Chapter 2

2.1 **a.** Spring constant, k**:** The change in the force per unit length change of the spring.

b. Coefficient of subgrade reaction, k':

Spring constant divided by the foundation contact area, $k' = \frac{k}{A}$

c. Undamped natural circular frequency: $\omega_n = \sqrt{\frac{k}{m}}$ rad/s

where m = mass = $\frac{W}{g}$

d. Undamped natural frequency: $f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ (in Hz)

Note: Circular frequency defines the rate of oscillation in term of radians per unit time; 2π radians being equal to one complete cycle of rotation.

e. Period, T: The time required for the motion to begin repeating itself.

f. Resonance: Resonance occurs when $\frac{\omega_n}{\omega} = 1$

g. Critical damping coefficient: $c_c = 2\sqrt{km}$

where k = spring constant; m = mass = $\frac{W}{g}$

h. Damping ratio: $D = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}$

where c = viscous damping coefficient; c_c = critical damping coefficient

i. Damped natural frequency:

$$\omega_d = \omega_n\sqrt{1-D^2}$$

$$f_d = \sqrt{1-D^2}\, f_n$$

2.2 Weight of machine + foundation, $W = 400$ kN
Spring constant, $k = 100{,}000$ kN/m

Mass of the machine + foundation, $m = \frac{W}{g} = \frac{400}{9.81} = 40.77\ \frac{\text{kN}}{\text{m/s}^2}$

Natural frequency of undamped free vibration is [Eq. (2.19)]

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{100{,}000}{40.77}} = \mathbf{7.88\ Hz}$$

From Eq. (2.18), $T = \frac{1}{f_n} = \frac{1}{7.88} = \mathbf{0.127\ s}$

2.3 Weight of machine + foundation, $W = 400$ kN
Spring constant, $k = 100{,}000$ kN/m

Static deflection of foundation is [Eq. (2.2)]

$$z_s = \frac{W}{k} = \frac{400}{100{,}000} = 4 \times 10^{-3}\ \text{m} = \mathbf{4\ mm}$$

2.4 External force to which the foundation is subjected, $Q = 35.6 \sin \omega t$ kN
$f = 13.33$ Hz
Weight of the machine + foundation, $W = 178$ kN
Spring constant, $k = 70{,}000$ kN/m

For this foundation, let time $t = 0$, $z = z_0 = 0$, $\dot{z} = v_0 = 0$

a. Mass of the machine + foundation, $m = \frac{W}{g} = \frac{178}{9.81} = 18.145\ \frac{\text{kN}}{\text{m/s}^2}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{70{,}000}{18.145}} = 62.11\ \text{rad/s}$$

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{62.11} = \mathbf{0.101\ s}$$

b. The frequency of loading, $f = 13.33$ Hz

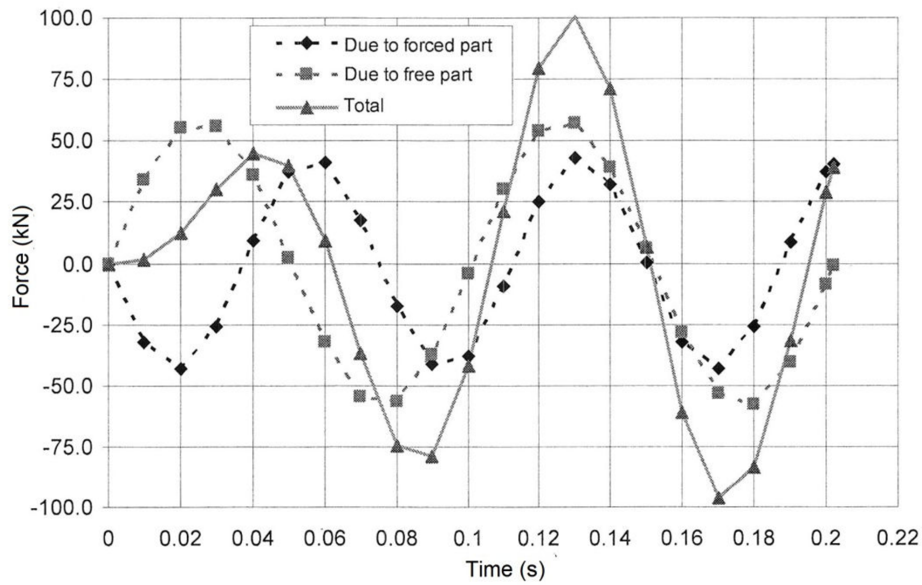
$$\omega = 2\pi f = 2\pi(13.33) = 83.75\ \text{rad/s}$$

Force due to forced part, $F_1 = k\left(\dfrac{Q_0/k}{1-\omega^2/\omega_n^2}\right)\sin\omega t$

$$= (70,000)\left(\frac{35.6/70,000}{1-83.75^2/62.11^2}\right)\sin(83.75t)$$

$$= \mathbf{43.51\sin(83.75\mathit{t})\ kN}$$

See the plot below for F_1 vs. t



c. Force due to free part, $F_2 = k\left(\dfrac{Q_0/k}{1-\omega^2/\omega_n^2}\right)\left(-\dfrac{\omega}{\omega_n}\sin\omega_n t\right)$

$$= 70,000\left(\frac{35.6/70,000}{1-83.75^2/62.11^2}\right)\left(-\frac{83.75}{62.11}\sin(62.11t)\right)$$

$$= \mathbf{58.67\sin(62.11\mathit{t})\ \ kN}$$

See the plot above in Part **b** for F_2 vs. t.

d. Total dynamic force on the subgrade:

$$F = F_1 + F_2 = \mathbf{-43.51\sin(83.75\mathit{t}) + 58.67(62.11\mathit{t})\ \ kN}$$

The plot of variation of the dynamic force on the subgrade of the foundation due to (a) forced part, (b) free part, and (c) total of the response for time $t = 0$ to $t = 2T$ is shown in the figure above (Part **b**).

2.5 The natural frequency of the undamped free vibration of the spring mass system is given by

$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}}$ where k_{eq} = equivalent stiffness of the spring system

For springs attached in series, the equivalent stiffness is given by

$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$, or $\frac{1}{k_{eq}} = \frac{k_1 k_2}{k_1 + k_2}$

The natural frequency of the given undamped free vibration spring mass system is

$$\boldsymbol{f_n = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{k_1 + k_2} \times \frac{1}{m}}}$$

2.6 The natural frequency of the undamped free vibration of the spring mass system is given by

$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}}$ where k_{eq} = equivalent stiffness of the spring system

For springs attached in parallel, the equivalent stiffness is given by

$k_{eq} = k_1 + k_2$

The natural frequency of the given undamped free vibration spring mass system is

$$\boldsymbol{f_n = \frac{1}{2\pi}\sqrt{\frac{(k_1 + k_2)}{m}}}$$

2.7 The natural frequency of the undamped free vibration of the spring mass system is given by

$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}}$ where k_{eq} = equivalent stiffness of the spring system

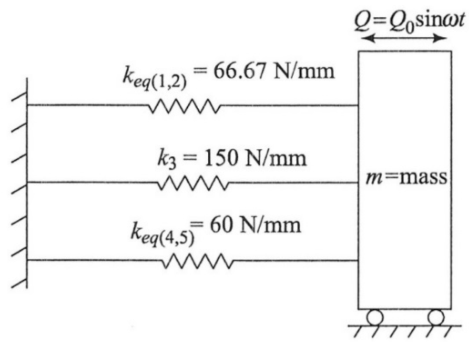
In the given spring-mass system, springs with stiffness k_1 and k_2 are in series.
Hence, their equivalent stiffness is

$$k_{eq(1,2)} = \frac{k_1 k_2}{k_1 + k_2} = \frac{100 \times 200}{100 + 200} = \frac{20{,}000}{300} = 66.67 \text{ N/mm}$$

Similarly, springs with stiffness k_4 and k_5 are in series. Hence, their equivalent stiffness is

$$k_{eq(4,5)} = \frac{k_4 k_5}{k_4 + k_5} = \frac{100 \times 150}{100 + 150} = 60 \text{ N/mm}$$

Now, the given spring system can be reduced to three springs in series.



The resulting system will be three springs in parallel. Their equivalent stiffness is given by

$$k_{eq} = k_{eq(1,2)} + k_3 + k_{eq(4,5)} = 66.67 + 150 + 60 = 276.67 \text{ N/mm} = 276.67 \text{ kN/m}$$

The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}} = \frac{1}{2\pi}\sqrt{\frac{267.67 \times 1000}{100}} = \mathbf{8.37\ Hz}$$

Time period $T = (1/f_n) = (1/8.37) = \mathbf{0.119\,s}$

2.8 Sinusoidal–varying force, $Q = 50 \sin \omega t$ N; $Q_0 = 50$ N; $\omega = 47$ rad/s

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{276.67 \times 1000}{100}} = 52.6 \text{ rad/s}$$

Amplitude of vibration = static deflection z_s × magnification M

$$z_s = \frac{Q_0}{k_{eq}} = \frac{50}{276.67} = 0.1807 \text{ mm}$$

From Eq. (2.34),

$$M = \frac{1}{1-(\omega/\omega_n)^2} = \frac{1}{1-(47/52.6)^2} = 4.96$$

Amplitude of vibration = 0.1807 × 4.96 = **0.896 mm**

2.9 Weight of the body, $W = 135$ N
Mass of the body, $m = W/g = 135/9.81 = 13.76$ kg
Spring constant, $k = 2600$ N/m
Dashpot resistance, $c = 0.7/(60/1000) = 11.67$ N-s/m

a. Damped natural frequency [Eq. (2.67)]

$$f_d = \sqrt{1-D^2}\, f_n$$

$$D = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{11.67}{2\sqrt{2600 \times 13.76}} = 0.031$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2600}{13.76}} = 2.19 \text{ Hz}$$

$$f_d = \sqrt{1-0.031^2} \times (2.19) = \mathbf{2.18\ Hz}$$

b. Damping ratio [Eq. (2.47b)],

$$D = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{11.67}{2\sqrt{2600 \times 13.76}} = \mathbf{0.031}$$

c. Ratio of successive amplitudes of the body is given by [Eq. (2.70)],

$$\frac{Z_n}{Z_{n+1}} = e^{\delta}$$

where $\delta = \ln\left(\frac{Z_n}{Z_{n+1}}\right) = \frac{2\pi D}{\sqrt{1-D^2}} = \frac{2\pi \times 0.031}{\sqrt{1-0.031^2}} = 0.195$

$$\frac{Z_n}{Z_{n+1}} = e^{\delta} = e^{0.195} = \mathbf{1.215}$$

d. At time $t = 0$ s, amplitude $Z_0 = 25$ mm.
After n cycles of disturbance

$$\frac{1}{n}\ln\frac{Z_0}{Z_n} = \frac{2\pi D}{\sqrt{1-D^2}}; \quad \ln\frac{Z_0}{Z_n} = \frac{2\pi n D}{\sqrt{1-D^2}}$$

With $n = 5$,

$$\ln\frac{Z_0}{Z_5} = \frac{2\pi \times 5 \times D}{\sqrt{1-D^2}} = \frac{2\pi \times 5 \times 0.031}{\sqrt{1-0.031^2}} = 0.974$$

$$\frac{Z_0}{Z_5} = e^{0.974} = 2.649; \quad Z_5 = \frac{25}{2.649} = 9.44 \text{ mm}$$

After 5 cycles of disturbance, the amplitude of vibration = **9.44 mm**

2.10 Q_0 = 6.7 kN
ω = 3100 rad/min = 51.67 rad/s
Weight of machine + foundation, W = 290 kN
Spring constant, k = 875 MN/m = 875,000 kN/m

Natural angular frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{875{,}000\times 10^3}{290\times 10^3/9.81}}$ = 172.04 rad/s

From Eq. (2.43), $F_{\text{dynam}} = \dfrac{Q_0}{1-(\omega/\omega_n)} = \dfrac{6.7}{1-(51.67/172.04)}$ = 9.58 kN

Maximum force on the subgrade = 290 + 9.58 = **299.58 kN**

Minimum force on the subgrade = 290 − 9.58 = **280.42 kN**

2.11 Q_0 = 200 kN
ω = 6000 rad/min = 100 rad/s
Weight of machine + foundation, W = 400 kN
Spring constant, k = 120,000 kN/m

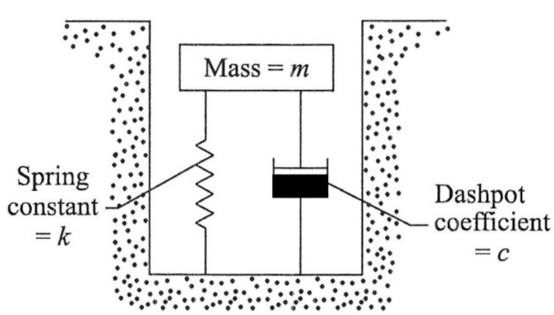
Natural angular frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{120{,}000\times 10^3}{400\times 10^3/9.81}}$ = 54.25 rad/s

Dynamic force, $F_{\text{dynam}} = \dfrac{Q_0}{1-\omega/\omega_n} = \dfrac{200}{1-(100/54.25)}$ = 237.16 kN

Maximum force on the subgrade = 400 + 237.16 = **637.16 kN**

Minimum force on the subgrade = 400 − 237.16 = **162.84 kN**

2.12 Weight of the body, W = 800 kN
Spring constant, k = 200,000 kN/m
Dashpot coefficient, c = 2340 kN-s/m



a. $c_c = 2\sqrt{km} = 2\sqrt{200{,}000 \times 800/9.81}$ = **8077.1 kN-s/m**

b. Damping ratio, $D = \dfrac{c}{c_c} = \dfrac{2340}{8077.1}$ = **0.29**

c. $\delta = \dfrac{2\pi D}{\sqrt{1-D^2}} = \dfrac{2\pi \times 0.29}{\sqrt{1-0.29^2}}$ = **1.9**

d. $f_d = \sqrt{1-D^2}\, f_n$; $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{200{,}000 \times 9.81}{800}}$ = 7.88 Hz

$f_d = \sqrt{1-0.29^2} \times 7.88$ = **7.54 Hz**

2.13 Weight of the body, W = 800 kN
Spring constant, k = 200,000 kN/m
Dashpot coefficient, c = 2340 kN-s/m
Q_0 = 25 kN
Operating frequency, ω = 100 rad/s

a. Natural circular frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{200{,}000 \times 9.81}{800}}$ = 49.52 rad/s

From Problem 2.12, damping ratio, D = 0.29

From Eq. (2.28), the amplitude of vertical vibration of the foundation is

$$Z = \frac{(Q_0/k)}{\sqrt{[1-(\omega^2/\omega_n^2)]^2 + 4D^2(\omega^2/\omega_n^2)}}$$

$$= \frac{(25/200{,}000)}{\sqrt{[1-(100^2/49.52^2)]^2 + 4 \times 0.29^2(100^2/49.52^2)}}$$

$$= \mathbf{3.795 \times 10^{-5}\,m = 3.795 \times 10^{-2}\ mm}$$

b. From Eq. (2.90), the maximum dynamic force transmitted to the subgrade is

$$Z\sqrt{k^2 + (c\omega)^2} = (3.795 \times 10^{-5})\sqrt{200{,}000^2 + (2340 \times 100)^2} = \mathbf{11.68\ kN}$$