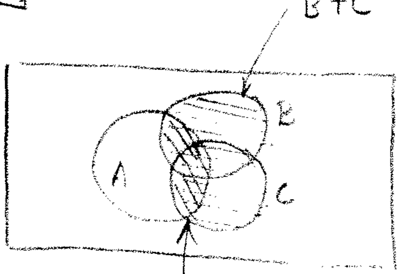
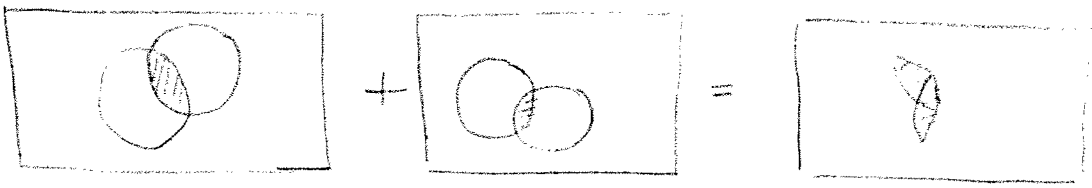
2.1

B + C

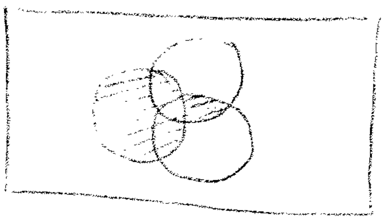


A · (B + C)

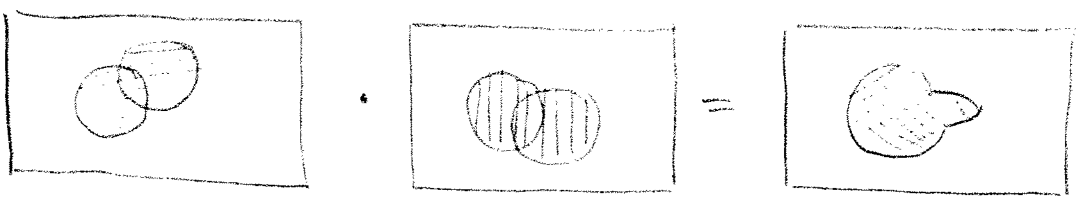


A · B + A · C = AB + AC

(b)



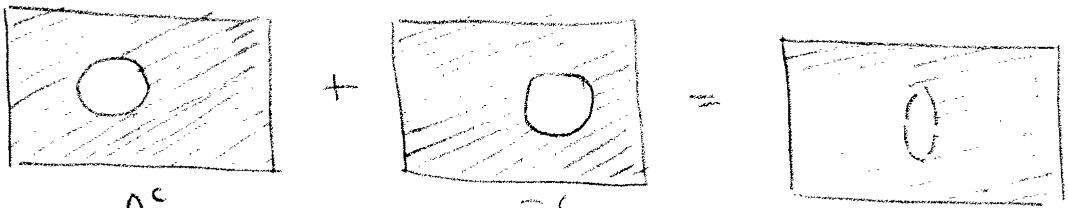
A + BC



A + C · A + C = (A + B)(B + C)

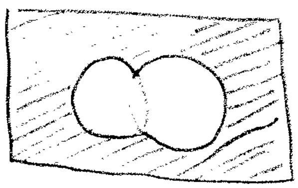
(c)

AB $(AB)^c$



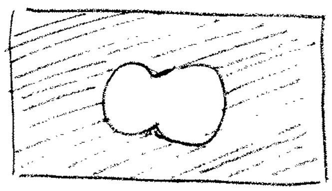
A^c + B^c = $A^c + B^c$

(d)



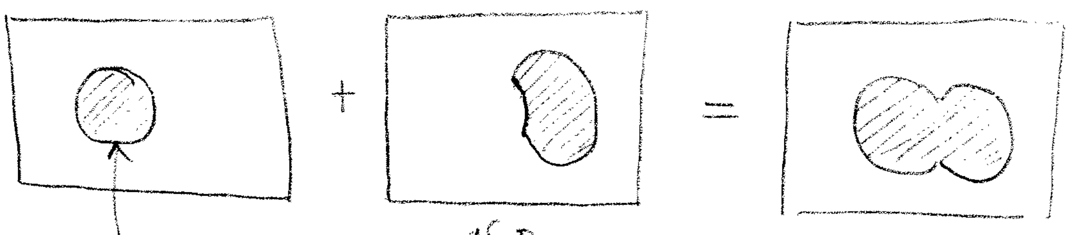
$(A+B)^c$

see above for diagrams of A^c and B^c $\Rightarrow$



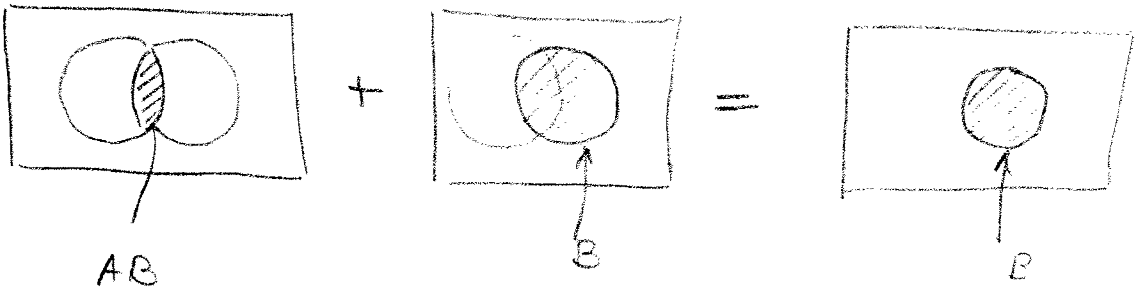
$A^c \cdot B^c$

(e)

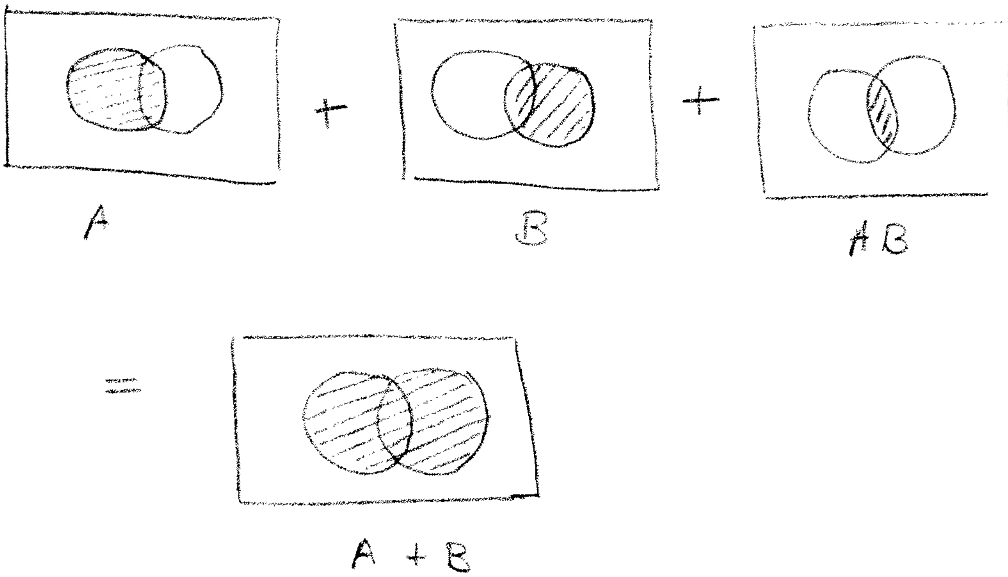


A + $A^c B$ = $A+B$

(f)



(g)



$\boxed{2.2}$

$$S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$$

$$A_1 = \{a_1, a_2, a_4\}, \quad A_2 = \{a_2, a_3, a_6\}, \quad A_3 = \{a_1, a_3, a_5\}$$

(a)

(i) $A_1 + A_2 = \{a_1, a_2, a_3, a_4, a_6\}$

(ii) $A_1 A_2 = \{a_2\}$

(iii) $A_3^c = \{a_2, a_4, a_6\}$

$A_1 + A_3^c = \{a_1, a_2, a_4, a_6\}$

$(A_1 + A_3^c) A_2 = \{a_2, a_6\}$

(b)(i) $A_2 + A_3 = \{a_1, a_2, a_3, a_5, a_6\}$

$A_1(A_2 + A_3) = \{a_1, a_2\}$ ①

$A_1 A_2 = \{a_2\}$, $A_1 A_3 = \{a_1\}$

$A_1 A_2 + A_1 A_3 = \{a_1, a_2\}$ ②

$\therefore$ ① = ②

$\boxed{2.2}$ Cont'd

(ii) $A_1 + A_2 A_3 = (A_1 + A_2)(A_1 + A_3)$

$A_2 A_3 = \{a_3\}$

$A_1 + A_2 A_3 = \{a_1, a_2, a_3, a_4\}$ ①

$A_1 + A_2 = \{a_1, a_2, a_3, a_4, a_6\}$

$A_1 + A_3 = \{a_1, a_2, a_3, a_4, a_5\}$

$(A_1 + A_2)(A_1 + A_3) = \{a_1, a_2, a_3, a_4\}$ ②

$\therefore$ ① = ②

(iii)

$A_1 + A_2 = \{a_1, a_2, a_3, a_4, a_6\}$

$(A_1 + A_2)^c = \{a_5\}$ ①

$A_1^c = \{a_3, a_5, a_6\}$, $A_2^c = \{a_1, a_4, a_5\}$

$A_1^c A_2^c = \{a_5\}$ ②

$\therefore$ ① = ②

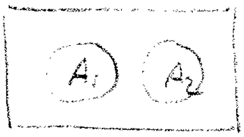
2.3

Axioms

(I) $\Pr[A] \geq 0$ (area always ≥ 0)

(II) $\Pr[S] = 1$ — we normalize the area of the Venn diagram so $\Pr[S] = 1$

(III)



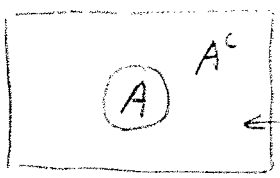
(no intersection)

The area of the union of two events is the sum of the areas

(IV) cannot be shown by Venn diagram

Corollaries (Table 2.4)

$\Pr[A^c] = 1 - \Pr[A]$

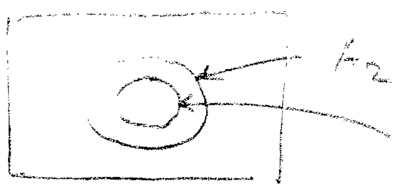


← S has area 1

$0 \leq \Pr[A] \leq 1$

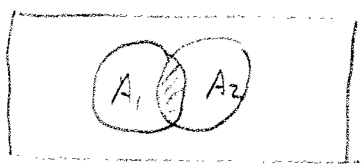
Area of A always ≥ 0
$A \subseteq S$ $\therefore$ area $\leq$ that of S.

If $A_1 \subseteq A_2$
then $\Pr[A_1] \leq \Pr[A_2]$



$A_1 A_2 = \phi \Rightarrow \Pr[A_1 A_2] = 0$ — Null set has area zero.

$$\Pr[A_1 + A_2] = \Pr[A_1] + \Pr[A_2] - \Pr[A_1 A_2]$$



Intersection is included twice if we sum the areas.

$\boxed{2.4}$

$$\Pr[A_1 + A_2 + A_3] = \Pr[A_1 + (A_2 + A_3]$$

$$= \Pr[A_1] + \Pr[A_2 + A_3] - \Pr[A_1(A_2 + A_3)]$$

$$= \Pr[A_1] + \Pr[A_2] + \Pr[A_3] - \Pr[A_2 A_3]$$

$$- \Pr\{A_1 A_2 + A_1 A_3\}$$

$$= \Pr[A_1] + \Pr[A_2] + \Pr[A_3] - \Pr[A_2 A_3]$$

$$- \left(\Pr[A_1 A_2] + \Pr[A_1 A_3] - \Pr[A_1 A_2 A_1 A_3]\right)$$

$$= \Pr[A_1] + \Pr[A_2] + \Pr(A_3] - \Pr[A_1 A_2] - \Pr\{A_2 A_3]$$

$$- \Pr\{A_1 A_3] + \Pr[A_1 A_2 A_3]$$

$\boxed{2.5}$

(ii) ~~(i)~~ $Pr[A_1 + A_2] = Pr[A_1] + Pr[A_2] - Pr[A_1 A_2]$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{6} = \frac{5}{6}$

(i) ~~(ii)~~ $Pr[A_1 A_2] = \frac{1}{6}$

(iii) ~~(iii)~~ $Pr[(A_1 + A_3^c) A_2] = Pr[\{a_2, a_6\}]$

$= \frac{2}{6}$

$\boxed{2.6}$

(a) The sample space has 8 outcomes

000 001 010 011 100 101 110 111

$S = \{0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7\}$

(b) All outcomes are equally likely.

$$A_1 = \{s > 5\} = \{6, 7\}$$

$$\Pr[A_1] = \frac{2}{8} = 1/4$$

(c)

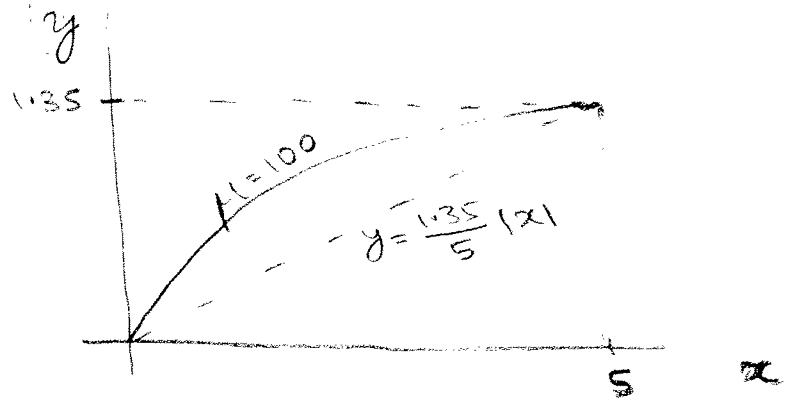
$$A_2 = \{3 \leq s \leq 6\} = \{3, 4, 5, 6\}$$

$$\Pr[A_2] = \frac{4}{8} = 1/2$$

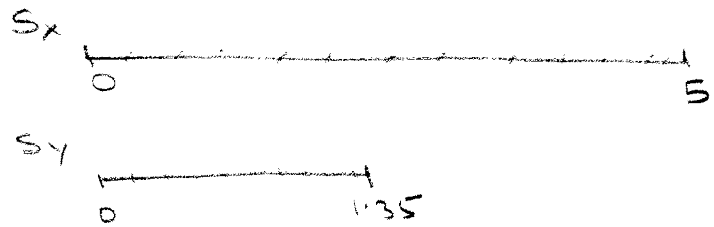
2.7

$$y = \frac{\log_e(1 + 100|x|)}{\log_e(1+100)} \qquad \underline{\text{for } \mu = 100}$$

Input- output transfer characteristic



(a) Sample spaces



(b) $A_1 = \{ y < 1 \}$

$Pr[Y < 1] = Pr[X < 1] = \frac{1}{5}$

(c) $Pr[\frac{1}{2} < X \le 1] = \frac{1/2}{5} = 1/10$

$Pr[\frac{1}{2} < Y \le 1] = Pr[0.0905 < X \le 1] = \frac{0.9095}{5}$

$= 0.1819$

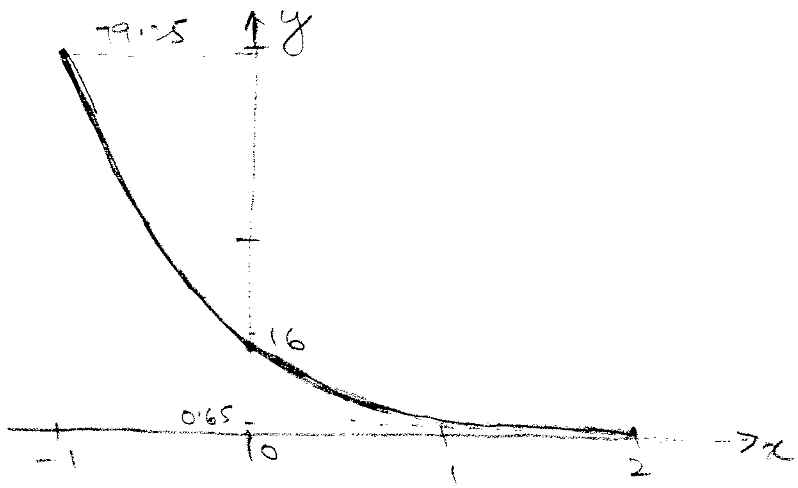
(d) $Pr[1 < Y \le 1.25] = Pr[1 < X \le 3.1919]$

$= \frac{2.1919}{5} = 0.4384$

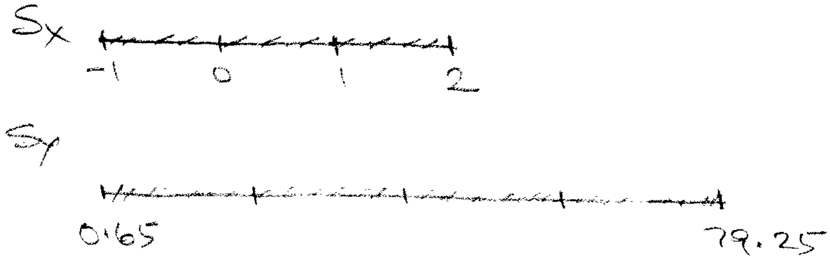
$\boxed{2.8}$

$$y = 16\, e^{-1.6x},$$

$$-1 \le x \le 2$$



(a) sample spaces (not to scale)



(b)

$$\Pr[Y \le 16] = \Pr[X > 0]$$

$$= \frac{2}{3}$$

$$\Pr[2 \le Y < 20] = \Pr[-0.1395 < X \le 1.2997]$$

$$= 0.4797$$

$\boxed{2.9}$

(a) $Pr[A+B] = Pr[A] + Pr[B] = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$

(b)

$$Pr[A+B] = Pr[A] + Pr[B] - Pr[A] \cdot Pr[B]$$

$$= \frac{1}{4} + \frac{1}{3} - \frac{1}{12} = \frac{6}{12} = \frac{1}{2}$$

(c) No. If it were possible then

$$P[A+B+C+D] = Pr[A] + Pr[B] + Pr[C] + Pr[D] > 1$$

$\therefore$ This is not possible.

$\boxed{2.10}$

Twelve-sided unfair die.

Events

$A = \{\text{odd numbered side}\}$

$B = \{4, 5, 6, 7, 8\}$

$Pr[1] = Pr[3] = Pr[5] = Pr[7] = Pr[9] = Pr[11]$

$Pr[2] = Pr[4] = Pr[6] = Pr[8] = Pr[10] = Pr[12]$

(a) $Pr[2] = 2\,Pr[1]$

$Pr[A] = \dfrac{6}{18} = 1/3$

(b) $Pr[B] = Pr[\{3 \text{ even}, 2 \text{ odd}\}]$

$= \dfrac{8}{18} = 4/9$

(c) $\{AB\} = \{1, 3, 5, 7, 9, 11\} \cap \{4, 5, 6, 7, 8\}$

$= \{5, 7\}$

$Pr[AB] = Pr[\{2 \text{ odd}\}]$

$= \dfrac{2}{18} = 1/9$

2.11

The sample space of the outcomes from this experiment:

$$S_I = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$$

$I \triangleq$ face value of 8-sided die − face value of 6-sided die

For different values of I, the number of arrangements are as follows:

−5	1−6						1
−4	1−5,	2−6					2
−3	1−4,	2−5,	3−6				3
−2	1−3,	2−4,	3−5,	4−6			4
−1	1−2,	2−3,	3−4,	4−5,	5−6		5
0	1−1,	2−2,	3−3,	4−4,	5−5,	6−6	6
1	2−1,	3−2,	4−3,	5−4,	6−5,	7−6	6
2	3−1,	4−2,	5−3,	6−4,	7−5,	8−6	6
3	4−1,	5−2,	6−3,	7−4,	8−5		5
4	5−1,	6−2,	7−3,	8−3			4
5	6−1,	7−2,	8−3				3
6	7−1,	8−2					2
7	8−1						1
						Total	48

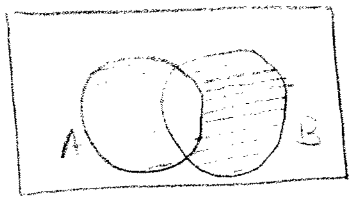
$$\Pr[\{I = -1\} \cup \{I = 5\}] = \frac{5+3}{48}$$

$$= 1/6$$

2.12

(a) $A =$ "hard drive fails" $\quad B =$ "memory fails"

We want the probability of the following event:



our event is:

$$AB^c + A^cB$$

$$A = A \cdot B + A \cdot B^c$$

$$Pr[A] = Pr[AB] + Pr[AB^c]$$

$$0.3 = 0.1 + Pr[AB^c] \Rightarrow Pr[AB^c] = 0.2$$

likewise

$$Pr[B] = Pr[AB] + Pr[A^cB]$$

$$0.2 = 0.1 + Pr[A^cB] \Rightarrow Pr[A^cB] = 0.1$$

Finally

$$Pr[AB^c + A^cB] = Pr[AB^c] + Pr[A^cB] = 0.2 + 0.1 = 0.3$$

(b) "No failures" $= A^cB^c$

Note that

$$A^cB^c = (A+B)^c$$

$$Pr[A+B] = Pr[A] + Pr[B] - Pr[AB]$$

$$= 0.3 + 0.2 - 0.1 = 0.4$$

$$Pr[\text{No failures}] = Pr[(A+B)^c] = 1 - Pr[A+B]$$

$$= 1 - 0.4 = 0.6$$

$\boxed{2\text{-}13}$

(a) $N = 26 \cdot 25 \cdot 24 \cdot 23$

$= 388,800$

$\text{Prob} = \dfrac{1}{388,800} = 2.787 \times 10^{-6}$

(b)

$\Pr[\text{S' in } 10^{th} \text{ position}] = \dfrac{1}{26}$

$= 0.038$

2.14

(a) Pr[one or more good]

$$= Pr[A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8]$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{7}{8}$$

Alternately,

$$Pr[\text{one or more good}] = 1 - Pr[\text{all bad}]$$

$$= 1 - Pr[A_1] = 1 - \frac{1}{8} = \frac{7}{8}$$

(b) In this case it is easiest to use the alternate approach above;

$$Pr[\text{one or more good}] = 1 - Pr[A_1]$$

However, in this case the probability of a bad disk is $\frac{3}{8}$, so

$$Pr[A_1] = \left(\frac{3}{8}\right)^3 = \frac{27}{512}$$

$$Pr[\text{one or more good}] = 1 - \frac{27}{512} = \frac{485}{512}$$

$$\approx 0.947$$

$\boxed{2.15}$

Draw the sample space and find the probabilities of each elementary event:

event	prob	
GGG	$(0.8)^3 =$	0.512
GGB	$(0.8)^2(0.2) =$	0.128
GBG	$(0.8)^2(0.2) =$	0.128
GBB	$(0.8)(0.2)^2 =$	0.032
BGG	$(0.8)^2(0.2) =$	0.128
BGB	$(0.8)(0.2)^2 =$	0.032
BBG	$(0.8)(0.2)^2 =$	0.032
BBB	$(0.2)^3 =$	0.008

(a) Event is: GGG probability = 0.512

(b) Event is: BBB probability = 0.008

(c) Event is:

GGB	GBG	BGG
0.128	0.128	0.128

probability = 0.128 + 0.128 + 0.128 = 0.384

(d) Event is:

GGB	GBG	BGG	GGG
0.128	0.128	0.128	0.512

probability = 0.128 + 0.128 + 0.128 + 0.512 = 0.896

$\boxed{2\text{-}15}$ (Alternate soln)

(a) $Pr[GGG] = \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} = \frac{64}{125} = 0.512$

(b) $Pr[BBB] = \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{125} = 0.008$

(c) $Pr[GGB + GBG + BGG]$

$$= 3\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right) = \frac{48}{125} = 0.384$$

(d) $Pr[GGB + GBG + BGG + GGG]$

$$= 3\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^3 = \frac{48}{125} + \frac{64}{125}$$

$$= \frac{112}{125} = 0.896$$

2-16

(a)

$$16 \cdot 16 \cdot 16 \cdot 16 = 16^4 = 65{,}536$$

(b)

$$16 \cdot 15 \cdot 14 \cdot 13 = \frac{16!}{12!} = 43{,}680$$

2.17

(a)

$$\binom{8}{2} = \frac{8!}{2!\,6!} = 28$$

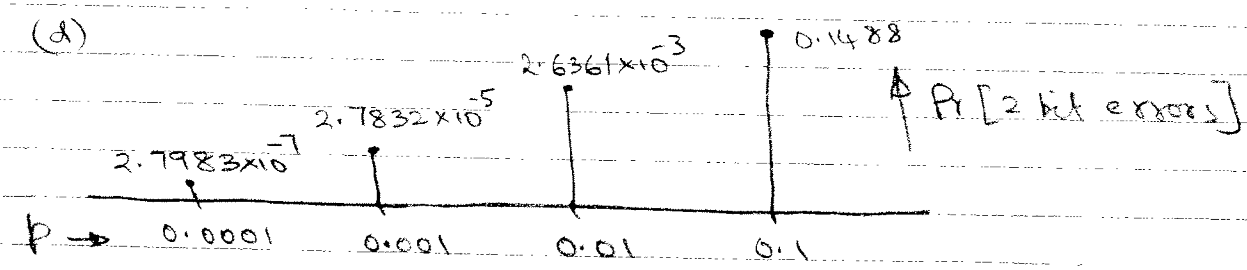
(b)

$$(1-p)\,p\,(1-p)(1-p)(1-p)\,p\,(1-p)(1-p) = p^2(1-p)^6$$

(c)

$$Pr[2 \text{ bit errors}] = \binom{8}{2} p^2 (1-p)^6 = 28\,p^2(1-p)^6$$

(d)



2-18

(a) $Pr[\text{Retrans. in 3rd packet}] = (1-p)^2 p = (1-0.01)^2 (0.01) = 9.801 \times 1$

(b) $Pr[\text{Retrans in 10th packet}] = (1-p)^9 p = 9.135 \times 10^{-3}$

(c)

$$Pr[\text{Retrans within the first 5 packets}]$$

$$= \sum_{k=0}^{4} (1-p)^k p = (1-p)^0 p + (1-p)^1 p + (1-p)^2 p + (1-p)^3 p + (1-p)^4 p$$

$$= 0.049$$

(d)

$$\sum_{k=0}^{N-1} (1-p)^k p \geq 0.1$$

$$p \cdot \frac{1-(1-p)^N}{1-(1-p)} = 1-(1-p)^N \geq 0.1$$

$$-(1-p)^N \geq -0.9 \quad \text{or} \quad (1-p)^N \leq 0.9$$

$$N \log_{10} 0.99 \leq \log_{10} 0.9$$

$$-N(4.3648) \leq -0.04576$$

$$\text{or} \quad N \geq 10.4833 \quad \text{or} \quad \underline{N > 10}$$

2-19

(a)

$$9Q(1-Q)^8$$

(b) $$Pr[\text{Errors and not detected}] = \sum_{k \text{ even}} Pr[k \text{ errors}]$$

$$= \sum_{k \text{ even}} \binom{9}{k} Q^k (1-Q)^{9-k}$$

$\boxed{2.20}$

$$F = ACD + ACE + BCE + BCD$$

$$= C\left(AD + AE + BE + BD\right)$$

$$= C \cdot (A+B) \cdot (D+E)$$

$\boxed{2.21}$

$$F = AD + BE + ACE + BCD$$

$$= A(D + CE) + B(E + CD)$$

Another possible answer is as follows:

$$F^c = A^cB^c + A^cC^cE^c + D^cE^c + D^cB^cC^c$$

$$= A^c(B^c + C^cE^c) + D^c(E^c + B^cC^c)$$

$$F = \left(A + B \cdot (C+E)\right) \cdot \left(D + E \cdot (B+C)\right)$$

$\boxed{2.22}$

$$F = C \cdot (A+B) \cdot (D+E)$$

$$Pr[F] = Pr[C] \cdot Pr[A+B] \cdot Pr[D+E]$$

$$Pr[A+B] = Pr[A] + Pr[B] - Pr[AB]$$
$$= Pr[A] + Pr[B] - Pr[A] \cdot Pr[B]$$

$$Pr[A+B] = 0.1 + 0.1 - (0.1)(0.1) = 0.19$$

like wise $Pr[D+E] = 0.19$

$$Pr[F] = (0.1)(0.19)(0.19) = 0.00361$$

An alternate procedure

Go to the sample space and identify all the elementary events that result in failure:

$\boxed{2.23}$

		F	F^c
$A_1 A_2 A_3 A_4$	p^4	✓	
$A_1 A_2 A_3 A_4^c$	$p^3(1-p)$	✓	
$A_1 A_2 A_3^c A_4$	$p^3(1-p)$	✓	
$A_1 A_2 A_3^c A_4^c$	$p^2(1-p)^2$		✓
$A_1 A_2^c A_3 A_4$	$p^3(1-p)$	✓	
$A_1 A_2^c A_3 A_4^c$	$p^2(1-p)^2$	✓	
$A_1 A_2^c A_3^c A_4$	$p^2(1-p)^2$	✓	
$A_1 A_2^c A_3^c A_4^c$	$p(1-p)^3$		✓
$A_1^c A_2 A_3 A_4$	$p^3(1-p)$	✓	
$A_1^c A_2 A_3 A_4^c$	$p^2(1-p)^2$	✓	
$A_1^c A_2 A_3^c A_4$	$p^2(1-p)^2$	✓	
$A_1^c A_2 A_3^c A_4^c$	$p(1-p)^3$		✓
$A_1^c A_2^c A_3 A_4$	$p^2(1-p)^2$	✓	
$A_1^c A_2^c A_3 A_4^c$	$p(1-p)^3$		✓
$A_1^c A_2^c A_3^c A_4$	$p(1-p)^3$	✓	
$A_1^c A_2^c A_3^c A_4^c$	$(1-p)^4$		✓

Then:

$$
\begin{aligned}
\Pr[F] = {} & p^4 + (p^3 - p^4) + (p^3 - p^4) \\
& + (p^3 - p^4) + (p^2 - 2p^3 + p^4) \\
& + (p^2 - 2p^3 + p^4) + (p^3 - p^4) \\
& + (p^2 - 2p^3 + p^4) + (p^2 - 2p^3 + p^4) \\
& + (p^2 - 2p^3 + p^4) + (p - 3p^2 + 3p^3 - p^4)
\end{aligned}
$$

$$= p + 2p^2 - 3p^3 + p^4$$