

Exercises

1.1 Numbers 101: The Very Basics

- The claim makes sense and is true.
 - The claim makes no sense; $\sqrt{8}$ isn't a subset.
 - The claim makes sense and is true.
 - The claim makes sense but is false; consider $a = 0$ and $b = \sqrt{2}$.
 - The claim makes sense and is true.
 - The claim makes sense but is false: consider $a = 0$.
- The claim is false; let $a = \sqrt{2}$.
 - The expression \mathbb{Q}^2 doesn't make sense.
 - The claim is true. Since $a^2 > 0$, some large n will work.
 - The claim is true; see Theorem ??.
 - If $a \in \mathbb{Q}$ and $a \neq 0$, then $a\sqrt{2} \notin \mathbb{Q}$.
- The number $1/a$ is an integer only if $a = \pm 1$. The number $1/a$ is rational for all nonzero integers a . The equation $1/a = a$ holds only if $a = \pm 1$.
- $1 \in S_1$ but $-1 \notin S_1$
 - $2 \in S_2$ but $1/2 \notin S_2$
 - $\sqrt{2} \in S_3$ but $1/\sqrt{2} = \sqrt{2}/2 \notin S_3$
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 - $\pi \in S_4$ but $\pi^2 \notin S_4$
- Theorem ?? is useful. Because \mathbb{Q} is closed under addition, multiplication, and division (if denominators aren't zero), expressions like those in (a) and (b) are rational—unless, for (b), $p = q = 0$. Expressions involving square roots are different. If $p = q = 1$, for instance, then $\sqrt{p^2 + q^2} = \sqrt{2}$ is irrational; the same expression is rational if $p = 3$ and $q = 4$. The quantity $\sqrt{p^2 + 2pq + q^2}$ is always rational, since $\sqrt{p^2 + 2pq + q^2} = \sqrt{(p+q)^2} = \pm(p+q)$. (Note that $\sqrt{(p+q)^2} = p+q$ may be false.)
- All of xy , $x+y$, $x-y$ and x/y can be either rational or irrational. Examples are easy to find.

8. The quantities in (a), (b), (c), (d) and (f) are all irrational; proofs are by contradiction. (E.g., if x/r were rational, then we could multiply by the rational number r . Then the product x is also rational, a contradiction.) \sqrt{r} can go either way.
9. Assume toward contradiction that $\sqrt{3} = a/b$ for integers a and b , where a/b is in reduced form. Then squaring both sides gives $3b^2 = a^2$. This implies (essentially as in the proof of Theorem ??) that 3 divides both a and b , which contradicts the assumption that a/b is in reduced form.
10. (a) Say $x^2 \notin \mathbb{Q}$. If $x \in \mathbb{Q}$, then (by Theorem ??) x^2 is rational, too, which contradicts our assumption.
- (b) Another proof by contradiction. If $x = \sqrt{2} + \sqrt{3}$ is rational, then $x^2 = 5 + 2\sqrt{6}$ is rational, too. This implies, in turn, that $\sqrt{6}$ is rational, which is absurd.
- (c) Yet another proof by contradiction. Let's write $x = \sqrt{2} + \sqrt{3} + \sqrt{5}$, then we have $x - \sqrt{5} = \sqrt{2} + \sqrt{3}$, and suppose x is rational. Squaring both sides of the last equation and simplifying gives

$$x^2 - 2x\sqrt{5} = 2\sqrt{6},$$

which is progress, since only *two* square roots remain. Squaring again gives

$$x^4 - 4x^3\sqrt{5} + 20x^2 = 24,$$

which is even better, as only *one* square root is left. The last equation implies that

$$\sqrt{5} = \frac{x^4 + 20x^2 - 24}{4x^3}.$$

Because x is rational, so is the right-hand side above, and thus so is the left. This absurdity completes the proof.

11. Parts (i) and (ii) follow from the fact that $1 < a/b < 2$. For part (iii), note that

$$\frac{a'^2}{b'^2} = \frac{(2b - a)^2}{(a - b)^2} = \frac{4b^2 - 4ab + a^2}{a^2 - 2ab + b^2},$$

and substituting $a^2 = 2b^2$ shows that the last fraction is 2.

This all shows that if $\sqrt{2} = a/b$ holds for *any* positive integers a and b , then we can find a new fraction a'/b' with $\sqrt{2} = a'/b'$ and $b' < b$, which is absurd.

12. (a) \mathbb{Z}_2 is not closed under addition: $1 + 1 = 2 \notin \mathbb{Z}_2$.
- (b) \mathbb{Z}_2 satisfies all the requirements in Theorem ??.

13. Matrix addition in $M_{2 \times 2}$ is commutative, but multiplication is not; examples are easy to find. Every matrix A in $M_{2 \times 2}$ has an additive inverse $-A$, but multiplicative inverses exist only for some nonzero matrices (those with nonzero determinant); again, examples are easy to find. Distributivity does indeed hold in $M_{2 \times 2}$.
14. If a and b are rational, then

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 + 2b^2}.$$

This shows that elements of F have multiplicative inverses in F . The rest is easier.

15. Since $\ln \ln \ln n$ tends to infinity, it must exceed two for large n . The well-ordering property guarantees that a smallest such n_0 exists. (Using a calculator we can see that n_0 has about 703 decimal digits.)
16. (a) The answer is no. For instance, the set $\{1, 1/2, 1/3, \dots\}$ is a subset of \mathbb{Q} , but has no least element.
- (b) The set $R = \{1, 10, 100, 1000, \dots\}$ does have the well-ordering property; every nonempty subset includes a *smallest* power of 10.
- (c) The set $T = \{-3, -2, -1, \dots, 41, 42\}$ (like all finite sets of real numbers) *does* have the well-ordering property, since every nonempty subset of T is also finite, and hence has a least element.
- (d) If we trade “least” for “greatest” in the well-ordering property, the result no longer holds for \mathbb{N} , since \mathbb{N} itself has no greatest element. The property *does* hold for the finite set T , and also for $\mathbb{Z} \setminus \mathbb{N} = \{\dots, -3, -2, -1, 0\}$.

1.2 Sets 101: Getting Started

1. (a) $D \subset I; D \in C$.
 - (b) $B = \{m \in A \mid m \text{ has 31 days}\}$.
 - (c) $A \times D$ is the set of ordered pairs (January, 2), (February, 2), ..., (December, 2), (January, 3), (February, 3), ..., (December, 3). There are 24 such pairs.
 - (d) $A \setminus B = \{\text{February, April, June, September, November}\}; B \setminus A = \emptyset;$
 $A \cap C = \{\text{November}\}; B \cap A = B; D \cap I = D; D \cup I = I$.
2. (a) $S = \{0, -1\}; T$ is the interval of numbers between $(-1 - \sqrt{21})/2 \approx -2.791$ and $(-1 + \sqrt{21})/2 \approx 1.791$.

- (b) Decide whether each of the following statements is true or false, and explain: $S \subset \mathbb{N}$ is false because $-1 \notin \mathbb{N}$; $S \subset T$ is true; $T \cap \mathbb{Q} \neq \emptyset$ is true, since $0 \in T \cap \mathbb{Q}$; $-2.8 \in \mathbb{Q} \setminus T$ is true.
- (c) The quadratic formula shows that $U = \{x \in \mathbb{R} \mid x^2 + x < 0\} = (-1, 0)$.
3. (a) $\mathbb{R} \setminus A = (-\infty, 1) \cup (3, \infty)$
 (c) $\mathbb{R} \setminus A = (-\infty, 1] \cup [2, 3] \cup [4, \infty)$
 (e) $\mathbb{R} \setminus A = \{0\}$
4. (a) $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$
 (b) $I = (-\infty, \infty)$ has empty complement; $I = (-\infty, 17)$ has closed complement $[17, \infty)$; $I = (0, 17)$ has complement $(-\infty, 0] \cup [17, \infty)$.
 (c) $\mathbb{R} \setminus \mathbb{Z} = (0, 1) \cup (-1, 0) \cup (1, 2) \cup (-2, -1) \cup \dots$
5. To say that a is in $\mathbb{R} \setminus (\mathbb{R} \setminus A)$ means that a is *not* in $\mathbb{R} \setminus A$; this means, in turn that $a \in A$.
6. (a) Claim (i) is false. As one example, take $A = \mathbb{R}$ and $B = \emptyset$. Then $R \setminus (A \cup B) = \emptyset$ but $(R \setminus A) \cup (R \setminus B) = \mathbb{R}$.
 (b) If $A = B$, then $A \cup B = A \cap B = A$, and both claims are clearly true.
 (c) To prove that (ii) holds, suppose $x \in \mathbb{R} \setminus (A \cup B)$. Thus $x \notin A \cup B$, so $x \notin A$ and $x \notin B$; in other words, $x \in \mathbb{R} \setminus A$ and $x \in \mathbb{R} \setminus B$. This is just another way of saying that $x \in (\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B)$. The “vice versa” implication is similar.
7. We know $\mathbb{R} \setminus A_1 = (-\infty, 1] \cup [3, \infty)$ and $\mathbb{R} \setminus A_2 = (-\infty, 2] \cup [5, \infty)$. Also, $\mathbb{R} \setminus (A_1 \cap A_2) = (-\infty, 2] \cup [3, \infty)$ and $\mathbb{R} \setminus (A_1 \cup A_2) = (-\infty, 1] \cup [5, \infty)$.
 It's easy to see that, as claimed, $\mathbb{R} \setminus (A_1 \cap A_2) = (-\infty, 2] \cup [3, \infty) = (-\infty, 1] \cup [3, \infty) \cup (-\infty, 2] \cup [5, \infty) = (\mathbb{R} \setminus A_1) \cup (\mathbb{R} \setminus A_2)$. Similarly, $\mathbb{R} \setminus (A_1 \cup A_2) = (-\infty, 1] \cup [5, \infty) = ((-\infty, 1] \cup [3, \infty)) \cap ((-\infty, 2] \cup [5, \infty)) = (\mathbb{R} \setminus A_1) \cap (\mathbb{R} \setminus A_2)$.
8. Here we have $A_1 \cup A_2 = (0, 1) \cup (2, \infty)$ and $A_1 \cap A_2 = \emptyset$. Thus $\mathbb{R} \setminus A_1 = (-\infty, 0] \cup [1, \infty)$ and $\mathbb{R} \setminus A_2 = (-\infty, 2]$. This implies that $(\mathbb{R} \setminus A_1) \cup (\mathbb{R} \setminus A_2) = (-\infty, \infty)$ and $(\mathbb{R} \setminus A_1) \cap (\mathbb{R} \setminus A_2) = (-\infty, 0] \cup [1, 2]$. Consistent with De Morgan, $\mathbb{R} \setminus (A_1 \cap A_2) = (-\infty, \text{infity})$; $\mathbb{R} \setminus (A_1 \cup A_2) = (-\infty, 0] \cup [1, 2]$.
9. If $x \in T'$ then $x \notin T$. Since $T \supset S$, we have $x \notin S$, which means $x \in S'$, as desired.

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10. Many possibilities exist for (a), (b), and (c). For (d) we could use $I = (0, \infty)$ and $J = (1, \infty)$; note that here, as in *all* possibilities for (d), one interval is contained in the other.
11. (a) $I = (-42, 0)$ and $J = (0, \infty)$ work.
(b) $I = (-42, 0)$ and $J = [0, \infty)$ work.
(c) The given conditions (draw a picture) mean that $a < c < 0 < b < d$, so $I \cup J = (a, d)$ which is indeed an open interval.
12. Suppose $a \in I \cup J$, $b \in I \cup J$, and $a < x < b$. We're done if we show that $x \in I \cup J$. This is trivial if $x = c$, so we assume $x \neq c$. Now if both $a \in I$ and $b \in I$, then $x \in I$ by Definition ??, and we're done. Similarly, we're done if both $a \in J$ and $b \in J$. So let's assume $a \in I$ and $b \in J$. If $x < c$, then we have $a < x < c$ with a and c in I ; by Definition ??, $x \in I$, too. Similarly, if $x > c$, then we have $c < x < b$ and so $x \in J$. We're done.
13. It's easy for I and $\mathbb{R} \setminus I$ to be intervals. For instance, if $I = (-\infty, 0)$, then $\mathbb{R} \setminus I = [0, \infty)$ is another interval. I and $\mathbb{R} \setminus I$ cannot both be bounded intervals; two bounded intervals can't "add up" to the unbounded set $(-\infty, \infty)$.
14. No. Any finite set I of numbers contains a smallest number, say a , and a second smallest, say b . If I were an interval, it would also have to contain the average, $(a + b)/2$, which lies (illegally) between a and b .
15. No. Suppose a and b are rational numbers in I , with $a < b$. Consider $c = a + (b - a)/\sqrt{2}$. Note that $c \in \mathbb{R} \setminus \mathbb{Q}$ and that $a < c < b$. If I were an interval, we'd have $c \in I$, which is impossible.
16. (a) $(1, 2) \cup (3, \infty)$ is the union of two open intervals, and hence also open. The complement, $(-\infty, 2] \cup [2, 3]$, is therefore closed.
(b) $\mathbb{R} \setminus \{a\} = (-\infty, a) \cup (a, \infty)$.
(c) $(-\infty, a)$ is itself an open interval, and therefore open. The complement of $(-\infty, a]$ is the open interval (a, ∞) , so $(-\infty, a]$ is closed.
(d) If $I = (0, 1)$ were closed, then $\mathbb{R} \setminus I$ would be open. This is false, because $1 \in \mathbb{R} \setminus I$, but no open interval containing 1 is contained in $\mathbb{R} \setminus I$.
17. (a) The complement of $\{1, 2, 3\}$ consists of four open intervals.
(b) $\mathbb{R} \setminus \mathbb{Z}$ is the union of all open intervals of the form $(n, n + 1)$, where $n \in \mathbb{Z}$.
(c) If \mathbb{Q} were open, we could find for each rational q an open interval I with $q \in I \subseteq \mathbb{Q}$. But $I \subseteq \mathbb{Q}$ is impossible.