x - E_y/A_y]^2 = \sin^2\delta$$

Using $\sin^2(\delta/2)+\cos^2(\delta/2) = 1$ and $\cos^2(\delta/2)-\sin^2(\delta/2) = \cos\delta$ lead to Eq. (1.6-12).

1.10 *Solution*:
Using the coordinate rotation of Figure 1.4, we have

$$E_x = E_{x'} \cos\phi - E_{y'} \sin\phi$$
$$E_y = E_{x'} \sin\phi + E_{y'} \cos\phi$$

Substitution of above equation into (1.6-12), we obtain, after multiplying both sides by $A_x^2 A_y^2$

$$A_y^2 (E_{x'} \cos\phi - E_{y'} \sin\phi)^2 + A_x^2 (E_{x'} \sin\phi + E_{y'} \cos\phi)^2$$
$$-2A_x A_y \cos\delta (E_{x'} \cos\phi - E_{y'} \sin\phi)(E_{x'} \sin\phi + E_{y'} \cos\phi) = A_x^2 A_y^2 \sin^2\delta$$

or

$$E_{x'}^2 (A_x^2 \sin^2\phi + A_y^2 \cos^2\phi - 2A_x A_y \cos\delta \cos\phi \sin\phi) +$$
$$E_{y'}^2 (A_x^2 \cos^2\phi + A_y^2 \sin^2\phi + 2A_x A_y \cos\delta \cos\phi \sin\phi) +$$
$$E_{x'} E_{y'} (-2A_y^2 \cos\phi \sin\phi + 2A_x^2 \cos\phi \sin\phi + 2A_x A_y \cos\delta \sin^2\phi - 2A_x A_y \cos\delta \cos^2\phi) = A_x^2 A_y^2 \sin^2\delta$$

In the principal coordinate, the equation must be of the following form

$E_{x'}^2 / a^2 + E_{y'}^2 / b^2 = 1$, or equivalently $b^2 E_{x'}^2 + a^2 E_{y'}^2 = a^2 b^2$

Thus, we obtain

$$\tan 2\phi = \frac{2A_x A_y}{A_x^2 - A_y^2} \cos\delta$$

$$a^2 = A_x^2 \cos^2\phi + A_y^2 \sin^2\phi + 2A_x A_y \cos\delta \cos\phi \sin\phi$$
$$b^2 = A_x^2 \sin^2\phi + A_y^2 \cos^2\phi - 2A_x A_y \cos\delta \cos\phi \sin\phi$$

The equality $a^2 b^2 = A_x^2 A_y^2 \sin^2\delta$ can be proven by using the above three equations.

We obtain

Eq.1 $\quad a^2 + b^2 = A_x^2 + A_y^2$

Eq.2 $\quad a^2 - b^2 = (A_x^2 - A_y^2)\cos 2\phi + 2A_x A_y \cos\delta \sin 2\phi$

Eq.3 $\quad 0 = (A_x^2 - A_y^2)\sin 2\phi - 2A_x A_y \cos\delta \cos 2\phi$

We now calculate (Eq. 1)^2 - (Eq. 2)^2 - (Eq. 3)^2. This leads to $a^2 b^2 = A_x^2 A_y^2 \sin^2\delta$.

1.11 *Solution*:
Without loss of generality, we assume

$$E_x = A_x \cos(\omega t), \qquad E_y = A_y \cos(\omega t + \delta)$$

We now examine the electric field vector at ωt=π/2−δ, and ωt= π/2−δ + Δt. We obtain

ωt=π/2−δ: $\quad E_x = A_x \sin\delta, \qquad E_y = 0$

ωt= π/2−δ + Δt: $\quad E_x = A_x \sin(\delta - \Delta t), \qquad E_y = -A_y \sin(\Delta t)$

We see that the polarization revolve in a clockwise direction if $\sin\delta > 0$.

1.12 *Solution*:
(b) We find the inclination angle of the major axis of the polarization ellipse.

$$\tan 2\phi_1 = \frac{2\cos\psi\sin\psi}{\cos^2\psi - \sin^2\psi}\cos\delta = \tan 2\psi\cos\delta$$

$$\tan 2\phi_2 = \frac{2\sin\psi\cos\psi}{-\cos^2\psi + \sin^2\psi}\cos(\pi+\delta) = \tan 2\psi\cos\delta = \tan 2\phi_1$$

So, $2\phi_2 = 2\phi_1 + m\pi$, where m is an integer. In other words, the major axes are either parallel or perpendicular. To show that the major axes of the polarization ellipses of the two states are mutually orthogonal, examine some special cases (e.g., ψ=0 or δ=0) and calculate the length *a* for the two states.
The length of the major axes and minor axes can be calculated by using Eq. (1.6-14).

$$a^2 = A_x^2\cos^2\phi + A_y^2\sin^2\phi + 2A_xA_y\cos\delta\cos\phi\sin\phi$$

$$b^2 = A_x^2\sin^2\phi + A_y^2\cos^2\phi - 2A_xA_y\cos\delta\cos\phi\sin\phi$$

$$a_1^2 = \cos^2\psi\cos^2\phi + \sin^2\psi\sin^2\phi + 2\cos\psi\sin\psi\cos\delta\cos\phi\sin\phi$$

$$b_1^2 = \cos^2\psi\sin^2\phi + \sin^2\psi\cos^2\phi - 2\cos\psi\sin\psi\cos\delta\cos\phi\sin\phi$$

$$a_2^2 = \sin^2\psi\cos^2\phi + \cos^2\psi\sin^2\phi + 2\cos\psi\sin\psi\cos(\delta+\pi)\cos\phi\sin\phi$$

$$= \sin^2\psi\cos^2\phi + \cos^2\psi\sin^2\phi - 2\cos\psi\sin\psi\cos\delta\cos\phi\sin\phi$$

$$b_2^2 = \sin^2\psi\sin^2\phi + \cos^2\psi\cos^2\phi - 2\cos\psi\sin\psi\cos(\delta+\pi)\cos\phi\sin\phi$$

$$= \sin^2\psi\sin^2\phi + \cos^2\psi\cos^2\phi + 2\cos\psi\sin\psi\cos\delta\cos\phi\sin\phi$$

We note $a_1^2 = b_2^2$ and $a_2^2 = b_1^2$. Thus, the major axes are indeed orthogonal.
The senses of revolution are opposite since $\sin\delta\sin(\pi+\delta) = -\sin^2\delta < 0$.

1.13 *Solution*:

From Problem 1.12, we have

$\tan 2\phi = \tan 2\psi \cos\delta = \dfrac{2\tan\psi}{1-\tan^2\psi}\cos\delta = \dfrac{\mathrm{Re}[\chi]}{1-|\chi|^2}$. This is Eq. (1.6-18).

Using the definition $\tan\theta = \pm b/a$, we have $\sin 2\theta = 2\sin\theta\cos\theta = 2ab/(a^2+b^2)$.

Using $a^2b^2 = A_x^2A_y^2\sin^2\delta$ and $a^2+b^2 = A_x^2+A_y^2$ from Problem 1.10, we obtain

$\sin 2\theta = 2ab/(a^2+b^2) = -2A_xA_y\sin\delta/(A_x^2+A_y^2) = -2\,\mathrm{Im}[\chi]/(1+|\chi|^2)$

where the choice of " - " sign is consistent with the sense of revolution of the polarization ellipse.

1.14 *Solution*:

(a) Without loss of generality, we assume

$$\mathbf{A} = \begin{pmatrix} \cos\psi_a \\ e^{i\delta_a}\sin\psi_a \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \cos\psi_b \\ e^{i\delta_b}\sin\psi_b \end{pmatrix}$$

$\mathbf{A}^*\cdot\mathbf{B} = 0$ leads to $\cos\psi_a\cos\psi_b + \sin\psi_a\sin\psi_b\exp[i(\delta_b-\delta_a)] = 0$ which leads to $\delta_a-\delta_b = \pm\pi$ and $\cos(\psi_a+\psi_b) = 0$.

(b) $\delta_a\delta_b < 0$ follows immediately from the condition that $-\pi < \delta < \pi$. If one of the phases is π, then the other phase must be zero. This proves $\delta_a\delta_b \le 0$.

(c)

$$\chi_a^*\chi_b = \frac{\sin\psi_a\sin\psi_b}{\cos\psi_a\cos\psi_b}\exp[i(\delta_b-\delta_a)] = -\frac{\sin\psi_a\sin\psi_b}{\cos\psi_a\cos\psi_b} = -1$$

(d) From Eq. (1.6-18), and $\delta_a-\delta_b = \pm\pi$ and $\cos(\psi_a+\psi_b) = 0$,

$\tan 2\phi_a = \tan 2\psi_a\cos\delta_a$

$\tan 2\phi_b = \tan 2\psi_b\cos\delta_b = \tan 2(\pi/2-\psi_a)\cos(\delta_a-\pi) = \tan 2\psi_a\cos\delta_a = \tan 2\phi_a$

So, the major axes are either parallel or orthogonal. To show that the major axes of the polarization ellipses of the two states are mutually orthogonal, examine some special cases (e.g., ψ=0 or δ=0) and calculate the length *a* for the two states.

Follow the same approach used in Problem 1.12(b).

1.15 *Solution*:

In the principal coordinate, the polarization ellipse can be written

$E_{x'} = a\cos(\omega t)$

$E_{y'} = b\cos(\omega t \pm \pi/2) = b\sin(\omega t)$

We note that in the principal coordinate the two orthogonal polarization components are out of phase by π/2. Align the wave plate with a phase retardation of π/2 so that its slow axis (or fast axis) is parallel (or perpendicular) to the one of the principal axes of the polarization ellipse. The output is a linear polarization state.

1.16 *Solution*:

Without loss of generality, we assume

$$E_x = A_x \cos(\omega t - kz), \qquad E_y = A_y \cos(\omega t - kz + \delta)$$

At z=0, the temporal variation is written $E_x = A_x \cos(\omega t)$, $\qquad E_y = A_y \cos(\omega t + \delta)$

At t=0, the spatial variation is written $E_x = A_x \cos(-kz)$, $\qquad E_y = A_y \cos(-kz + \delta)$

A direct comparison shows that the spatial variation is equivalent to time-reversed variation. Thus, the E-vector of right-hand circular polarized light will appear left-handed in the space domain, and vice versa.

(a) Let $\mathbf{E} = \mathbf{R} + e^{i\delta}\mathbf{L}$, where δ is an arbitrary phase shift. The (x, y) components of the complex field amplitudes can be written

$$E_x = \frac{1}{\sqrt{2}}(1 + e^{i\delta})\exp[i(\omega t - kz)], \qquad E_y = \frac{1}{\sqrt{2}}(-i + ie^{i\delta})\exp[i(\omega t - kz)]$$

We now examine the ratio of the complex amplitudes,

$$\frac{(-i + ie^{i\delta})}{(1 + e^{i\delta})} = \frac{(-i + ie^{i\delta})(1 + e^{-i\delta})}{(1 + e^{i\delta})(1 + e^{-i\delta})} = \frac{-i + i + i(e^{i\delta} - e^{-i\delta})}{2 + 2\cos\delta} = \frac{-2\sin\delta}{2 + 2\cos\delta}$$

This is a real number. In other words, the (x, y) components are in phase. So, the resultant is a linearly polarized wave, regardless of the phase shift.

(b) Let a polarized wave be written

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = c_1\mathbf{E}_1 + c_2\mathbf{E}_2 = c_1\begin{pmatrix} a \\ ib \end{pmatrix} + c_2\begin{pmatrix} b \\ -ia \end{pmatrix}$$

where we assume that both *a* and *b* are real.

The constants c_1 and c_2 can be easily obtained by using the orthogonal property of the basis. They are given by

$$c_1 = \frac{a\alpha - ib\beta}{a^2 + b^2}, \qquad c_2 = \frac{b\alpha + ia\beta}{a^2 + b^2}$$

For a beam of linearly polarized light, α, β are real. So, both c_1 and c_2 are complex.

For a beam of right-hand circularly polarized light with $\alpha = 1/\sqrt{2}$, $\beta = -i/\sqrt{2}$, the expansion coefficients are

$$c_1 = \frac{1}{\sqrt{2}}(a - b), \qquad c_2 = \frac{1}{\sqrt{2}}(a + b)$$

We note that both c_1 and c_2 are real.

1.17 *Solution*:

(a) Let the circularly polarized state be written $\mathbf{E}_C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$, and the unpolarized state be written $\mathbf{E}_U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\delta} \end{pmatrix}$, where δ is a random phase. The projection along the transmission axis of a polarizer (oriented at azimuth angle ψ) is given by

$$\mathbf{p}\cdot\mathbf{E}_C = \frac{1}{\sqrt{2}}(\cos\psi - i\sin\psi), \quad \mathbf{p}\cdot\mathbf{E}_U = \frac{1}{\sqrt{2}}(\cos\psi + e^{i\delta}\sin\psi)$$

It follows that

$$|\mathbf{p}\cdot\mathbf{E}_C|^2 = 1/2 = \left\langle |\mathbf{p}\cdot\mathbf{E}_U|^2 \right\rangle = 1/2$$

where $\langle\ \rangle$ represents statistical average over the random phase δ.

Thus, a polarizer alone can not distinguish the difference between circularly polarized light and unpolarized light.

(b) Let the elliptically polarized state be written $\mathbf{E} = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ ib \end{pmatrix}$. The projection along the transmission axis of a polarizer (oriented at azimuth angle ψ) is given by

$|\mathbf{p}\cdot\mathbf{E}|^2 = \frac{1}{a^2+b^2}|a\cos\psi + ib\sin\psi|^2 = \frac{a^2\cos^2\psi + b^2\sin^2\psi}{a^2+b^2}$. A measurement of $|\mathbf{p}\cdot\mathbf{E}|^2$ as a function of the azimuth angle ψ yields the major axis and the minor axis of the ellipse, provided $a \neq b$.

It is important to note, a beam of partially polarized light can yield similar result.

1.18 *Solution:*

(a) Using the orthogonal relation, $\mathbf{E}_2^* \cdot \mathbf{E}_1 = 0$, we can obtain

$c_1 = (a\cos\psi - ib\sin\psi)/(a^2+b^2)$, $c_2 = (b\cos\psi + ia\sin\psi)/(a^2+b^2)$

(b) We write a general linearly polarized light as:

$$\mathbf{E}_0 = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} = \frac{1}{2}e^{i\psi}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1}{2}e^{-i\psi}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

After propagating through the optically active medium, the polarization state becomes

$$\mathbf{E}_L = \frac{1}{2}e^{i\psi}\begin{pmatrix} 1 \\ -i \end{pmatrix}e^{i\alpha} + \frac{1}{2}e^{-i\psi}\begin{pmatrix} 1 \\ i \end{pmatrix}e^{-i\alpha} = \begin{pmatrix} \cos(\psi+\alpha) \\ \sin(\psi+\alpha) \end{pmatrix}$$

We note that the polarization rotates an angle of α, where

$$\alpha = \frac{\pi}{\lambda}(n_r - n_l)L$$

We can also write a general elliptical polarization state as a sum of two orthogonal linear polarization states:

$$\mathbf{E} = \begin{pmatrix} a \\ ib \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + ib\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

After propagation through the medium, each linearly polarized basis is rotated by an angle α

$$\mathbf{E}' = a\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + ib\begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

These two rotated bases remain the principal axes of the ellipse, as the phase shift between then remains π/2.

So, the major axis is rotated by an angle of α.

1.19 *Solution:*

(a) Using table 1.5 with Γ=π

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} -i\cos 2\psi & -i\sin 2\psi \\ -i\sin 2\psi & i\cos 2\psi \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i\sin 2\psi \\ i\cos 2\psi \end{pmatrix} = \begin{pmatrix} -\sin 2\psi \\ \cos 2\psi \end{pmatrix}$$

We note the polarization state is rotated by an angle of 2ψ.

(b)

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} -i\cos 2\psi & -i\sin 2\psi \\ -i\sin 2\psi & i\cos 2\psi \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin 2\psi - i\cos 2\psi \\ \cos 2\psi - i\sin 2\psi \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i\exp(i2\psi) \\ \exp(i2\psi) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\exp(i2\psi)$$

We note the output polarization state is LHC, regardless of the azimuth angle ψ.

(c) $d = \dfrac{\lambda}{2\,|n_e - n_o|} = 1254\ \text{nm}$, or odd integral multiples.

1.20 *Solution:*

(a) Using table 1.5

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi & -i\sin(\Gamma/2)\sin(2\psi) \\ -i\sin(\Gamma/2)\sin(2\psi) & e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i\sin(\Gamma/2)\sin(2\psi) \\ e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix} = \begin{pmatrix} -i\sin 2\psi \\ 1+i\cos 2\psi \end{pmatrix}$$

(b) By definition, $\chi = \dfrac{1+i\cos 2\psi}{-i\sin 2\psi} \equiv x + iy$. This leads to $x = -\cos 2\psi / \sin 2\psi$, $y = 1/\sin 2\psi$, and then $y^2 - x^2 = 1$ which is exactly a hyperbola. When ψ varies from 0 to $\pi/2$, y remains positive, so the locus is the upper branch of the hyperbola.

(c) $d = \dfrac{\lambda}{4\,|n_e - n_o|} = 1841\ \text{nm}$, or odd integral multiples.

1.21 *Solution:*

(a) Using table 1.5

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi & -i\sin(\Gamma/2)\sin(2\psi) \\ -i\sin(\Gamma/2)\sin(2\psi) & e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i\sin(\Gamma/2)\sin(2\psi) \\ e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix}$$

$$= \begin{pmatrix} -i\sin(\Gamma/2)\sin(2\psi) \\ \cos(\Gamma/2) + i\sin(\Gamma/2)\cos 2\psi \end{pmatrix}$$

(b), (c), (d)

By definition, $\chi = \dfrac{\cos(\Gamma/2) + i\sin(\Gamma/2)\cos 2\psi}{-i\sin(\Gamma/2)\sin 2\psi} \equiv x + iy$. This leads to $x = -\cos 2\psi/\sin 2\psi$, $y = 1/[\tan(\Gamma/2)\sin 2\psi]$, and then $y^2 - \tan^2(\Gamma/2)x^2 = 1$ which is exactly a hyperbola. When ψ varies from 0 to $\pi/2$, and Γ varies from 0 to 2π, the points (x, y) cover the entire complex plane.

(e) Using Eq. (1.9-11), the matrix is written $W = R(-\psi)W_0R(\psi)$, the Hermitian coniugate can be written

$W^\dagger = [R(-\psi)W_0R(\psi)]^\dagger = R(\psi)^\dagger W_0{}^\dagger R(-\psi)^\dagger = R(-\psi)W_0{}^\dagger R(\psi)$. Thus,

$W^\dagger W = R(-\psi)W_0{}^\dagger R(\psi)R(-\psi)W_0R(\psi) = R(-\psi)W_0{}^\dagger W_0R(\psi) = R(-\psi)R(\psi) = I$

(f) $\mathbf{V}_1^*{}'\cdot\mathbf{V}_2{}' = (W\mathbf{V}_1)^\dagger \cdot (W\mathbf{V}_2) = \mathbf{V}_1^* \cdot (W^\dagger W)\mathbf{V}_2 = \mathbf{V}_1^* \cdot \mathbf{V}_2$

Scalar product is invariant under unitary transformation.

1.22 *Solution:*

(a)

$$P_0^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_0,$$

$$P^2 = R(-\psi)P_0R(\psi)R(-\psi)P_0R(\psi) = R(-\psi)P_0P_0R(\psi) = R(-\psi)P_0R(\psi) = P$$

Using Dirac notation of linear algebra, a projection operator can be written $P = |p\rangle\langle p|$

(b) The transmitted state through a polarizer is obtained by operating the projection operator on the input polarization state. Thus we have

$$|E_t\rangle = P|E_i\rangle = |p\rangle\langle p|E_i\rangle$$

(c) The amplitude of transmission is given by

$\langle x|u\rangle\langle u|y\rangle = \cos\psi\cos\psi$, where $\psi = 45^\circ$.

(d) The transmitted amplitude is given by

$$\langle x|u_N\rangle\langle u_N|u_{N-1}\rangle\cdots\langle u_4|u_3\rangle\langle u_2|u_1\rangle\langle u_1|y\rangle = \cos\psi\cos\psi\cos\psi\cdots\cos\psi\cos\psi = (\cos\psi)^N$$

where $\psi=\pi/(2N)$.

For large N, $\pi/(2N) \ll 1$,

$$\cos\left(\frac{\pi}{2N}\right) \approx 1 - \frac{1}{2}\left(\frac{\pi}{2N}\right)^2$$

Using

$$\lim_{N\to\infty}\left(1-\frac{x}{N}\right)^N = \exp(-x)$$

we obtain

$$\lim_{N\to\infty}\left[\cos\left(\frac{\pi}{2N}\right)\right]^N = \lim_{N\to\infty}\left[1-\frac{1}{2}\left(\frac{\pi}{2N}\right)^2\right]^N = \lim_{N\to\infty}\left[1-\frac{1}{N}\frac{\pi^2}{8N}\right]^N = \lim_{N\to\infty}\exp\left(-\frac{\pi^2}{8N}\right) = 1$$

1.23 *Solution:*

(a) Using Eq. (1.9-39), the transmission of unpolarized light through the first stage (polarizer, wave plate, polarizer) is

$T=\frac{1}{2}\cos^2 x$. The transmission of polarized light through later stages is then $T=\cos^2 2^m x$, m=1, 2, 3, .. N-1. This leads to the overall transmission.

(b) Using $\cos\theta=(e^{i\theta}+e^{-i\theta})/2$, the transmission can be written

$$T=\frac{1}{2^{2N+1}}\left(e^{ix}+e^{-ix}\right)^2\left(e^{i2x}+e^{-i2x}\right)^2\left(e^{i4x}+e^{-i4x}\right)^2\ldots\left(e^{i2^{N-2}x}+e^{-i2^{N-2}x}\right)^2\left(e^{i2^{N-1}x}+e^{-i2^{N-1}x}\right)^2.$$

Carrying out the multiplications, we obtain

$$T=\frac{1}{2^{2N+1}}\left(e^{i(2^N-1)x}+e^{i(2^N-3)x}+e^{i(2^N-5)x}+.....+e^{-i(2^N-7)x}+e^{-i(2^N-5)x}+e^{-i(2^N-3)x}+e^{-i(2^N-1)x}\right)^2$$

Notice that the left side is a geometric series. Thus, we obtain

$$T=\frac{1}{2^{2N+1}}\left(\frac{e^{i(2^N-1)x}-e^{-i(2^N+1)x}}{1-e^{-i2x}}\right)^2=\frac{1}{2^{2N+1}}\left(\frac{\sin 2^N x}{\sin x}\right)^2$$

(c) For the thin plate, the transmission spectrum is $\cos^2 x$. The separation between peaks is $\Delta x=\pi$, which corresponds $\Delta\nu=\frac{c}{d(n_e-n_o)}$. The FWHM of each peak is $\Delta x_{1/2}=\pi/2$ and $\Delta\nu_{1/2}=\frac{c}{2d(n_e-n_o)}$. For the thickest plate the transmission spectrum is $\cos^2 2^{N-1}x$. The FWHM of each peak is $\Delta x_{1/2}=\pi/(2^N)$, which corresponds to $\Delta\nu_{1/2}=\frac{c}{2^N d(n_e-n_o)}$.

So, the overall transmission spectrum consists of peaks separated at $\Delta\nu=\frac{c}{d(n_e-n_o)}$, with the FWHM of each peak given by $\Delta\nu_{1/2}=\frac{c}{2^N d(n_e-n_o)}$. The finesse is thus $F\sim 2^N$.

(d) Using $\Delta\nu_{1/2}=\frac{c}{2^N d(n_e-n_o)}=\frac{c}{2D(n_e-n_o)}$, where D is the thickness of the thickest plate, we obtain

$$D=\frac{c}{2\Delta\nu_{1/2}(n_e-n_o)}=\frac{\lambda c}{2\Delta\lambda_{1/2}(n_e-n_o)\nu}=\frac{\lambda^2}{2\Delta\lambda_{1/2}(n_e-n_o)}=\frac{(6563)^2}{2(1.5506-1.5416)}\text{Angstrom}=24$$

cm

(e) The spectra feature of the function

$f(x)=\frac{\sin Mx}{\sin x}$, also appears in grating diffraction, is dominated by the numerator when M >>1. The function is periodic with a period of 2π, and peak values of $f(0)=M$. The function drops to zero at $x=\pi/(M)$. At $x=\pi/(2M)$, the function is approximately $f=2M/\pi$, with $f^2=0.405M$. At $x=0.886\pi/(2M)$, $f^2=0.5M$.

1.24 *Solution:*

Using Table 1.7, the Jones matrix is given by (with $\psi=\theta$)

$$\begin{pmatrix} e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi & -i\sin(\Gamma/2)\sin(2\psi) \\ -i\sin(\Gamma/2)\sin(2\psi) & e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix}$$

With crossed polarizers, the transmission is given by

$$T = |M_{12}|^2/2$$

1.25 *Solution:*

(a) Using Table 1.7, the Jones matrix for the wave plate followed by a rotator is given by

$$M = \begin{pmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{pmatrix}\begin{pmatrix} e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi & -i\sin(\Gamma/2)\sin(2\psi) \\ -i\sin(\Gamma/2)\sin(2\psi) & e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix} \equiv RW \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Carrying out the matrix multiplication, we obtain

$$M_{11} = \cos\rho[e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi] + i\sin\rho\sin(\Gamma/2)\sin(2\psi)$$

$$M_{12} = -\sin\rho[e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi] - i\cos\rho\sin(\Gamma/2)\sin(2\psi)$$

$$M_{21} = \sin\rho[e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi] - i\cos\rho\sin(\Gamma/2)\sin(2\psi)$$

$$M_{22} = \cos\rho[e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi] - i\sin\rho\sin(\Gamma/2)\sin(2\psi)$$

To show unitary property, we examine

$$M^\dagger M = (RW)^\dagger(RW) = W^\dagger R^\dagger RW = W^\dagger IW = W^\dagger W = I$$

(b) $a = \mathrm{Re}[M_{11}] = \cos(\Gamma/2)\cos\rho$,

$b = \mathrm{Im}[M_{11}] = \sin\rho\sin(\Gamma/2)\sin 2\psi - \cos\rho\sin(\Gamma/2)\cos 2\psi = -\cos(\rho+2\psi)\sin(\Gamma/2)$

$c = \mathrm{Re}[M_{12}] = -\cos(\Gamma/2)\sin\rho$,

$d = \mathrm{Im}[M_{12}] = -\cos\rho\sin(\Gamma/2)\sin 2\psi - \sin\rho\sin(\Gamma/2)\cos 2\psi = -\sin(\rho+2\psi)\sin(\Gamma/2)$

Thus, we obtain

$$\cos^2\frac{\Gamma}{2} = a^2 + c^2 \qquad \sin^2\frac{\Gamma}{2} = b^2 + d^2$$

$$\tan(\rho+2\psi) = \frac{d}{b} \qquad \tan\rho = -\frac{c}{a}$$

1.26 *Solution:*

(a) Using the Jones matrix method, the output state can be written

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{+i\Gamma/2} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} e^{-i\Gamma/2}\cos\theta \\ e^{+i\Gamma/2}\sin\theta \end{pmatrix}$$

We now examine the Stokes parameter of this state of polarization.

$S_1 = \cos 2\theta$, $S_2 = \sin 2\theta \cos\Gamma$, $S_3 = \sin 2\theta \sin\Gamma$

We now keep Γ fixed and let θ vary from 0 to π. The points (S_1, S_2, S_3) form a circle on the plane defined by $S_3 / S_2 = \tan\Gamma$

This is a great circle formed by the intersection of the Poincare sphere(a unit circle) with the plane $S_3 / S_2 = \tan\Gamma$.

If we rotate the equatorial plane by an angle Γ around S_1-axis, we obtain the same great circle.

We now keep θ fixed and let Γ vary from 0 to 2π. The points (S_1, S_2, S_3) form a circle around S_1-axis with S_1 fixed at $\cos 2\theta$.

(b) If the wave plate is oriented at an azimuth angle ψ and the input linear polarization state maintains the same angle θ relative to the c-axis (slow axis) of the wave plate, then the output polarization ellipse can be obtained from the case of ψ=0 by a rotation of an angle of ψ. The rotation of a polarization ellipse by an angle ψ can be represented by the rotator matrix described in Problem 1.25. On the Poincare sphere, the effect of a rotator by an angle ψ in the xy-plane is a rotation around the polar axis (S_3-axis) by an angle of 2ψ. This is proven as follows. Let the rotated state be written

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{pmatrix} \begin{pmatrix} a \\ be^{i\delta} \end{pmatrix} = \begin{pmatrix} a\cos\rho - be^{i\delta}\sin\rho \\ a\sin\rho + be^{i\delta}\cos\rho \end{pmatrix}$$

where ρ is the angle of rotation. The Stokes parameters of the state before rotation is

$S_1 = a^2 - b^2, \quad S_2 = 2ab\cos\delta, \quad S_3 = 2ab\sin\delta$

whereas those of the state after the rotation is given by

$$S_1' = |V_x|^2 - |V_y|^2, \quad S_2' = V_x V_y^* + V_x^* V_y, \quad S_3' = i(V_x V_y^* - V_x^* V_y)$$

$$S_1' = (a\cos\rho - b\sin\rho\cos\delta)^2 + (b\sin\rho\sin\delta)^2 - (a\sin\rho + b\cos\rho\cos\delta)^2 - (b\cos\rho\sin\delta)^2$$

$$= (a^2 - b^2)\cos 2\rho - 2ab\cos\delta\sin 2\rho = S_1\cos 2\rho - S_2\sin 2\rho$$

$$S_2' = (a\cos\rho - be^{i\delta}\sin\rho)(a\sin\rho + be^{i\delta}\cos\rho)^* + (a\cos\rho - be^{i\delta}\sin\rho)^*(a\sin\rho + be^{i\delta}\cos\rho)$$

$$= 2(a\cos\rho - b\sin\rho\cos\delta)(a\sin\rho + b\cos\rho\cos\delta) - 2b^2\sin\rho\cos\rho\sin^2\delta$$

$$= (a^2 - b^2)\sin 2\rho + 2ab\cos\delta\cos 2\rho = S_1\sin 2\rho + S_2\cos 2\rho$$

$$S_3' = i(a\cos\rho - be^{i\delta}\sin\rho)(a\sin\rho + be^{i\delta}\cos\rho)^* - i(a\cos\rho - be^{i\delta}\sin\rho)^*(a\sin\rho + be^{i\delta}\cos\rho)$$

$$= 2ab\sin\delta = S_3$$

We note that (S_1', S_2', S_3') is obtained by a rotation of (S_1, S_2, S_3) by an angle 2ρ around the polar axis (S_3-axis). A great circle remains a great circle after the rotation.

(c) According to the results from Problem 1.25(b), a general birefringent network is equivalent to a wave plate followed by a rotator. If the input linear state is parallel to the slow (or fast) axis of the wave plate, then the output state will remain linear after transmitting through the wave plate. The rotator mere rotates the linear state by an angle ρ.

1.27 *Solution:*

From the basics of eigenvalue problem in linear algebra, the eigenvectors of the following equation

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

can be written (in terms of row vectors)

$$(x \quad y \quad z) = \left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right), \qquad \text{or} \qquad \left(\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \quad \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \right), \qquad \text{or}$$

$$\left(\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} \quad \begin{vmatrix} c_3 & a_3 \\ c_1 & a_1 \end{vmatrix} \quad \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} \right)$$

From Eq. (1.7-9), $\begin{pmatrix} \omega^2\mu\varepsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2\mu\varepsilon_z - k_x^2 - k_y^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$

Note, we are interested in the direction of the eigenvectors. So, for simplicity, we evaluate the ratios of the components. This avoids having to deal with terms involving ω^4. Thus

$$x : y = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \qquad y : z = \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} : \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

From Eq. (1.7-9), we have

$$E_x : E_y = \begin{vmatrix} k_x k_y & k_x k_z \\ \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 & k_y k_z \end{vmatrix} : \begin{vmatrix} k_x k_z & \omega^2\mu\varepsilon_x - k_y^2 - k_z^2 \\ k_y k_z & k_y k_x \end{vmatrix} = k_x k_z (k^2 - \omega^2\mu\varepsilon_y) : k_y k_z (k^2 - \omega^2\mu\varepsilon_x)$$

$$E_y : E_z = \begin{vmatrix} k_y k_z & k_y k_x \\ \omega^2\mu\varepsilon_z - k_x^2 - k_y^2 & k_z k_x \end{vmatrix} : \begin{vmatrix} k_y k_x & \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 \\ k_z k_x & k_z k_y \end{vmatrix} = k_y k_x (k^2 - \omega^2\mu\varepsilon_z) : k_z k_x (k^2 - \omega^2\mu\varepsilon_y)$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. From the above two equations, we obtain

$$(E_x \quad E_y \quad E_z) = \left(\frac{k_x}{(k^2 - \omega^2\mu\varepsilon_x)} \quad \frac{k_y}{(k^2 - \omega^2\mu\varepsilon_y)} \quad \frac{k_z}{(k^2 - \omega^2\mu\varepsilon_z)} \right)$$

If we define $\mathbf{k} = ns\omega / c$, then

$$(E_x \quad E_y \quad E_z) = \left(\frac{s_x}{(n^2 - \varepsilon_x / \varepsilon_0)} \quad \frac{s_y}{(n^2 - \varepsilon_y / \varepsilon_0)} \quad \frac{s_z}{(n^2 - \varepsilon_z / \varepsilon_0)} \right)$$

(b) Using $\mathbf{D} = \varepsilon\mathbf{E}$, we have

$$(D_x \quad D_y \quad D_z) = \left(\frac{\varepsilon_x s_x}{(n^2 - \varepsilon_x / \varepsilon_0)} \quad \frac{\varepsilon_y s_y}{(n^2 - \varepsilon_y / \varepsilon_0)} \quad \frac{\varepsilon_z s_z}{(n^2 - \varepsilon_z / \varepsilon_0)} \right)$$