# Chapter Problems

## Notes on Maple and Mathematica commands

Several solutions in this manual have added notes on how to set up expressions under the symbolic mathematics programs Maple (Waterloo Maple, Inc.) and Mathematica (Wolfram Research) to assist in solving the problem. Please note, however, that a symbolic math program is useful only after we have overcome the conceptual challenges of the problem. Where these programs have their greatest use is when we need to solve simultaneous or transcendental equations, or obtain numerical values or algebraic expressions for integrals. We still have to know how to set up an integral, and we have to decide where and how to apply approximations (which turns out to be more important than one might think). Once we have taken these steps, the program may be able to take over.

Even then, however, the math in the majority of our problems is limited to relatively straightforward algebraic manipulations, however complex the concepts may be. For most of the cases where the Maple and Mathematica syntax is not shown, little would differ from the written mathematics already appearing in the solution. This is largely the goal of symbolic math programs in the first place: to accept input in the form that one would write the problem on paper.

The following problems in this volume give specific commands for use with Maple or Mathematica:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A.10 | A.18 | 1.12 | 1.13 | 2.11 | 2.15 | 3.12 | 3.16 |
| 3.22 | 3.27 | 3.38 | 3.49 | 3.50 | 4.34 | 5.21 | 5.45 |
| 6.5 | 7.9 | 7.36 | 9.56 | 10.34 | | | |

## Chapter A

**A.1** This problem uses a common manipulation, one of the features of logarithms that makes them so useful:

$$\begin{aligned} \mathrm{pK_a} = -\log_{10} K_\mathrm{a} &= -\log_{10}\ e^{-\Delta G/(RT)} \\ &= -\left[-\frac{\Delta G}{RT}\right]\ \log_{10}\ e \qquad\qquad \log x^a = a\log x \\ &= \frac{\Delta G}{RT}(0.434). \end{aligned}$$

This shows, if you don't mind us getting ahead of ourselves a little, that the $\mathrm{pK_a}$ is directly proportional to the free energy of dissociation, $\Delta G$, and inversely proportional to the temperature, $T$.

**A.2** The idea here is that, even if we think at first we have no idea what the number ought to be, a closer look at the available choices makes it clear that we can spot some potentially ridiculous answers:

a. $2 \cdot 10^{10}\,\mathrm{m\,s^{-1}}$ is faster than the speed of light.

b. $2 \cdot 10^5\,\mathrm{m\,s^{-1}}$ has no obvious objections.

c. $2\,\mathrm{m\,s^{-1}}$ is the speed of a slow walk, and would imply, for example, that you could send an e-mail message over a cable connection to a friend half a mile away, and then run the half-mile to arrive and deliver the message in person before the e-mail finishes traveling through the wires.

When we have calculations that toss around factors of $10^{-34}$, for one example, this is a significant skill. The correct answer is $\boxed{2 \cdot 10^5\,\mathrm{m\,s^{-1}}.}$

**A.3** The volume is roughly $125\,\text{Å}^3$, which we can show is not big enough to hold more than about 15 atoms. Chemical bonds, formed between *overlapping* atoms, are roughly 1 Å long, and so typical atomic diameters are roughly 2 Å or more, and occupy a volume on the order of $(2\,\text{Å})^3 = 8\,\text{Å}^3$. A volume of $125\,\text{Å}^3$, therefore, cannot hold more than about $125/8 = 15.6$ atoms. Among the choices, the only reasonable value is $\boxed{8.}$

**A.4** a. Chemical bond lengths in molecules are always in the range 0.6–4.0 Å, or $0.6 \cdot 10^{-10}$ to $4.0 \cdot 10^{-10}$ m. $25 \cdot 10^{-8}$ m is much too large for a bond length. $\boxed{\text{no}}$

b. Six carbon atoms have a mass of $6 \cdot 12 = 72$ amu. With the added mass of a few hydrogen atoms at 1 amu each, 78 amu is a reasonable value. $\boxed{\text{yes}}$

**A.5** a. The derivatives $d[\mathrm{A}]$ and $dt$ have the same units as the parameters [A] and $t$, respectively. Both sides of the equation should therefore have units of $\mathrm{mol\,L^{-1}\,s^{-1}}$. That means that $k$ needs to provide the units of $\mathrm{s^{-1}}$ and cancel one factor of concentration units on the righthand side. $k$ has units of $\boxed{\mathrm{L\,s^{-1}\,mol^{-1}}.}$

b. The argument of the exponential function must be unitless, so $k_\mathrm{B}$ must cancel units of energy (J) in the numerator and temperature (K) in the denominator. The correct units are $\boxed{\mathrm{J\,K^{-1}}.}$

c. The units all cancel, and $K_\mathrm{eq}$ is $\boxed{\text{unitless.}}$

d. Squaring both sides of the equation, we can solve for $k$: $\mu\omega^2 = k$. $k$ must therefore have units of $\boxed{\mathrm{kg\,s^{-2}}.}$

**A.6** There are two factors on the lefthand side, $(2x+1)^2$ and $e^{-ax^2}$. For the product to be zero, at least one of these factors must be zero. If $(2x+1) = 0$, then $\boxed{x = -\frac{1}{2}\ .}$ If $e^{-ax^2} = 0$, then $\boxed{x \to\ \pm\ \infty\ .}$ All three are valid solutions.

**A.7** In general, for any complex number $(a + ib)$, the complex conjugate is $(a + b)^* = a - ib$. We look for the imaginary component and and invert its sign:

a. $x - iy : a = x \quad b = -y,\quad \boxed{x\ +\ iy.}$

b. $ix^2y^2 : a = 0 \quad b = x^2y^2,\quad \boxed{-ix^2y^2\ .}$

c. $xy(x + iy + z) :\ a = x^2y + xyz \quad b = xy^2,\ \ x^2y + xyz - ixy^2,\quad \boxed{xy(x - iy + z).}$

d. $a = x/z \quad b = y/z,\quad \boxed{(x - iy)/z\ .}$

e.

$$\begin{aligned} e^{ix} &= 1 + ix - x^2 - ix^3 + x^4 + ix^5 - \ldots \\ a &= 1 - x^2 + x^4 - \ldots \\ b &= x - x^3 + x^5 - \ldots \\ a - ib &= 1 - ix - x^2 + ix^3 + x^4 - ix^5 - \ldots = \boxed{e^{-ix}.} \end{aligned}$$

f. 54.3: $a = 54.3 \quad b = 0,$ $\boxed{54.3.}$

**A.8** This problem tests a few algebraic operations involving vectors, particularly useful to know when we look at angular momentum and (often related) magnetic field effects.

a. The length of a vector is calculated using the Pythagorean theorem: $|\vec{C}| = \sqrt{0^2 + 2^2 + 1^2} = \boxed{\sqrt{5}\ .}$

b. We add vectors one coordinate at a time: $\vec{A} + \vec{B} = (1+1, 0+0, 0+1) = \boxed{(2,\ 0,\ 1).}$

c. The dot product of two vectors multiplies the values for each coordinate of the two vectors and sums the results: $\vec{A} \cdot \vec{B} = (1 \cdot 1) + (0 \cdot 0) + (0 \cdot 1) = \boxed{1.}$

d. In the case of perpendicular vectors, this gives us zero: $\vec{A} \cdot \vec{C} = (1 \cdot 0) + (0 \cdot 2) + (0 \cdot 1) = \boxed{0.}$

e. The cross product involves a little more work, and yields a new vector, perpendicular to the two original vectors: $\vec{A} \times \vec{B} = (0 \cdot 1 - 0 \cdot 0, 0 \cdot 1 - 1 \cdot 1, 1 \cdot 0 - 0 \cdot 1) = \boxed{(0, -1, 0).}$

**A.9** If we accept that the Taylor series expansion is exact if we take it to infinite order, then the Euler formula can be proven by the expansions of $e^x$ (Eq. A.25), $\sin x$ (Eq. A.26), and $\cos x$ (Eq. A.27):

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{1}{n!}(ix)^n \\ &= 1 + ix - \tfrac{1}{2}x^2 - \frac{i}{6}x^3 + \frac{1}{24}x^4 + \frac{i}{120}x^5 - \ldots \\ &= (1 - \tfrac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots) + i(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots) \\ &= \boxed{\cos x + i \sin x\ .} \end{aligned}$$

This equation is of practical importance to us, and is famous among mathematicians for tying together three fundamental mathematical values—$\pi$, $i$, and $e$—in one equation:

$$e^{i\pi} = 1.$$

**A.10**

- **Maple**: We can have Maple solve the equation

$$\frac{\left(P - \frac{a}{V_{\mathrm{m}}^2}\right)(V_{\mathrm{m}} - b)}{RT} = 1$$

directly using the **solve** command. After checking that all of the units are indeed consistent, enter the Maple command
**solve((1.000-(3.716/V^2))*(V-0.0408)/(0.083145*298.15)=1,V);**
The resulting solution, 24.98, is in the same units as $b$, so our final value to three significant digits is $\boxed{25.0\ \mathrm{L\ mol^{-1}}.}$

- **Mathematica**: In Mathematica we use the **Solve** command to find the value of a variable in an algebraic equation:
**Solve[(1.000 - (3.716/V^2))*(V - 0.0408)/(0.083145*298.15) == 1, V]**
Note that we need to use a double equals sign in the command. This command gives 24.979, which rounds to $\boxed{25.0\ \mathrm{L\ mol^{-1}}.}$

- **Successive approximation**: There are several ways to solve this, corresponding to different forms of the equation that leaves $V_{\rm m}$ on one side. One way to set up the equation quickly is to recognize that $(V_{\rm m} - b)$ will vary rapidly compared to $P - (a/V_{\rm m}^2)$, so we can isolate $V_{\rm m}$ as follows:

$$
\begin{aligned}
\frac{\left(P - \frac{a}{V_{\rm m}^2}\right)(V_{\rm m} - b)}{RT} &= 1 \\
\left(P - \frac{a}{V_{\rm m}^2}\right)(V_{\rm m} - b) &= RT \\
V_{\rm m} - b &= \frac{RT}{\left(P - \frac{a}{V_{\rm m}^2}\right)} \\
V_{\rm m} &= \frac{RT}{\left(P - \frac{a}{V_{\rm m}^2}\right)} + b.
\end{aligned}
$$

Substituting in the values for $P$, $a$, $b$, $R$, and $T$ (making sure that the units are all compatible), we can reduce the equation to the following:

$$V_{\rm m}(\,{\rm L\,mol^{-1}}) = \frac{(0.083145)(298.15)}{1 - \frac{3.716}{V_{\rm m}^2}} + 0.0408 = \frac{24.790}{1 - \frac{3.716}{V_{\rm m}^2}} + 0.0408.$$

Guessing an initial value of $1\,{\rm L\,mol^{-1}}$ yields the following series of approximations:

$$
\begin{aligned}
V_{\rm m} &= \frac{24.790}{1 + \frac{3.716}{1^2}} + 0.0408 = -9.0864 \\
V_{\rm m} &= \frac{24.790}{1 + \frac{3.716}{5.330^2}} + 0.0408 = 25.999 \\
V_{\rm m} &= \frac{24.790}{1 + \frac{3.716}{22.099^2}} + 0.0408 = 24.967 \\
V_{\rm m} &= \frac{24.790}{1 + \frac{3.716}{24.796^2}} + 0.0408 = 24.979 \\
V_{\rm m} &= \frac{24.790}{1 + \frac{3.716}{24.834^2}} + 0.0408 = 24.979.
\end{aligned}
$$

The series has converged to the three significant digits requested. The final value for $V_{\rm m}$ is $\boxed{25.0\ {\rm L\ mol^{-1}}.}$

**A.11** Here we apply the rules of differentiation summarized in Table A.3.

a.

$$
\begin{aligned}
f(x) &= (x+1)^{1/2} \\
\frac{df}{dx} &= \tfrac{1}{2}(x+1)^{-1/2}.
\end{aligned}
$$

b.

$$
\begin{aligned}
f(x) &= [x/(x+1)]^{1/2} \\
\frac{df}{dx} &= \tfrac{1}{2}\left(\frac{x}{x+1}\right)^{-1/2}\left[\frac{1}{x+1}\,\frac{dx}{dx} + x\frac{d}{dx}\left(\frac{1}{x+1}\right)\right] \\
&= \tfrac{1}{2}\left(\frac{x}{x+1}\right)^{-1/2}\left[\frac{1}{x+1} - \frac{x}{(x+1)^2}\right].
\end{aligned}
$$

c.

$$\frac{df}{dx} = \exp\left[x^{1/2}\right] \quad \frac{d}{dx}\left(x^{1/2}\right)$$
$$= \tfrac{1}{2}x^{-1/2}\exp\left[x^{1/2}\right].$$

d.

$$\frac{df}{dx} = \exp\left[\cos\ x^2\right]\frac{d}{dx}(\cos\ x^2)$$
$$= \exp\left[\cos\ x^2\right](-\sin\ x^2)\frac{d}{dx}(x^2)$$
$$= -2x\ \sin\ x^2\ \exp\left[\cos\ x^2\right].$$

**A.12** This problem tests our ability to use a few of the analytic integration results given in Table A.5.

a. $\int_0^\infty\ e^{-ax}dx = -\frac{1}{a}e^{-ax}|_0^\infty = -\frac{1}{a}(0-1) = \boxed{\frac{1}{a}.}$

b. $\int_1^5\ x^2 dx = \frac{1}{3}x^3|_1^5 = \frac{1}{3}(125-1) = \boxed{\frac{124}{3}.}$

c. $\int_1^5\ x^{-3/2}dx = -2x^{-1/2}|_1^5 = \boxed{-2(\frac{1}{\sqrt{5}}-1).}$

d. $r^2\int_0^{2\pi}\ d\phi\ \int_0^\pi\ \sin\theta d\theta = r^2\ (\phi)|_0^{2\pi}\ (-\cos\theta)|_0^\pi = r^2(2\pi-0)[-(-1)-(-1)] = \boxed{4\pi r^2.}$

**A.13** We use the Coulomb force law, Eq. A.41, using the charge of the electron $-e$ for both charges and $r_{12}$ set to 1.00 Å:

$$F_{\text{Coulomb}} = \frac{e^2}{4\pi\epsilon_0 r^2}$$
$$= \frac{(1.602\cdot 10^{-19}\,\text{C})^2}{(1.113\cdot 10^{-10}\,\text{C}^2\,\text{J}^{-1}\,\text{m}^{-1})(1.00\,\text{Å})^2(10^{-10}\,\text{m}\,\text{Å}^{-1})^2} = \boxed{2.31\cdot 10^{-8}\,\text{N}\ .}$$

**A.14** This problem relies on the definitions of the linear momentum $p$ and the kinetic energy $K$ (Eq. A.36):

$$p = mv$$
$$K = \tfrac{1}{2}mv^2 = \boxed{\frac{p^2}{2m}.}$$

**A.15** We're calling the altitude $r$. Because the acceleration is downward but $r$ increases in the upward direction, the acceleration is negative: $-9.80\,\text{m}\,\text{s}^{-2}$. We invoke the relationship between force and the potential energy, and find that we have to solve an integral:

$$U(r) = -\int_0^r F(r')\,dr' = -\int_0^r (-mg)\,dr' = \boxed{mgr.}$$

**A.16**

$$|F_{\text{Coulomb}}| = \frac{e^2}{4\pi\epsilon_0 r^2} = \frac{(1.602 \cdot 10^{-19}\,\text{C})^2}{(1.113 \cdot 10^{-10}\,\text{C}^2\,\text{J}^{-1}\,\text{m}^{-1})(0.529\,\text{Å})^2(10^{-10}\,\text{m}\,\text{Å}^{-1})^2}$$
$$= \boxed{8.23 \cdot 10^{-8}\,\text{N}}$$
$$|F_{\text{gravity}}| = m_H g$$
$$= (1.008\,\text{amu})(1.661 \cdot 10^{-27}\,\text{kg}\,\text{amu}^{-1})\left(9.80\,\text{m}\,\text{s}^2\right)$$
$$= \boxed{1.64 \cdot 10^{-26}\,\text{N.}}$$

Sure enough, the gravitational force is smaller than the Coulomb force by orders of magnitude, and the motions of these particles will be dictated—as well as we can measure them—exclusively by the Coulomb force.

**A.17** We are proving an equation that depends on $L$ and $x$ and $t$ and $v_x$, which may look like too many variables. If we use the definition of $L$ to put this equation in terms of $K$ and $U$, then we can at least put $K$ in terms of speed. Then, because speed itself is a function of position and time, the number of variables is quite manageable. Nonetheless, keeping things in terms of $K$ and $U$ is useful, because of their straightforward dependence on only $v$ and $x$, respectively.

To prove the equation, we could try working from both sides and seeing if the results meet in the middle. First the lefthand side:

$$\frac{\partial L}{\partial x} = \underbrace{\frac{\partial K}{\partial x}}_{=0} - \frac{\partial U}{\partial x} \qquad K \text{ not a function of } x$$
$$= F_x = ma \qquad F_x = -dU/dx$$
$$= m\frac{d^2x}{dt^2} \qquad \text{acceleration} = d^2x/dt^2$$

Next the righthand side:

$$\frac{d}{dt}\frac{\partial L}{\partial v_x} = \frac{d}{dt}\left[\frac{1}{2}m\frac{\partial v_x^2}{\partial v_x} - \frac{\partial U}{\partial v_x}\right]$$
$$= \frac{d}{dt}\left[mv_x - \underbrace{\frac{\partial U}{\partial v_x}}_{=0}\right] \qquad U \text{ not a function of } v_x$$
$$= m\frac{dv_x}{dt} = m\frac{d^2x}{dt^2}.$$

And there we are. One of the useful features of the Lagrangian is that the equation proved here can be made to hold for different choices of coordinates. This enables the mechanics problems to be written in coordinates that take advantage of symmetry (for example, if the only force is a radial one, attracting or repelling particles from a single point), and the Lagrangian then provides a starting point to develop relationships between the positions and velocities of the particles.

**A.18** The overall energy before the collision is the sum of the two kinetic energies:

$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2,$$

and this must equal the energy after the collision:

$$K = \tfrac{1}{2}m_1{v_1'}^2 + \tfrac{1}{2}m_2{v_2'}^2.$$

Similarly, we may set the expressions for the linear momentum before and after the collision equal to each other:

$$p = m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'.$$

So there are two equations and two unknowns. At this point, the problem is ready to solve with a symbolic math program.

**Maple.** The problem can be solved in a single step by asking Maple to solve the conservation of energy and conservation of momentum equations simultaneously to get the final speeds (here $vf[1]$ and $vf[2]$ in terms of the masses and initial speeds:

**solve({m[1]*v[1]+m[2]*v[2] = m[1]*vf[1]+m[2]*vf[2], (1/2) * m[1] * v[1]^2+(1/2) * m[2] * v[2]^2 = (1/2) * m[1] * vf[1]^2+(1/2) * m[2]*vf[2]^2}, [vf[1], vf[2]]);**

**Mathematica.** We are looking for an algebraic expression, rather than a numerical value, in the solution to this problem. We can use the **Solve** command for either case:

**Solve[m1*v1 + m2*v2 == m1*vf1 + m2*vf2 && (1/2)*m1*v1^2 + (1/2)*m2*v2^2 == (1/2)*m1*vf1^2 + (1/2)*m2*vf2^2, vf1, vf2]**

**On paper.** This last equation lets us eliminate one variable by writing, for example, the final speed $v_2'$ in terms of $v_1'$:

$$v_2' = \frac{m_1v_1 + m_2v_2 - m_1v_1'}{m_2}.$$

Now we can put this value into the equation for $K$, and solve for $v_1'$:

$$\begin{aligned} K &= \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 \qquad (a)\\ &= \tfrac{1}{2}m_1{v_1'}^2 + \tfrac{1}{2}m_2{v_2'}^2\\ &= \tfrac{1}{2}m_1{v_1'}^2 + \tfrac{1}{2}m_2\left(\frac{m_1v_1 + m_2v_2 - m_1v_1'}{m_2}\right)^2. \end{aligned}$$

This is going to be an equation that depends on ${v_1'}^2$ and $v_1'$, so we can solve it using the quadratic formula. In that case, it's easiest to put all the quantities on one side of the equation:

$$\begin{aligned} 0 &= \tfrac{1}{2}m_1{v_1'}^2 + \tfrac{1}{2}m_2\left(\frac{m_1v_1 + m_2v_2 - m_1v_1'}{m_2}\right)^2 - \left(\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2\right) && \text{subtract } (a) \text{ above}\\ &= \tfrac{1}{2}m_1{v_1'}^2 + \tfrac{1}{2}m_2\left(\frac{m_1^2v_1^2 + m_2^2v_2^2 + m_1^2{v_1'}^2 + 2m_1m_2v_1v_2 - 2m_1^2v_1v_1' - 2m_1m_2v_1'v_2}{m_2^2}\right) && \text{expand the square}\\ &- \tfrac{1}{2}m_1v_1^2 - \tfrac{1}{2}m_2v_2^2\\ &= m_1{v_1'}^2 + \frac{m_1^2}{m_2}v_1^2 + m_2v_2^2 + \frac{m_1^2}{m_2}{v_1'}^2 + 2m_1v_1v_2 && \text{divide by } 1/2\\ &- 2\frac{m_1^2}{m_2}v_1v_1' - 2m_1v_1'v_2 - m_1v_1^2 - m_2v_2^2\\ &= {v_1'}^2\left(m_1 + \frac{m_1^2}{m_2}\right) + v_1'\left(-2\frac{m_1^2}{m_2}v_1 - 2m_1v_2\right) && \text{group by power of } v_1' \end{aligned}$$

$$
\begin{aligned}
&+ \left( \frac{m_1^2}{m_2} v_1^2 + m_2 v_2^2 + 2m_1 v_1 v_2 - m_1 v_1^2 - m_2 v_2^2 \right) \\
&= {v_1'}^2 \left( m_1 + \frac{m_1^2}{m_2} \right) + v_1' \left( -2\frac{m_1^2}{m_2} v_1 - 2m_1 v_2 \right) \\
&+ \left[ \left( \frac{m_1^2}{m_2} - m_1 \right) v_1^2 + 2m_1 v_1 v_2 \right] \\
v_1' &= \left( 2m_1 + 2\frac{m_1^2}{m_2} \right)^{-1} \left\{ \left( 2\frac{m_1^2}{m_2} v_1 + 2m_1 v_2 \right) \right. && \text{quadratic formula} \\
&\left. \pm \left[ \left( 2\frac{m_1^2}{m_2} v_1 + 2m_1 v_2 \right)^2 - 4\left( m_1 + \frac{m_1^2}{m_2} \right) \left[ \left( \frac{m_1^2}{m_2} - m_1 \right) v_1^2 + 2m_1 v_1 v_2 \right] \right]^{1/2} \right\}.
\end{aligned}
$$

To deal with this equation, we can expand the multiplication inside the square brackets:

$$
\begin{aligned}
\left( 2\frac{m_1^2}{m_2} v_1 + 2m_1 v_2 \right)^2 &= 4\frac{m_1^4}{m_2^2} v_1^2 + 8\frac{m_1^3}{m_2} v_1 v_2 + 4m_1^2 v_2^2 \\
-4\left( m_1 + \frac{m_1^2}{m_2} \right) \left[ \left( \frac{m_1^2}{m_2} - m_1 \right) v_1^2 + 2m_1 v_1 v_2 \right] &= -4\frac{m_1^3}{m_2} v_1^2 - 4\frac{m_1^4}{m_2^2} v_1^2 + 4m_1^2 v_1^2 + 4\frac{m_1^3}{m_2} v_1^2 \\
&\quad - 8m_1^2 v_1 v_2 - 8\frac{m_1^3}{m_2} v_1 v_2.
\end{aligned}
$$

Nearly all of these terms cancel when we add these two expressions together, leaving:

$$
4m_1^2 v_2^2 + 4m_1^2 v_1^2 - 8m_1^2 v_1 v_2.
$$

In the quadratic equation, we have to take the square root of this, but that turns out to be easy:

$$
\begin{aligned}
\left[ 4m_1^2 v_2^2 + 4m_1^2 v_1^2 - 8m_1^2 v_1 v_2 \right]^{1/2} &= 2m_1 \left[ v_2^2 + v_1^2 - 2v_1 v_2 \right]^{1/2} \\
&= 2m_1 \left( v_2 - v_1 \right).
\end{aligned}
$$

Finally, putting this back into our equation for $v_1'$, we get

$$
\begin{aligned}
v_1' &= \left( 2m_1 + 2\frac{m_1^2}{m_2} \right)^{-1} \left\{ \left( 2\frac{m_1^2}{m_2} v_1 + 2m_1 v_2 \right) \pm 2m_1 \left( v_2 - v_1 \right) \right\} \\
&= \left( 1 + \frac{m_1}{m_2} \right)^{-1} \left\{ \left( \frac{m_1}{m_2} v_1 + v_2 \right) \pm \left( v_2 - v_1 \right) \right\}. && \text{divide out } 2m_1
\end{aligned}
$$

This is correct as far as it goes, but we have two solutions, corresponding to either the + or − sign. If we use the − sign, then we get

$$
v_1' = \left( 1 + \frac{m_1}{m_2} \right)^{-1} \left\{ \frac{m_1}{m_2} v_1 + v_2 - v_2 + v_1 \right\} = v_1.
$$

This is the solution if the collision *doesn't* occur; particle 1 just keeps moving at the same speed as before. The + sign gives us the correct solution:

$$
\begin{aligned}
v_1' &= \left(1+\frac{m_1}{m_2}\right)^{-1}\left\{\frac{m_1}{m_2}v_1+v_2+v_2-v_1\right\} \\
&= \left(1+\frac{m_1}{m_2}\right)^{-1}\left[\left(\frac{m_1}{m_2}-1\right)v_1+2v_2\right] \\
&= \frac{1}{m_1+m_2}\left[\left(m_1-m_2\right)v_1+2m_2v_2\right].
\end{aligned}
$$

We can now use the conservation of momentum to solve for $v_2'$. I'm going to factor out a $1/(m_1+m_2)$ to get an equation similar to the one for $v_1'$:

$$
\begin{aligned}
v_2' &= \frac{m_1v_1+m_2v_2-m_1v_1'}{m_2} \\
&= \frac{1}{m_2}\left\{m_1v_1+m_2v_2-\frac{m_1}{m_1+m_2}\left[\left(m_1-m_2\right)v_1+2m_2v_2\right]\right\} \\
&= \left(\frac{m_1}{m_2}\right)v_1+v_2-\left(\frac{m_1-m_2}{m_1+m_2}\right)\left(\frac{m_1}{m_2}\right)v_1-2\left(\frac{m_1}{m_1+m_2}\right)v_2 \\
&= \left(\frac{1}{m_1+m_2}\right)\left[\frac{m_1(m_1+m_2)}{m_2}v_1+(m_1+m_2)v_2-\left(\frac{m_1(m_1-m_2)}{m_2}\right)v_1-2m_1v_2\right] \\
&= \frac{1}{m_1+m_2}\left[\left(m_2-m_1\right)v_2+2m_1v_1\right].
\end{aligned}
$$

Because there is nothing in the problem that determines which particle is labeled 1 and which is labeled 2, the equations for $v_1'$ and $v_2'$ must be exactly the same, with all the labels 1 and 2 switched.

If you haven't seen this result or simply don't remember it, it's worthwhile to check a few values. For example, if the two particles have equal mass ($m_1 = m_2$), then the final speeds are $v_1' = v_2$ and $v_2' = v_1$; *i.e.*, the particles simply exchange speeds. Another example: if particle 1 is initially at rest ($v_1 = 0$), then it picks up a speed $2m_2v_2/(m_1+m_2)$ from the collision. In that case, if particle 2 dominates the mass ($m_2 \gg m_1$), then particle 1 will find itself with a final speed equal to $2v_2$. In contrast, if particle 1 is much more massive than 2, then the collision will hardly affect it ($v_1' \approx 0$) and particle 2 will simply reverse direction ($v_1' \approx -v_2$).

Note that the two particles don't have to be moving in opposite directions. If particle 1 is behind 2 but moving faster and in the same direction, then they will strike each other, and particle 2 will acquire particle 1's higher speed.

**A.19**

$$
\begin{aligned}
K = U &= -\frac{e^2}{4\pi\epsilon_0 r} = \frac{(1.602\cdot 10^{-19}\,\mathrm{C})^2}{(1.113\cdot 10^{-10}\,\mathrm{C^2\,J^{-1}\,m^{-1}})(1.0\,\text{Å})(10^{-10}\,\mathrm{m}\,\text{Å}^{-1})} = 2.31\cdot 10^{-18}\,\mathrm{J} \\
L &= |\vec{r}\times\vec{p}| = rp,\ \text{since } \vec{r}\perp\vec{p}. \\
p &= \sqrt{2m_eK} = \left[2(9.109\cdot 10^{-31}\,\mathrm{kg})\cdot(2.31\cdot 10^{-18}\,\mathrm{J})\right]^{1/2} = 2.05\cdot 10^{-24}\,\mathrm{kg\,m\,s^{-1}} \\
L &= (1.0\,\text{Å})(10^{-10}\,\mathrm{m}\,\text{Å}^{-1})(2.05\cdot 10^{-24}\,\mathrm{kg\,m\,s^{-1}}) = \boxed{2.05\cdot 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}}\ .}
\end{aligned}
$$

**A.20** a. *Find the center of mass positions $\vec{r}_i^{(0)}$ at collision.* Let's call the center of mass of the entire system the origin. The particles have equal mass, so the origin will always lie exactly in between the two particles. At the time of the collision, we may draw a right triangle for each particle, connecting the particle's center of mass, the origin, and with the right angle resting on the $z$ axis. The hypotenuse of the triangle connects the center of mass to the point of contact between the two particles, and must be of length $d/2$ (the radius of the particle). The other two sides are of length $(d/2)\cos\theta$ (along the $z$ axis) and $(d/2)\sin\theta$ (along the $x$ axis), based on the definitions of the sine and cosine functions in Eqs. A.5. These correspond to the magnitudes of the $z$ and $x$ coordinates, respectively, of the particle centers of mass at the collision. The signs of the values may be determined by inspection of the figure: at the time of the collision, $x_1$ and $z_2$ are positive while $x_2$ and $z_1$ are negative, so the position vectors are:

$$
\begin{aligned}
\vec{r}_1^{(0)} &= ((d/2)\sin\theta, 0, -(d/2)\cos\theta) \\
\vec{r}_2^{(0)} &= (-(d/2)\sin\theta, 0, (d/2)\cos\theta)\,.
\end{aligned}
$$

b. *Find the velocities $\vec{v}_i'$ after collision.* Simple collisions obey a simple reflection law: the angle of incidence is equal to the angle of reflection. These are the angles between the velocity vectors and the normal vector—the line at angle $\theta$ from the $z$ axis. (This is the normal vector because it lies perpendicular to the plane that lies between the two spheres at the point of collision; this plane is effectively the surface of reflection for the collision.) Therefore, the velocity vector after the collision is at an angle $2\theta$ from the $z$ axis, and the velocity vectors after the collision are

$$
\begin{aligned}
\vec{v}_1' &= v_0(\sin 2\theta, 0, -\cos 2\theta) \\
\vec{v}_2' &= v_0(-\sin 2\theta, 0, \cos 2\theta).
\end{aligned}
$$

Notice that the speed after the collision is still $v_0$ for each particle. Because they each began with the same magnitude of linear momentum, the momentum transfer that takes place only affects the trajectories.

c. *Show that $\vec{L}$ is conserved before and after the collision.* We now have position and velocity vectors before and after the collision:

$$
\begin{aligned}
\vec{r}_1 &= ((d/2)\sin\theta, 0, -(d/2)\cos\theta) \;+\; \vec{v}_1 t & \qquad \vec{r}_2 &= (-(d/2)\sin\theta, 0, (d/2)\cos\theta) \;+\; \vec{v}_1 t \\
\vec{v}_1'' &= v_0(0,0,1) & \vec{v}_2'' &= v_0(0,0,-1) \\
\vec{v}_1' &= v_0(\sin 2\theta, 0, -\cos 2\theta) & \vec{v}_2' &= v_0(-\sin 2\theta, 0, \cos 2\theta).
\end{aligned}
$$

We take the cross products of these for each particle to get $\vec{L}$ for each particle, and we add these together to get the total angular momentum for the system. Before the collision,

$$
\begin{aligned}
\vec{r}_1'' &= ((d/2)\sin\theta, 0, -(d/2)\cos\theta) + v_0 t(0,0,1) \\
\vec{L}_1'' &= m\vec{r}_1'' \times \vec{v}_1'' \\
&= m\left(y_1''v_{z1}'' - z_1''v_{y1}, z_1''v_{x1}'' - x_1''v_{z1}, x_1''v_{y1}'' - y_1''v_{x1}\right) \\
&= m\,(0, -(dv_0/2)\sin\theta, 0)
\end{aligned}
$$

and similarly for $\vec{L}_2''$:

$$
\vec{L}_2'' = m\,(0, -(dv_0/2)\sin\theta, 0)
$$

and combining these yields:

$$
\vec{L}'' = \vec{L}_1'' + \vec{L}_2'' = -mdv_0\ (0, \sin\theta, 0).
$$

All of the position or velocity vectors have only zero $y$ components, and therefore only the $y$ component of the cross product survives. After the collision,

$$\begin{aligned}\vec{r}_1' &= ((d/2)\sin\theta, 0, -(d/2)\cos\theta) + v_0 t(\sin 2\theta, 0, -\cos 2\theta)\\ \vec{L}_1' &= m\vec{r}_1' \times \vec{v}_1'\end{aligned}$$

which has a $y$ component

$$\vec{L}_{y1}' = m\left\{-(d/2)\cos\theta\sin 2\theta - v_0 t\cos 2\theta\sin 2\theta - [(d/2)\sin\theta(-\cos 2\theta) + v_0 t\sin 2\theta(-\cos 2\theta)]\right\}$$

and similarly for $\vec{L}_{y2}'$:

$$\vec{L}_{y1}' = m\left\{(d/2)\cos\theta(-\sin 2\theta) + v_0 t\cos 2\theta(-\sin 2\theta) - [-(d/2)\sin\theta\cos 2\theta - (-v_0 t\sin 2\theta)\cos 2\theta]\right\}.$$

Adding the two components together we find that all the $t$-dependent terms cancel, and trigonometric identities from Table A.2 simplify the rest:

$$\begin{aligned}\vec{L}_y' &= \vec{L}_{1y}' + \vec{L}_{2y}'\\ &= \frac{mdv_0}{2}\left[-2\cos\theta\sin 2\theta + 2\sin\theta\cos 2\theta\right]\\ \sin 2\theta &= 2\sin\theta\cos\theta\\ \cos 2\theta &= 2\cos^2\theta - 1\\ \vec{L}_y' &= \frac{2mdv_0}{2}\left[-\cos\theta\left(2\sin\theta\cos\theta\right) + \sin\theta\left(2\cos^2\theta - 1\right)\right]\\ &= mdv_0\left[-2\cos^2\theta\sin\theta + 2\cos^2\theta\sin\theta - \sin\theta\right]\\ &= -mdv_0\sin\theta.\end{aligned}$$

This is the $y$ component of $\vec{L}'$, and the $x$ and $z$ components are again zero in the cross products, so we have shown that both $\vec{L}'$ and $\vec{L}''$ are equal to

$$\vec{L} = mdv_0\,(0, \sin\theta, 0).$$

If the particles hit head-on, then $\theta = 0$ and the angular momentum is zero. As $\theta$ increases, $L$ increases to a maximum value of $mdv_0$ when the two particles just barely touch each other in passing.

If we had used the conservation of $L$ at the outset, we could have found this solution quickly. Because the angular momentum does not depend on the size of the particles, we can replace our two objects here with point masses. It won't matter that they now won't collide, because if $L$ is conserved we have to get the same answer before the collision takes place anyway. In fact, because $L$ is conserved, we can pick any point in time that's convenient for us to calculate $L$, so I would pick the time when the two particles reach $z = 0$. At this time, both particles are traveling on trajectories that are exactly perpendicular to their position vectors ($\vec{v}_i$ is perpendicular to $\vec{r}_i$). This makes the cross product for each particle easy to evaluate:

$$\vec{L}_i = m\vec{r}_i \times \vec{v}_i = m(\pm(d/2)\sin\theta, 0, 0) \times (0, 0, \pm v_0) = mdv_0/2(0, \sin\theta, 0),$$

where the minus sign applies to particle 2. There are two particles, so we multiply this vector by 2, arriving at the same $\vec{L}$ as above.

**A.21** a. *Write $\vec{\mathcal{E}}_1$ in vector form.* The magnitude of the electric field generated by particle 1 is given by $F = q_2\mathcal{E}_1$, and this force must be equal to the Coulomb force $F = -q_1 q_2/(4\pi\epsilon_0 r^2)$. The force vector points along the axis separating the two particles, and we can include this direction-dependence by multiplying the magnitude of the vector by $\vec{r}/r$. The Cartesian form of the vector $\vec{r}$ from particle 1 to 2, just working off part (b) of the figure, may be written $(rv_1/c, y_2, 0)$ and has length

$$r = \left[\left(\frac{rv_1}{c}\right)^2 + y_2^2\right]^{1/2} .$$

Therefore, the force vector is

$$\vec{F} = \left(\frac{q_1 q_2}{4\pi\epsilon_0 r^2}\right) \frac{\vec{r}}{r} = \frac{q_1 q_2}{4\pi\epsilon_0 r^3}(rv_1/c, y_2, 0)$$

and the electric field vector is

$$\vec{\mathcal{E}}_1 = \frac{\vec{F}}{q_2} = \frac{q_1}{4\pi\epsilon_0 r^3}(rv_1/c, y_2, 0).$$

b. *Write $\vec{B}$ in vector form.* Here we just have to be careful to correctly evaluate the cross product. We are using the equation $\vec{B} = \frac{1}{c^2}\vec{\mathcal{E}}_1 \times \vec{v}_1$, and we have an equation for $\vec{\mathcal{E}}_2$ already. The velocity vector consists only of an $x$-velocity component: $\vec{v}_1 = (v_1, 0, 0)$. Notice that because these two vectors lie in the $xy$ plane, their cross product—which is perpendicular to both vectors—will lie along the $z$ axis. The $z$ component of the cross product $\vec{a} \times \vec{b}$ is equal to $a_x b_y - a_y b_x$, so we have

$$\vec{B} = \frac{1}{c^2}\vec{\mathcal{E}}_1 \times \vec{v}_1 = \frac{q_1}{4\pi\epsilon_0 c^2 r^3}\ (0, 0, v_1 y_2)\ .$$

c. *Find the magnetic force vector.* Again, we take a cross product with the velocity. This time, the $\vec{B}$ vector lies along $z$, and $\vec{v}_1$ lies along $x$, so the cross product lies along $y$:

$$\vec{F}_{\text{mag}} = q_2 \vec{v}_1 \times \vec{B} = \frac{q_1 q_2}{4\pi\epsilon_0 c^2 r^3}\ (0, v_1^2 y_2, 0)\ .$$

d. *Calculate the difference between the actual and classical values of the Coulomb force.* To compute the actual Coulomb force, we use the distance $r$, so $\vec{F}$ has a magnitude

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}.$$

The classical Coulomb force would be

$$F' = \frac{q_1 q_2}{4\pi\epsilon_0 y_2^2}\ ,$$

and the difference between the two forces is

$$F - F' = \frac{q_1 q_2}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{1}{y_2^2}\right].$$

We can simplify this by relating $r^2$ and $y^2$:

$$r^2 = \left(\frac{rv_1}{c}\right)^2 + y_2^2$$
$$r^2 = y_2^2\left[1 - \left(\frac{v_1}{c}\right)^2\right]^{-1} .$$

So, finally, we have

$$F - F' = \frac{q_1 q_2}{4\pi\epsilon_0}\left[\frac{1}{y^2}\left[1 - \left(\frac{v_1}{c}\right)^2\right] - \frac{1}{y_2^2}\right]$$
$$= \frac{q_1 q_2}{4\pi\epsilon_0 y_2^2}\left[-\left(\frac{v_1}{c}\right)^2\right] = -\frac{q_1 q_2 v_1^2}{4\pi\epsilon_0 c^2 y_2^2}.$$

In comparison, the magnitude of the magnetic force we calculated from the standard equations is

$$F_{\text{mag}} = q_1 q_2 v_1^2 y_2/(4\pi\epsilon_0 c^2 r^3),$$

and for $v \ll c$, we can allow $r \approx y_2$, so that

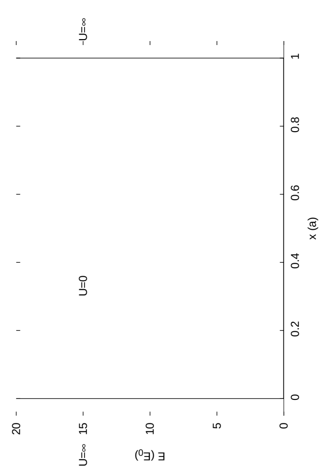
$$F_{\text{mag}} = q_1 q_2 v_1^2/(4\pi\epsilon_0 c^2 y_2^2).$$

Magnetic forces are a natural result of the motion of electrical charge when special relativity is taken into account. It was this relationship between electric and magnetic forces that was the basis of Einstein's original paper on special relativity.

# Chapter 1

**1.1** Briefly, the nucleus of any atom except hydrogen has multiple protons, which repel each other, coexisting at very small distances. With only protons and neutrons present, there is no negative charge to counter the proton-proton repulsion, and the gravitational attraction between nuclear particles is much too weak to play a role in holding the nucleus together. If our theories of mass and charge do not explain the binding of positively charged protons into a nucleus, that suggests that there is some other property that explains it. This reasoning led to the concepts of quark *color* and the strong force.

**1.2**




The potential energy climbs to infinity at the walls and is zero in between. We know this because the walls are impenetrable—the particles always bounce off the wall and transfer no energy into the wall. In this idealized limit, no amount of energy in the particle will get it to occupy the region of the wall, so the potential energy of the wall (the energy it would take to occupy that location) must be infinite. At the wall, the slope of the potential energy is also infinite, so the force $F = dU/dx$ is infinite, but pushing back in the negative $x$ direction, so $\boxed{F(x/a) = -\infty.}$ Within the container, there are no forces working on the particles, so $F$ and $U = dF/dx$ are both equal to zero.

**1.3** One approach would be the following:

1. Invent the scale as described in the chapter, using a spring to find forces by measuring the displacement of the spring with our ruler.

2. Find the acceleration due to gravity $g$ by measuring changes in speed of falling objects, using the ruler and clock to compare $\Delta$(distance)/$\Delta$(time) at different times. Once $g$ is known, we can convert the weight of a water sample to a mass.

3. Finally, measure the volume of a sample of water with the ruler and a rectangular container for the water.

The ratio of the mass to the volume will be the density.

**1.4** We will need to figure out how the pressure at the bottom of the column varies with the mass of water above it, and convert the mass to height. This problem can be started from either end, but let's start from how the mass determines the pressure:

$$P = \frac{F}{A} = \frac{Mg}{A},$$

where $F$ is the force exerted at the base of the column, $M$ is the mass of the water in the column, and $g = 9.8\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity near the Earth's surface. The mass is related to the height through the density. The volume of the water is equal to the area $A$ times the height $z$ (which is what we wish to solve), and the mass within a volume $V$ is equal to the volume times the mass density $\rho_m = 1.00\,\mathrm{g\,cm^{-3}} = 1.00 \cdot 10^3\,\mathrm{kg\,m^{-3}}$. So we can set $P = 1.00\,\mathrm{bar} = 1.00 \cdot 10^5\,\mathrm{Pa}$ and solve for $z$:

$$\begin{aligned} P &= \frac{Mg}{A} = \frac{\rho_m V g}{A} = \frac{\rho_m (Az) g}{A} = \rho_m g z \\ z &= \frac{P}{\rho_m g} = \frac{1.00 \cdot 10^5\,\mathrm{Pa}}{(1.00 \cdot 10^3\,\mathrm{kg\,m^{-3}})(9.8\,\mathrm{m\,s^{-2}})} \\ &= \boxed{10.\,\mathrm{m.}} \end{aligned}$$

**1.5** Pressure is related to force through Eq. 1.4, and here we need to solve for the force:

$$\begin{aligned} F &= PA \\ &= (0.010\,\mathrm{bar})(10^5\,\mathrm{Pa\,bar^{-1}})(78 \times 30)(2.54\,\mathrm{cm})^2(10^{-2}\,\mathrm{m/\,cm})^2 = \boxed{1.5 \cdot 10^3\,\mathrm{N.}} \end{aligned}$$

This force is equivalent to lifting a weight of

$$\frac{F}{g} = \frac{1.5 \cdot 10^3\,\mathrm{N}}{9.81\,\mathrm{m\,s^{-2}}} = 150\,\mathrm{kg}.$$

**1.6** The area of the water drop is

$$A_{\mathrm{water}} = \pi(d/2)^2 = 0.785\,\mathrm{cm}^2 = 7.85 \cdot 10^{-5}\,\mathrm{m}^2.$$

The approximate number of molecules that can fit in this area is given by the ratio of this area to the effective area of a single molecule:

$$N \approx \frac{A_{\mathrm{water}}}{A_{\mathrm{butanol}}} = \frac{7.85 \cdot 10^{-5}\,\mathrm{m}^2}{(33\,\text{Å}^2)(10^{-10}\,\mathrm{m}\,\text{Å}^{-1})^2} = 2.38 \cdot 10^{14}$$

We use Avogadro's number to convert this value to the number of moles:

$$n = \frac{N}{\mathcal{N}_A} = \boxed{3.9 \cdot 10^{-10}\,\mathrm{mol.}}$$