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About this manual

This manual contains solutions to the problems set at the end of each chapter of “Quantum Mechanics”. It is divided into sections corresponding to the chapters in the text and titled accordingly. Bracketed numbers refer to equations in the main text.



The Physics and Mathematics of Waves

1.1 Use Euler's formula to find a purely real expression for i^i .

Solution

$$i^i = (e^{i\pi/2})^i = e^{-\pi/2}$$

1.2 Show that (1.3) can be written as

$$x(t) = A \cos(\omega t + \phi)$$

and derive expressions for A and ϕ in terms of B and C . Assuming the oscillator starts out at position $x(0) = x_0$ with velocity $v(0) = v_0$, determine A and ϕ in terms of x_0 and v_0 . Note: We call A the *amplitude* and ϕ the *phase* of the oscillation.

Solution

Replace the constants B and C in $x(t) = B \cos \omega t + C \sin \omega t$ with two different constants A and ϕ which solve $B = A \cos \phi$ and $C = -A \sin \phi$. This results in $x(t) = A \cos(\omega t + \phi)$. Now $x(0) = A \cos \phi = x_0$ and $\dot{x}(0) = -\omega A \sin \phi = v_0$ so $A = (x_0^2 + v_0^2/\omega^2)^{1/2}$ and $\phi = -\tan^{-1}(v_0/x_0\omega)$.

1.3 A spring with stiffness k hangs vertically from point on the ceiling. A mass m is attached to the lower end of the spring without stretching it, and then is released from rest. Show that when the gravitational force mg is taken into account, the motion is still sinusoidal with $\omega = (k/m)^{1/2}$ but with an equilibrium position shifted to a lower point. Find the new equilibrium position in terms of m , k , and g .

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Solution

Let y measure the vertical position of the mass, with $y = 0$ the unstretched string. Then $m\ddot{y} = -ky - mg = -k(y + mg/k)$. Defining $x \equiv y + mg/k$, get $m\ddot{x} = -kx$, so once again $\omega^2 = k/m$. The equilibrium point is $x = 0$ or $y = -mg/k$.

1.4 Consider a system of a mass and spring, such as in Figure 1.1, but with an additional force $F_{\text{damp}} = -bv$ proportional to velocity but acting in the direction opposite to the motion. Reformulate the equation of motion, and find the solution for $x(0) = x_0$ and $v(0) = v_0$. Use the ansatz $x(t) = \exp(iat)$ to solve for α . You may assume that b^2/m^2 is less than $4k/m$.

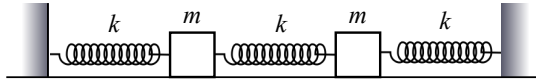
Solution

The equation of motion is $m\ddot{x} = -kx - b\dot{x}$ so $-m\alpha^2 = -k - i b\alpha$. Defining $\omega_0^2 \equiv k/m$ and $\beta \equiv b/2m$, yields the equation $\alpha^2 - 2i\beta\alpha - \omega_0^2 = 0$. Solving this,

$$\alpha = \frac{1}{2} \left[2i\beta \pm (-4\beta^2 + 4\omega_0^2)^{1/2} \right] = i\beta \pm \omega$$

where $\omega^2 \equiv \omega_0^2 - \beta^2$. (Note that $\beta^2/\omega_0^2 = (b^2/m^2)/(4k/m) < 1$ so ω is real.) The solution becomes $x(t) = Ae^{-\beta t} \cos(\omega t + \phi)$ where $A \cos \phi = x_0$ and $-A(\beta \cos \phi + \omega \sin \phi) = v_0$. These can be solved in principle, but the interesting solution is when $\beta \ll \omega$, i.e. “lightly damped” motion. The result is oscillation which slowly damps.

1.5 Two equal masses m move in one dimension and are each connected to fixed walls by springs with stiffness k . The masses are also connected to each other by a third, identical spring, as shown:



Write the (differential) equations of motion for the positions $x_1(t)$ and $x_2(t)$ of the two masses. Solve those equations with the ansatz $x_1(t) = A_1 \exp(iat)$ and $x_2(t) = A_2 \exp(iat)$; you will discover nontrivial solutions only for two values of ω^2 . (Those two values are called *eigenfrequencies*.) What kind of motion corresponds to each of these two eigenfrequencies?

Solution

Label the two masses #1 and #2 from left to right. The force on m #1 is $-kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$, and the force on m #2 is $-kx_2 - k(x_2 - x_1) = -2kx_2 + kx_1$, so, defining $\omega_0^2 \equiv k/m$, the equations of motion are

$$\ddot{x}_1 = -2\omega_0^2 x_1 + \omega_0^2 x_2 \quad \text{and} \quad \ddot{x}_2 = -2\omega_0^2 x_2 + \omega_0^2 x_1$$

Now insert the ansatz solution. After a little rearranging, you find

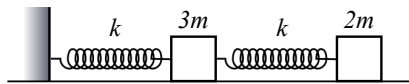
$$(2\omega_0^2 - \omega^2)A_1 - \omega_0^2 A_2 = 0 \quad \text{and} \quad -\omega_0^2 A_1 + (2\omega_0^2 - \omega^2)A_2 = 0$$

These are two homogenous equations for A_1 and A_2 . The only solution is $A_1 = A_2 = 0$, that is no motion, unless the determinant vanishes:

$$\omega_0^4 = (\omega^2 - 2\omega_0^2)^2 \quad \text{so} \quad \omega^2 = \omega_0^2 \quad \text{or} \quad \omega^2 = 3\omega_0^2$$

For $\omega^2 = \omega_0^2$, find $A_1 = A_2$ so the two masses oscillate in phase with the same amplitude. For $\omega^2 = 3\omega_0^2$, find $A_1 = -A_2$ so the two masses oscillate out of phase with the same amplitude.

1.6 Find the eigenfrequencies for the two-mass, two-spring system shown here:



Solution

Label mass $3m$ #1 and mass $2m$ #2. Then the equations of motion are

$$3m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \quad \text{and} \quad 2m\ddot{x}_2 = -k(x_2 - x_1) = kx_1 - kx_2$$

Using the standard definitions and ansatz,

$$(3\omega^2 - 2\omega_0^2)A_1 + \omega_0^2 A_2 = 0 \quad \text{and} \quad \omega_0^2 A_1 + (2\omega^2 - \omega_0^2)A_2 = 0$$

Next, set the determinant equal to zero to find

$$(3\omega^2 - 2\omega_0^2)(2\omega^2 - \omega_0^2) - \omega_0^4 = 6\omega^4 - 7\omega_0^2\omega^2 + \omega_0^2 = (6\omega^2 - \omega_0^2)(\omega^2 - \omega_0^2) = 0$$

so the eigenfrequencies are $\omega^2 = \omega_0^2$, in which case the two masses oscillate with equal amplitude but out of phase, and $\omega^2 = \omega_0^2/6$, in which case the oscillations are in phase with $A_1/A_2 = 2/3$.

1.7 For the two-mass, three-spring system discussed in Problem 1.5, find expressions for $x_1(t)$ and $x_2(t)$ subject to the initial conditions $x_1(0) = A$ and $x_2(0) = v_1(0) = v_2(0) = 0$. Make a plot of $x_1(t)$ and $x_2(t)$, and also plot the quantities $x_1(t) + x_2(t)$ and $x_1(t) - x_2(t)$. Comment on your observations.

Solution

Now we need to write the the general solution for the motion of the two masses:

$$\begin{aligned} x_1(t) &= ae^{i\omega_0 t} + be^{-i\omega_0 t} + ce^{\sqrt{3}i\omega_0 t} + de^{-\sqrt{3}i\omega_0 t} \\ x_2(t) &= ae^{i\omega_0 t} + be^{-i\omega_0 t} - ce^{\sqrt{3}i\omega_0 t} - de^{-\sqrt{3}i\omega_0 t} \end{aligned}$$

Note that we have maintained the amplitude ratios and relative phases between the different solutions for the particular eigenfrequencies. This is necessary in order

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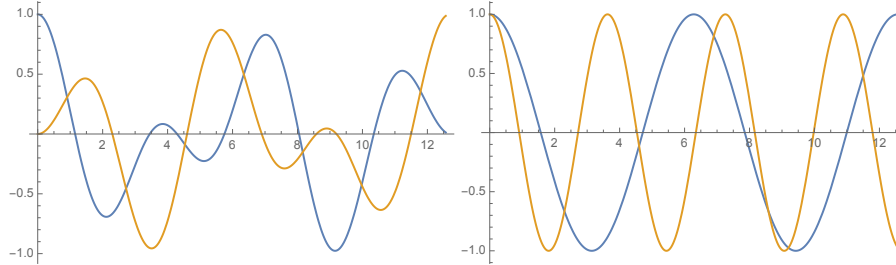
to make sure that each of the four terms separately solves the coupled differential equations. Now we can apply the initial conditions:

$$\begin{aligned} A &= a + b + c + d \\ 0 &= a - b + \sqrt{3}(c - d) \\ 0 &= a + b - c - d \\ 0 &= a - b - \sqrt{3}(c - d) \end{aligned}$$

Adding first and third gives $a + b = A/2$ and adding second and fourth gives $a - b = 0$, so $a = b = A/4$. Subtracting third from first gives $c + d = A/2$ and subtracting fourth from second gives $c - d = 0$ so $c = d = A/4$. Therefore

$$\begin{aligned} x_1(t) &= \frac{A}{2} [\cos(\omega_0 t) + \cos(\sqrt{3}\omega_0 t)] \\ x_2(t) &= \frac{A}{2} [\cos(\omega_0 t) - \cos(\sqrt{3}\omega_0 t)] \end{aligned}$$

Below left, plots of $x_1(t)$ and $x_2(t)$. Below right, plots of $x_1(t) + x_2(t)$ and $x_1(t) - x_2(t)$.



The wiggling motion is a superposition of different eigenfrequencies, but the sum and difference show the individual isolated eigenfrequencies.

1.8 Repeat Problem 1.7, but this time let the “coupling” spring between the two masses have a spring constant $k_c = k/100$. Show that the overall motion “oscillates” between cases where the first mass is in simple harmonic motion by itself, to one where the second mass is in simple harmonic motion, and then back again. What is the frequency of these low frequency oscillations between the two masses?

Solution

First, go back to Problem 5. Now, the force on $m \#1$ is $-kx_1 + k_c(x_2 - x_1) = -(k + k_c)x_1 + k_c x_2$, and the force on $m \#2$ is $-kx_2 - k_c(x_2 - x_1) = -(k + k_c)x_2 + k_c x_1$, so, defining $\omega_0^2 \equiv k/m$ and $\alpha^2 = 2k_c/m = 2(k_c/k)\omega_0^2$, the equations of motion are

$$\ddot{x}_1 = -(\omega_0^2 + \alpha^2/2)x_1 + (\alpha^2/2)x_2 \quad \text{and} \quad \ddot{x}_2 = -(\omega_0^2 + \alpha^2/2)x_2 + (\alpha^2/2)x_1$$

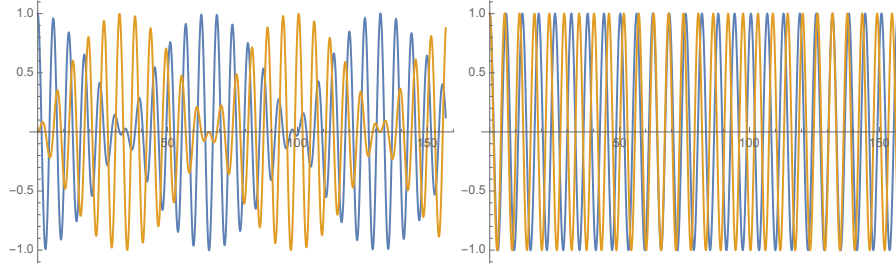
Now insert the ansatz solution. After a little rearranging, you find

$$(\omega_0^2 + \alpha^2/2 - \omega^2)A_1 - (\alpha^2/2)A_2 = 0 \quad \text{and} \quad -(\alpha^2/2)A_1 + (\omega_0^2 + \alpha^2/2 - \omega^2)A_2 = 0$$

so $\omega_0^2 + \alpha^2/2 - \omega^2 = \pm\alpha^2/2$ and the solutions are $\omega^2 = \omega_0^2$ and $\omega^2 = \omega_0^2 + \alpha^2$. It is easy to see, as in Problem 5, that these two solutions correspond to equal amplitude oscillations in phase and out of phase, respectively. At this point, the motions of the two masses work out just as in Problem 7, and we have

$$\begin{aligned}x_1(t) &= \frac{A}{2} \left[\cos(\omega_0 t) + \cos(\sqrt{\omega_0^2 + \alpha^2} t) \right] \\x_2(t) &= \frac{A}{2} \left[\cos(\omega_0 t) - \cos(\sqrt{\omega_0^2 + \alpha^2} t) \right]\end{aligned}$$

When $k = k_c$, $\alpha^2 = 2\omega_0^2$ and we get the correct solution to Problem 7. Following are the same plots, but for $k_c = k/10$, that is, $\alpha^2 = \omega_0^2/5$ (which plot more nicely than $k_c = k/100$):



The right plot shows that the eigenfrequencies are very close to each other, resulting in the beat pattern shown in the left plot. With $\alpha^2 \ll \omega_0^2$, and the trigonometric identities

$$\begin{aligned}\cos u + \cos v &= 2 \cos\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right) \\ \cos u - \cos v &= -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)\end{aligned}$$

it is clear that the plots are the product of a high frequency component

$$\frac{1}{2} \left(\omega_0 + \sqrt{\omega_0^2 + \alpha^2} \right) \approx \omega_0$$

with an envelope with low frequency

$$\frac{1}{2} \left(\sqrt{\omega_0^2 + \alpha^2} - \omega_0 \right) \approx \frac{\omega_0}{2} \left(1 + \frac{\alpha^2}{2\omega_0^2} - 1 \right) = \frac{\omega_0}{2} \frac{k_c}{k}$$

That is, for the left plot above, there are ≈ 20 crests within one envelope wavelength.

1.9 Derive the solution (1.13) to the wave equation (1.12) by going through the following steps. Consider a change of variables from x and y to $\xi = x - vt$ and $\eta = x + vt$. Then use the chain rule to rewrite the wave equation in terms of ξ and η . You should find that

$$\frac{\partial^2 y}{\partial \xi \partial \eta} = 0$$

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Then argue that this means that y is a function of either ξ or η , but not both at the same time. In other words, the solution is (1.13). If you are not familiar with the chain rule for partial differentiation, it means that if w and z are functions of x and y , then

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x}$$

and similarly for $\partial/\partial y$. You can assume that you get the same result regardless of the order in which the partial derivatives are taken.

Solution

As directed, apply the chain rule to the wave equation

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta} \\ \frac{\partial^2 y}{\partial x^2} &= \frac{\partial^2 y}{\partial \xi^2} + 2 \frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial \eta^2} \\ \frac{\partial y}{\partial t} &= -v \frac{\partial y}{\partial \xi} + v \frac{\partial y}{\partial \eta} \\ \frac{\partial^2 y}{\partial t^2} &= v^2 \frac{\partial^2 y}{\partial \xi^2} - 2v^2 \frac{\partial^2 y}{\partial \xi \partial \eta} + v^2 \frac{\partial^2 y}{\partial \eta^2} \\ \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} &= -4v^2 \frac{\partial^2 y}{\partial \xi \partial \eta} = 0 \end{aligned}$$

leading to the expression we sought. The obvious solution to this differential equation is the sum of two “constants” in ξ and η , respectively, that is $y(\xi, \eta) = f(\xi) + g(\eta)$.

1.10 Prove the principle of linear superposition for the wave equation (1.12). That is, show that if $y_1(x, t)$ and $y_2(x, t)$ are solutions of the wave equation, then $y(x, t) = ay_1(x, t) + by_2(x, t)$ is also a solution, where a and b are arbitrary constants.

Solution

All you have to do is plug it in and it falls out easily:

$$\begin{aligned} \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} &= a \frac{1}{v^2} \frac{\partial^2 y_1}{\partial t^2} + b \frac{1}{v^2} \frac{\partial^2 y_2}{\partial t^2} - a \frac{\partial^2 y_1}{\partial x^2} - b \frac{\partial^2 y_2}{\partial x^2} \\ &= a \left[\frac{1}{v^2} \frac{\partial^2 y_1}{\partial t^2} - \frac{\partial^2 y_1}{\partial x^2} \right] + b \left[\frac{1}{v^2} \frac{\partial^2 y_2}{\partial t^2} - \frac{\partial^2 y_2}{\partial x^2} \right] \\ &= 0 + 0 = 0 \end{aligned}$$

1.11 A string with linear mass density μ hangs motionless between two fixed points $(x, y) = (\pm a, 0)$ where y measures the vertical direction. The length of the string is greater than $2a$, so the lowest point is at $(x, y) = (0, b)$. Derive the differential equation