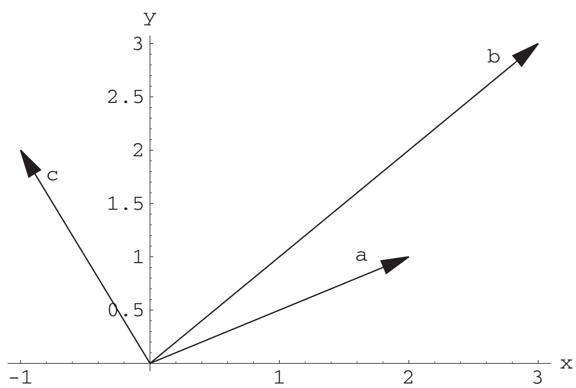
Chapter 1

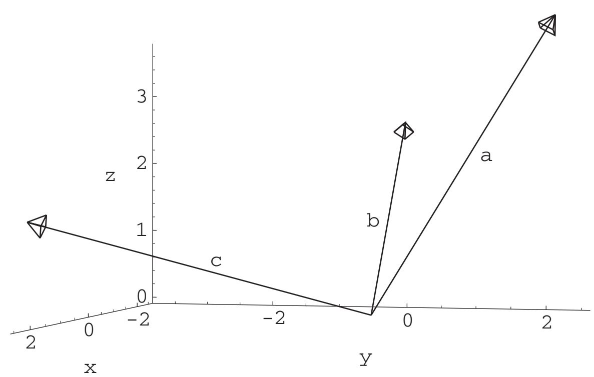
Vectors

1.1 Vectors in Two and Three Dimensions

1. *Here we just connect the point (0, 0) to the points indicated*:



2. *Although more difficult for students to represent this on paper, the figures should look something like the following. Note that the origin is not at a corner of the frame box but is at the tails of the three vectors*.

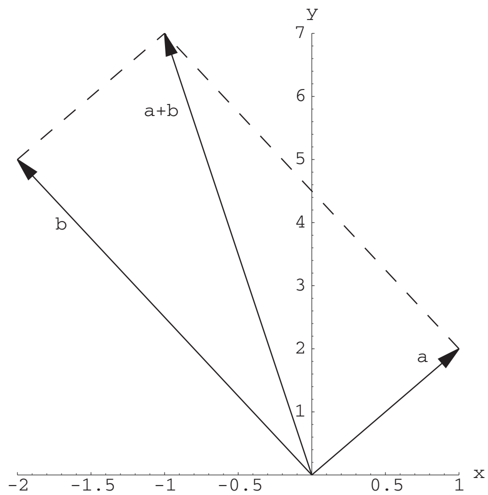


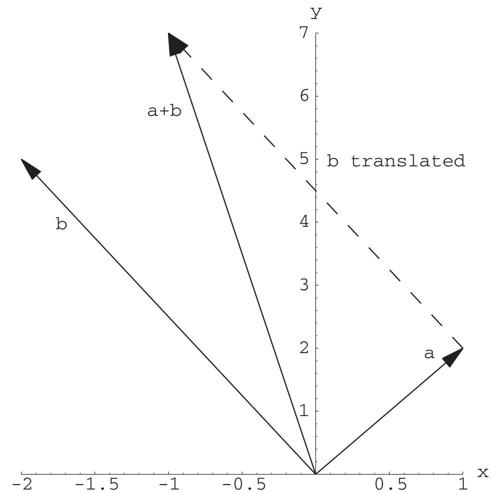
In problems 3 and 4, we supply more detail than is necessary to stress to students what properties are being used:

3. **(a)** $(3,1)+(-1,7)=(3+[-1],1+7)=(2,8)$.
(b) $-2(8,12)=(-2\cdot 8,-2\cdot 12)=(-16,-24)$.
(c) $(8,9)+3(-1,2)=(8+3(-1),9+3(2))=(5,15)$.
(d) $(1,1)+5(2,6)-3(10,2)=(1+5\cdot 2-3\cdot 10,1+5\cdot 6-3\cdot 2)=(-19,25)$.
(e) $(8,10)+3((8,-2)-2(4,5))=(8+3(8-2\cdot 4),10+3(-2-2\cdot 5))=(8,-26)$.

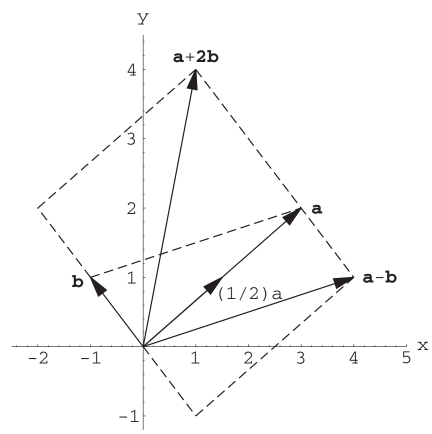
4. **(a)** $(2,1,2)+(-3,9,7)=(2-3,1+9,2+7)=(-1,10,9)$.
(b) $\frac{1}{2}(8,4,1)+2\left(5,-7,\frac{1}{4}\right)=\left(4,2,\frac{1}{2}\right)+\left(10,-14,\frac{1}{2}\right)=(14,-12,1)$.
(c) $-2\left((2,0,1)-6\left(\frac{1}{2},-4,1\right)\right)=-2((2,0,1)-(3,-24,6))=-2(-1,24,-5)=(2,-48,10)$.

5. We start with the two vectors **a** and **b**. We can complete the parallelogram as in the figure on the left. The vector from the origin to this new vertex is the vector $\mathbf{a}+\mathbf{b}$. In the figure on the right we have translated vector **b** so that its tail is the head of vector **a**. The sum $\mathbf{a}+\mathbf{b}$ is the directed third side of this triangle.

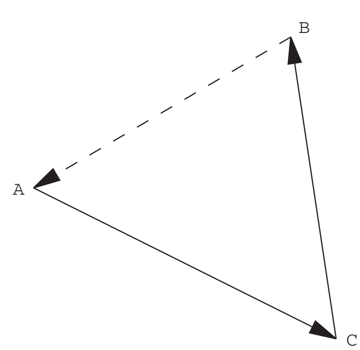




6. $\mathbf{a} = (3, 2) \quad \mathbf{b} = (-1, 1)$
$\mathbf{a} - \mathbf{b} = (3 - (-1), 2 - 1) = (4, 1) \qquad \frac{1}{2}\mathbf{a} = \left(\frac{3}{2}, 1\right) \quad \mathbf{a} + 2\mathbf{b} = (1, 4)$



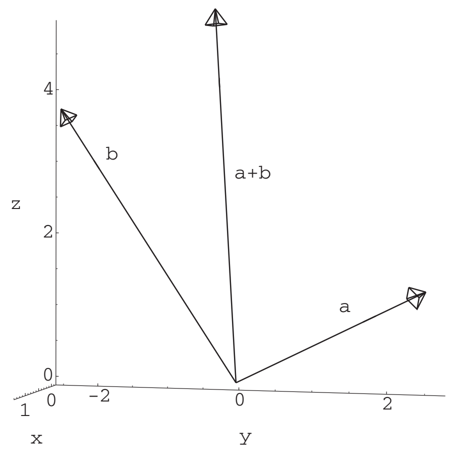
7. **(a)** $\overrightarrow{AB} = (-3 - 1, 3 - 0, 1 - 2) = (-4, 3, -1) \quad \overrightarrow{BA} = -\overrightarrow{AB} = (4, -3, 1)$
(b) $\overrightarrow{AC} = (2 - 1, 1 - 0, 5 - 2) = (1, 1, 3)$
$\overrightarrow{BC} = (2 - (-3), 1 - 3, 5 - 1) = (5, -2, 4)$
$\overrightarrow{AC} + \overrightarrow{CB} = (1, 1, 3) - (5, -2, 4) = (-4, 3, -1)$
(c) This result is true in general:



Head-to-tail addition demonstrates this.

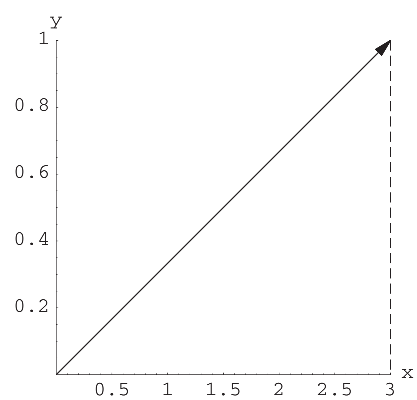
8. The vectors $\mathbf{a} = (1, 2, 1)$, $\mathbf{b} = (0, -2, 3)$ and $\mathbf{a} + \mathbf{b} = (1, 2, 1) + (0, -2, 3) = (1, 0, 4)$ are graphed below. *Again note that the origin is at the tails of the vectors in the figure*.

Also, $-1(1, 2, 1) = (-1, -2, -1)$. This would be pictured by drawing the vector (1, 2, 1) in the opposite direction. Finally, $4(1, 2, 1) = (4, 8, 4)$ which is four times vector **a** and so is vector **a** stretched four times as long in the same direction.

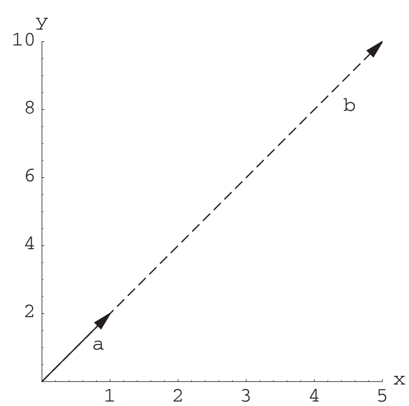


9. *Since the sum on the left must equal the vector on the right componentwise*:
$-12 + x = 2$, $9 + 7 = y$, and $z + -3 = 5$. Therefore, $x = 14$, $y = 16$, and $z = 8$.

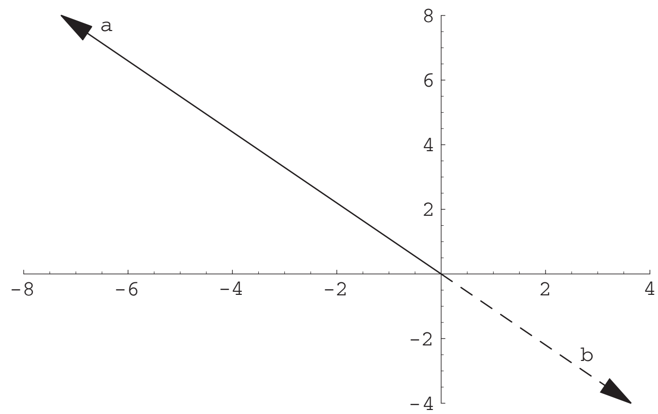
10. If we drop a perpendicular from (3, 1) to the x-axis we see that by the Pythagorean Theorem the length of the vector $(3, 1) = \sqrt{3^2 + 1^2} = \sqrt{10}$.



11. Notice that $\mathbf{b}$ (represented by the dotted line) $= 5\mathbf{a}$ (represented by the solid line).



12. Here the picture has been projected into two dimensions so that you can more clearly see that **a** (represented by the solid line) $= -2\mathbf{b}$ (represented by the dotted line).



13. The natural extension to higher dimensions is that we still add componentwise and that multiplying a scalar by a vector means that we multiply each component of the vector by the scalar. In symbols this means that:

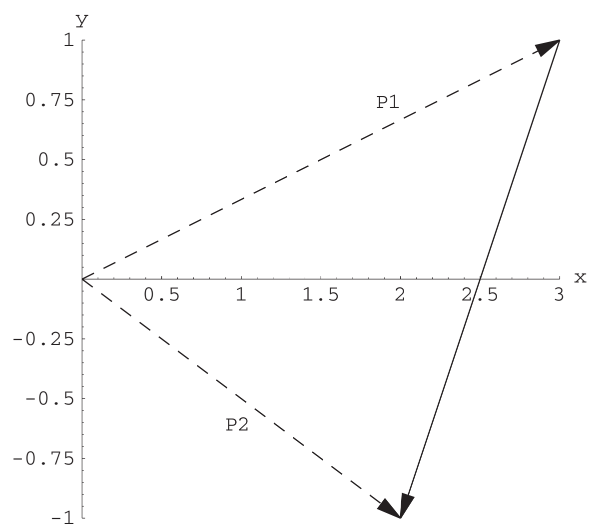
$\mathbf{a}+\mathbf{b} = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ and $k\mathbf{a} = (ka_1, ka_2, \ldots, ka_n)$.

In our particular examples, $(1, 2, 3, 4) + (5, -1, 2, 0) = (6, 1, 5, 4)$, and $2(7, 6, -3, 1) = (14, 12, -6, 2)$.

14. The diagrams for parts (a), (b) and (c) are similar to Figure 1.12 from the text. The displacement vectors are:

(a) $(1, 1, 5)$
(b) $(-1, -2, 3)$
(c) $(1, 2, -3)$
(d) $(-1, -2)$

Note: The displacement vectors for (b) and (c) are the same but in opposite directions (i.e., one is the negative of the other). The displacement vector in the diagram for (d) is represented by the solid line in the figure below:

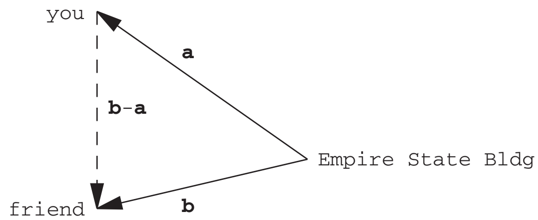


15. In general, we would define the displacement vector from $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_n)$ to be $(b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)$.

In this specific problem the displacement vector from P_1 to P_2 is $(1, -4, -1, 1)$.

16. Let B have coordinates (x, y, z). Then $\overrightarrow{AB} = (x - 2, y - 5, z + 6) = (12, -3, 7)$ so $x = 14$, $y = 2$, $z = 1$ so B has coordinates (14, 2, 1).

17. If $\mathbf{a}$ is your displacement vector from the Empire State Building and $\mathbf{b}$ your friend's, then the displacement vector from you to your friend is $\mathbf{b} - \mathbf{a}$.



18. Property 2 follows immediately from the associative property of the reals:

$$\begin{aligned}(\mathbf{a}+\mathbf{b})+\mathbf{c} &= ((a_1,a_2,a_3)+(b_1,b_2,b_3))+(c_1,c_2,c_3)\\ &= ((a_1+b_1,a_2+b_2,a_3+b_3)+(c_1,c_2,c_3)\\ &= ((a_1+b_1)+c_1,(a_2+b_2)+c_2,(a_3+b_3)+c_3)\\ &= (a_1+(b_1+c_1),a_2+(b_2+c_2),a_3+(b_3+c_3))\\ &= (a_1,a_2,a_3)+((b_1+c_1),(b_2+c_2),(b_3+c_3))\\ &= \mathbf{a}+(\mathbf{b}+\mathbf{c}).\end{aligned}$$

Property 3 also follows from the corresponding componentwise observation:

$$\mathbf{a}+\mathbf{0}=(a_1+0,a_2+0,a_3+0)=(a_1,a_2,a_3)=\mathbf{a}.$$

19. *We provide the proofs for* $\mathbf{R}^3$:

$$\begin{aligned}(1)\ (k+l)\mathbf{a} &= (k+l)(a_1,a_2,a_3)=((k+l)a_1,(k+l)a_2,(k+l)a_3)\\ &= (ka_1+la_1,ka_2+la_2,ka_3+la_3)=k\mathbf{a}+l\mathbf{a}.\\ (2)\ k(\mathbf{a}+\mathbf{b}) &= k((a_1,a_2,a_3)+(b_1,b_2,b_3))=k(a_1+b_1,a_2+b_2,a_3+b_3)\\ &= (k(a_1+b_1),k(a_2+b_2),k(a_3+b_3))=(ka_1+kb_1,ka_2+kb_2,ka_3+kb_3)\\ &= (ka_1,ka_2,ka_3)+(kb_1,kb_2,kb_3)=k\mathbf{a}+k\mathbf{b}.\\ (3)\ k(l\mathbf{a}) &= k(l(a_1,a_2,a_3))=k(la_1,la_2,la_3)\\ &= (kla_1,kla_2,kla_3)=(lka_1,lka_2,lka_3)\\ &= l(ka_1,ka_2,ka_3)=l(k\mathbf{a}).\end{aligned}$$

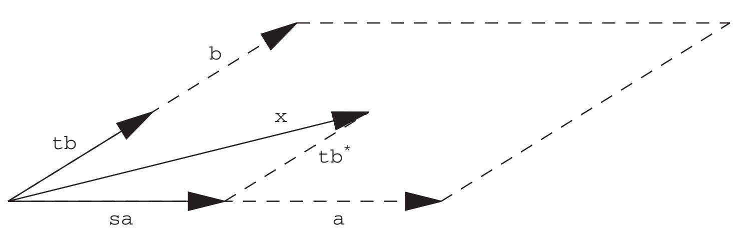
20. (a) $0\mathbf{a}$ is the zero vector. For example, in $\mathbf{R}^3$:

$$0\mathbf{a}=0(a_1,a_2,a_3)=(0\cdot a_1,0\cdot a_2,0\cdot a_3)=(0,0,0).$$

(b) $1\mathbf{a}=\mathbf{a}$. Again in $\mathbf{R}^3$:

$$1\mathbf{a}=1(a_1,a_2,a_3)=(1\cdot a_1,1\cdot a_2,1\cdot a_3)=(a_1,a_2,a_3)=\mathbf{a}.$$

21. (a) The head of the vector $s\mathbf{a}$ is on the x-axis between 0 and 2. Similarly the head of the vector $t\mathbf{b}$ lies somewhere on the vector **b**. Using the head-to-tail method, $s\mathbf{a}+t\mathbf{b}$ is the result of translating the vector $t\mathbf{b}$, in this case, to the right by $2s$ (represented in the figure by $t\mathbf{b}$*****). The result is clearly inside the parallelogram determined by **a** and **b** (and is only on the boundary of the parallelogram if either t or s is 0 or 1.



(b) Again the vectors **a** and **b** will determine a parallelogram (with vertices at the origin, and at the heads of **a**, **b**, and $\mathbf{a} + \mathbf{b}$. The vectors $s\mathbf{a} + t\mathbf{b}$ will be the position vectors for all points in that parallelogram determined by (2, 2, 1) and (0, 3, 2).

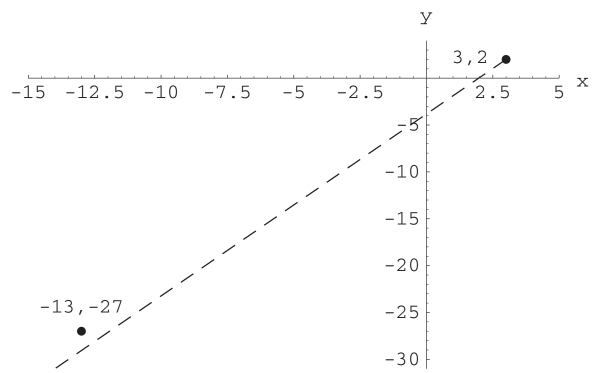
22. Here we are translating the situation in Exercise 21 by the vector $\overrightarrow{OP_0}$. The vectors will all be of the form $\overrightarrow{OP_0} + s\mathbf{a} + t\mathbf{b}$ for $0 \leq s, t \leq 1$.

23. **(a)** The speed of the flea is the length of the velocity vector $= \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$ units per minute.

(b) After 3 minutes the flea is at $(3, 2) + 3(-1, -2) = (0, -4)$.

(c) We solve $(3, 2)+t(-1, -2) = (-4, -12)$ for t and get that $t = 7$ minutes. Note that *both* $3-7 = -4$ and $2-14 = -12$.

(d) We can see this algebraically or geometrically: Solving the x part of $(3, 2) + t(-1, -2) = (-13, -27)$ we get that $t = 16$. But when $t = 16$, $y = -30$ not -27. Also in the figure below we see the path taken by the flea will miss the point $(-13, -27)$.



24. **(a)** The plane is climbing at a rate of 4 miles per hour.

(b) To make sure that the axes are oriented so that the plane passes over the building, the positive x direction is east and the positive y direction is north. Then we are heading east at a rate of 50 miles per hour at the same time we're heading north at a rate of 100 miles per hour. We are directly over the skyscraper in 1/10 of an hour or 6 minutes.

(c) Using our answer in (b), we have traveled for 1/10 of an hour and so we've climbed 4/10 of a mile or 2112 feet. The plane is $2112 - 1250$ or 862 feet about the skyscraper.

25. **(a)** Adding we get: $\mathbf{F}_1 + \mathbf{F}_2 = (2, 7, -1) + (3, -2, 5) = (5, 5, 4)$.

(b) You need a force of the same magnitude in the opposite direction, so $\mathbf{F}_3 = -(5, 5, 4) = (-5, -5, -4)$.

26. **(a)** Measuring the force in pounds we get $(0, 0, -50)$.

(b) The z components of the two vectors along the ropes must be equal and their sum must be opposite of the z component in part (a). Their y components must also be opposite each other. Since the vector points in the direction (0, ±2, 1), the y component will be twice the z component. Together this means that the vector in the direction of $(0, -2, 1)$ is $(0, -50, 25)$ and the vector in the direction (0, 2, 1) is (0, 50, 25).

27. The force $\mathbf{F}$ due to gravity on the weight is given by $\mathbf{F} = (0, 0, -10)$. The forces along the ropes are each parallel to the displacement vectors from the weight to the respective anchor points. That is, the tension vectors along the ropes are

$$\mathbf{F}_1 = k((3, 0, 4) - (1, 2, 3)) = k(2, -2, 1)$$
$$\mathbf{F}_2 = l((0, 3, 5) - (1, 2, 3)) = l(-1, 1, 2),$$

where k and l are appropriate scalars. For the weight to remain in equilibrium, we must have $\mathbf{F}_1+\mathbf{F}_2+\mathbf{F} = \mathbf{0}$, or, equivalently, that

$$k(2, -2, 1) + l(-1, 1, 2) + (0, 0, -10) = (0, 0, 0).$$

Taking components, we obtain a system of three equations:

$$\begin{cases} 2k - l = 0 \\ -2k + l = 0 \\ k + 2l = 10. \end{cases}$$

Solving, we find that $k = 2$ and $l = 4$, so that

$$\mathbf{F}_1 = (4, -4, 2) \text{ and } \mathbf{F}_2 = (-4, 4, 8).$$

1.2 More about Vectors

It may be useful to point out that the answers to Exercises 1 and 5 are the "same", but that in Exercise 1, $\mathbf{i} = (1, 0)$ *and in Exercise 5,* $\mathbf{i} = (1, 0, 0)$. *This comes up when going the other direction in Exercises 9 and 10. In other words, it's not always clear whether the exercise "lives" in* $\mathbf{R}^2$ *or* $\mathbf{R}^3$.

1. $(2, 4) = 2(1, 0) + 4(0, 1) = 2\mathbf{i} + 4\mathbf{j}$.

2. $(9, -6) = 9(1, 0) - 6(0, 1) = 9\mathbf{i} - 6\mathbf{j}$.

3. $(3, \pi, -7) = 3(1, 0, 0) + \pi(0, 1, 0) - 7(0, 0, 1) = 3\mathbf{i} + \pi\mathbf{j} - 7\mathbf{k}$.

4. $(-1, 2, 5) = -1(1, 0, 0) + 2(0, 1, 0) + 5(0, 0, 1) = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$.

5. $(2, 4, 0) = 2(1, 0, 0) + 4(0, 1, 0) = 2\mathbf{i} + 4\mathbf{j}$.

6. $\mathbf{i} + \mathbf{j} - 3\mathbf{k} = (1, 0, 0) + (0, 1, 0) - 3(0, 0, 1) = (1, 1, -3)$.

7. $9\mathbf{i} - 2\mathbf{j} + \sqrt{2}\mathbf{k} = 9(1, 0, 0) - 2(0, 1, 0) + \sqrt{2}(0, 0, 1) = (9, -2, \sqrt{2})$.

8. $-3(2\mathbf{i} - 7\mathbf{k}) = -6\mathbf{i} + 21\mathbf{k} = -6(1, 0, 0) + 21(0, 0, 1) = (-6, 0, 21)$.

9. $\pi\mathbf{i} - \mathbf{j} = \pi(1, 0) - (0, 1) = (\pi, -1)$.

10. $\pi\mathbf{i} - \mathbf{j} = \pi(1, 0, 0) - (0, 1, 0) = (\pi, -1, 0)$.

Note: You may want to assign both Exercises 11 and 12 together so that the students may see the difference. You should stress that the reason the results are different has nothing to do with the fact that Exercise 11 is a question about $\mathbf{R}^2$ *while Exercise 12 is a question about* $\mathbf{R}^3$.

11. (a) $(3, 1) = c_1(1, 1) + c_2(1, -1) = (c_1 + c_2, c_1 - c_2)$, so $\begin{cases} c_1 + c_2 = 3, \text{ and} \\ c_1 - c_2 = 1. \end{cases}$

Solving simultaneously (for instance by adding the two equations), we find that $2c_1 = 4$, so $c_1 = 2$ and $c_2 = 1$. So $\mathbf{b} = 2\mathbf{a}_1 + \mathbf{a}_2$.

(b) Here $c_1 + c_2 = 3$ and $c_1 - c_2 = -5$, so $c_1 = -1$ and $c_2 = 4$. So $\mathbf{b} = -\mathbf{a}_1 + 4\mathbf{a}_2$.

(c) More generally, $(b_1, b_2) = (c_1 + c_2, c_1 - c_2)$, so $\begin{cases} c_1 + c_2 = b_1, \text{ and} \\ c_1 - c_2 = b_2. \end{cases}$

Again solving simultaneously, $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$. So

$$\mathbf{b} = \left(\frac{b_1 + b_2}{2}\right)\mathbf{a}_1 + \left(\frac{b_1 - b_2}{2}\right)\mathbf{a}_2.$$

12. Note that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$, so really we are only working with two (linearly independent) vectors.

(a) $(5, 6, -5) = c_1(1, 0, -1) + c_2(0, 1, 0) + c_3(1, 1, -1)$; this gives us the equations:

$$\begin{cases} 5 = c_1 + c_3 \\ 6 = c_2 + c_3 \\ -5 = -c_1 - c_3. \end{cases}$$

The first and last equations contain the same information and so we have infinitely many solutions. You will quickly see one by letting $c_3 = 0$. Then $c_1 = 5$ and $c_2 = 6$. So we could write $\mathbf{b} = 5\mathbf{a}_1 + 6\mathbf{a}_2$. More generally, you can choose any value for c_1 and then let $c_2 = c_1 + 1$ and $c_3 = 5 - c_1$.

(b) We cannot write (2, 3, 4) as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. Here we get the equations:

$$\begin{cases} c_1 + c_3 = 2 \\ c_2 + c_3 = 3 \\ -c_1 - c_3 = 4. \end{cases}$$

The first and last equations are inconsistent and so the system cannot be solved.

(c) As we saw in part (b), not all vectors in $\mathbf{R}^3$ can be written in terms of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. In fact, only vectors of the form $(a, b, -a)$ can be written in terms of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. For your students who have had linear algebra, this is because the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are not linearly independent.

Note: As pointed out in the text, the answers for 13–21 are not unique.

13. $\mathbf{r}(t) = (2, -1, 5) + t(1, 3, -6)$ so $\begin{cases} x = 2 + t \\ y = -1 + 3t \\ z = 5 - 6t. \end{cases}$

14. $\mathbf{r}(t) = (12, -2, 0) + t(5, -12, 1)$ so $\begin{cases} x = 12 + 5t \\ y = -2 - 12t \\ z = t. \end{cases}$

15. $\mathbf{r}(t) = (2, -1) + t(1, -7)$ so $\begin{cases} x = 2 + t \\ y = -1 - 7t. \end{cases}$

16. $\mathbf{r}(t) = (2, 1, 2) + t(3 - 2, -1 - 1, 5 - 2)$ so $\begin{cases} x = 2 + t \\ y = 1 - 2t \\ z = 2 + 3t. \end{cases}$

17. $\mathbf{r}(t) = (1, 4, 5) + t(2 - 1, 4 - 4, -1 - 5)$ so $\begin{cases} x = 1 + t \\ y = 4 \\ z = 5 - 6t. \end{cases}$

18. $\mathbf{r}(t) = (8, 5) + t(1 - 8, 7 - 5)$ so $\begin{cases} x = 8 - 7t \\ y = 5 + 2t. \end{cases}$

Note: In higher dimensions, we switch our notation to x_i.

19. $\mathbf{r}(t) = (1, 2, 0, 4) + t(-2, 5, 3, 7)$ so $\begin{cases} x_1 = 1 - 2t \\ x_2 = 2 + 5t \\ x_3 = 3t \\ x_4 = 4 + 7t. \end{cases}$

20. $\mathbf{r}(t) = (9, \pi, -1, 5, 2) + t(-1 - 9, 1 - \pi, \sqrt{2} + 1, 7 - 5, 1 - 2)$ so $\begin{cases} x_1 = 9 - 10t \\ x_2 = \pi + (1 - \pi)t \\ x_3 = -1 + (\sqrt{2} + 1)t \\ x_4 = 5 + 2t \\ x_5 = 2 - t. \end{cases}$

21. **(a)** $\mathbf{r}(t) = (-1, 7, 3) + t(2, -1, 5)$ so $\begin{cases} x = -1 + 2t \\ y = 7 - t \\ z = 3 + 5t. \end{cases}$

(b) $\mathbf{r}(t) = (5, -3, 4) + t(0 - 5, 1 + 3, 9 - 4)$ so $\begin{cases} x = 5 - 5t \\ y = -3 + 4t \\ z = 4 + 5t. \end{cases}$

(c) Of course, there are infinitely many solutions. For our variation on the answer to (a) we note that a line parallel to the vector $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ is also parallel to the vector $-(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ so another set of equations for part (a) is:

$$\begin{cases} x = -1 - 2t \\ y = 7 + t \\ z = 3 - 5t. \end{cases}$$

For our variation on the answer to (b) we note that the line passes through both points so we can set up the equation with respect to the other point:

$$\begin{cases} x = -5t \\ y = 1 + 4t \\ z = 9 + 5t. \end{cases}$$

(d) The symmetric forms are:

$$\frac{x+1}{2} = 7 - y = \frac{z-3}{5} \qquad \text{(for (a))}$$

$$\frac{5-x}{5} = \frac{y+3}{4} = \frac{z-4}{5} \qquad \text{(for (b))}$$

$$\frac{x+1}{-2} = y - 7 = \frac{z-3}{-5} \qquad \text{(for the variation of (a))}$$

$$\frac{x}{-5} = \frac{y-1}{4} = \frac{z-9}{5} \qquad \text{(for the variation of (b))}$$

22. Solve for t in each of the parametric equations. Thus

$$t = \frac{x-5}{-2}, t = \frac{y-1}{3}, t = \frac{z+4}{6}$$

and the symmetric form is

$$\frac{x-5}{-2} = \frac{y-1}{3} = \frac{z+4}{6}.$$

23. Solving for t in each of the parametric equations gives $t = x-7$, $t = (y+9)/3$, and $t = (z-6)/(-8)$, so that the symmetric form is

$$\frac{x-7}{1} = \frac{y+9}{3} = \frac{z-6}{-8}.$$

24. Set each piece of the equation equal to t and solve:

$$\frac{x-2}{5} = t \Rightarrow x-2 = 5t \Rightarrow x = 2+5t$$
$$\frac{y-3}{-2} = t \Rightarrow y-3 = -2t \Rightarrow y = 3-2t$$
$$\frac{z+1}{4} = t \Rightarrow z+1 = 4t \Rightarrow z = -1+4t.$$

25. Let $t = (x+5)/3$. Then $x = 3t-5$. In view of the symmetric form, we also have that $t = (y-1)/7$ and $t = (z+10)/(-2)$. Hence a set of parametric equations is $x = 3t-5$, $y = 7t+1$, and $z = -2t-10$.

Note: In Exercises 26–29, we could say for certain that two lines are not the same if the vectors were not multiples of each other. In other words, it takes two pieces of information to specify a line. You either need two points, or a point and a direction (or in the case of $\mathbf{R}^2$*, equivalently, a slope).*

26. The first line is parallel to the vector $\mathbf{a}_1 = (5,-3,4)$, while the second is parallel to $\mathbf{a}_2 = (10,-5,8)$. Since $\mathbf{a}_1$ and $\mathbf{a}_2$ are not parallel, the lines cannot be the same.

27. If we multiply each of the pieces in the second symmetric form by -2, we are effectively just traversing the same path at a different speed and with the opposite orientation. So the second set of equations becomes:

$$\frac{x+1}{3} = \frac{y+6}{7} = \frac{z+5}{5}.$$

This looks a lot more like the first set of equations. If we now subtract one from each piece of the second set of equations (as suggested in the text), we are effectively just changing our initial point but we are still on the same line:

$$\frac{x+1}{3} - \frac{3}{3} = \frac{y+6}{7} - \frac{7}{7} = \frac{z+5}{5} - \frac{5}{5}.$$

We have transformed the second set of equations into the first and therefore see that they both represent the same line in $\mathbf{R}^3$.

28. If you first write the equation of the two lines in vector form, we can see immediately that their direction vectors are the same so either they are parallel or they are the same line:

$$\mathbf{r_1}(t) = (-5,2,1) + t(2,3,-6)$$
$$\mathbf{r_2}(t) = (1,11,-17) - t(2,3,-6).$$

The first line contains the point $(-5,2,1)$. If the second line contains $(-5,2,1)$, then the equations represent the same line. Solve just the x component to get that $-5 = 1-2t \Rightarrow t = 3$. Checking we see that $\mathbf{r_2}(3) = (1,11,-17) - 3(2,3,-6) = (-5,2,1)$ so the lines are the same.

29. Here again the vector forms of the two lines can be written so that we see their headings are the same:

$$\mathbf{r_1}(t) = (2,-7,1) + t(3,1,5)$$
$$\mathbf{r_2}(t) = (-1,-8,-3) + 2t(3,1,5).$$

The point $(2,-7,1)$ is on line one, so we will check to see if it is also on line two. As in Exercise 28 we check the equation for the x component and see that $-1+6t = 2 \Rightarrow t = 1/2$. Checking we see that $\mathbf{r_2}(1/2) = (-1,-8,-3)+(1/2)(2)(3,1,5) = (2,-7,2) \neq (2,-7,1)$ so the equations do not represent the same lines.

Note: It is a good idea to assign both Exercises 30 and 31 together. Although they look similar, there is a difference that students might miss.

30. If you make the substitution $u = t^3$, the equations become: $\begin{cases} x = 3u + 7, \\ y = -u + 2, \text{ and} \\ z = 5u + 1. \end{cases}$

The map $u = t^3$ is a bijection. The important fact is that u takes on exactly the same values that t does, just at different times. Since u takes on all reals, the parametric equations do determine a line (it's just that the speed along the line is not constant).

31. This time if you make the substitution $u = t^2$, the equations become: $\begin{cases} x = 5u - 1, \\ y = 2u + 3, \text{ and} \\ z = -u + 1. \end{cases}$

The problem is that u cannot take on negative values so these parametric equations are for a ray with endpoint $(-1, 3, 1)$ and heading $(5, 2, -1)$.

32. **(a)** The vector form of the equations is: $\mathbf{r}(t) = (7, -2, 1) + t(2, 1, -3)$. The initial point is then $\mathbf{r}(0) = (7, -2, 1)$, and after 3 minutes the bird is at $\mathbf{r}(3) = (7, -2, 1) + 3(2, 1, -3) = (13, 1, -8)$.

(b) $(2, 1, -3)$

(c) We only need to check one component (say the x): $7 + 2t = 34/3 \Rightarrow t = 13/6$. Checking we see that $\mathbf{r}\left(\frac{13}{6}\right) = (7, -2, 1) + \left(\frac{13}{6}\right)(2, 1, -3) = \left(\frac{34}{3}, \frac{1}{6}, -\frac{11}{2}\right)$.

(d) As in part (c), we'll check the x component and see that $7 + 2t = 17$ when $t = 5$. We then check to see that $\mathbf{r}(5) = (7, -2, 1) + 5(2, 1, -3) = (17, 3, -14) \neq (17, 4, -14)$ so, no, the bird doesn't reach $(17, 4, -14)$.

33. We can substitute the parametric forms of x, y, and z into the equation for the plane and solve for t. So $(3t - 5) + 3(2 - t) - (6t) = 19$ which gives us $t = -3$. Substituting back in the parametric equations, we find that the point of intersection is $(-14, 5, -18)$.

34. Using the same technique as in Exercise 33, $5(1 - 4t) - 2(t - 3/2) + (2t + 1) = 1$ which simplifies to $t = 2/5$. This means the point of intersection is $(-3/5, -11/10, 9/5)$.

35. We will set each of the coordinate equations equal to zero in turn and substitute that value of t into the other two equations.

$$x = 2t - 3 = 0 \Rightarrow t = 3/2. \text{ When } t = 3/2, y = 13/2 \text{ and } z = 7/2.$$

$$y = 3t + 2 = 0 \Rightarrow t = -2/3, \text{ so } x = -13/3 \text{ and } z = 17/3.$$

$$z = 5 - t = 0 \Rightarrow t = 5, \text{ so } x = 7 \text{ and } y = 17.$$

The points are (0, 13/2, 7/2), $(-13/3, 0, 17/3)$, and (7, 17, 0).

36. We could show that two points on the line are also in the plane or that for points on the line: $2x - y + 4z = 2(5 - t) - (2t - 7) + 4(t - 3) = 5$, so they are in the plane.

37. For points on the line we see that $x - 3y + z = (5 - t) - 3(2t - 3) + (7t + 1) = 15$, so the line does not intersect the plane.

38. First we parametrize the line by setting $t = (x - 3)/6$, which gives us $x = 6t + 3$, $y = 3t - 2$, $z = 5t$. Plugging these parametric values into the equation for the plane gives

$$2(6t + 3) - 5(3t - 2) + 3(5t) + 8 = 0 \iff 12t + 24 = 0 \iff t = -2.$$

The parameter value $t = -2$ yields the point $(6(-2) + 3, 3(-2) - 2, 5(-2)) = (-9, -8, -10)$.

39. We find parametric equations for the line by setting $t = (x - 3)/(-2)$, so that $x = 3 - 2t$, $y = t + 5$, $z = 3t - 2$. Plugging these parametric values into the equation for the plane, we find that

$$3(3 - 2t) + 3(t + 5) + (3t - 2) = 9 - 6t + 3t + 15 + 3t - 2 = 22$$

for *all* values of t. Hence the line is contained in the plane.

40. Again we find parametric equations for the line. Set $t = (x + 4)/3$, so that $x = 3t - 4$, $y = 2 - t$, $z = 1 - 9t$. Plugging these parametric values into the equation for the plane, we find that

$$2(3t - 4) - 3(2 - t) + (1 - 9t) = 7 \iff 6t - 8 - 6 + 3t + 1 - 9t = 7 \iff -13 = 7.$$

Hence we have a contradiction; that is, *no* value of t will yield a point on the line that is also on the plane. Thus the line and the plane do *not* intersect.

41. We just plug the parametric expressions for x, y, z into the equation for the surface:

$$\frac{(at+a)^2}{a^2}+\frac{b^2}{b^2}-\frac{(ct+c)^2}{c^2}=\frac{c^2(t+1)^2}{a^2}+1-\frac{c^2(t+1)^2}{c^2}=1$$

for all values of $t \in \mathbf{R}$. Hence all points on the line satisfy the equation for the surface.

42. As explained in the text, we can't just set the two sets of equations equal to each other and solve. If the two lines intersect at a point, we may get to that point at two different times. Let's call these times t_1 and t_2 and solve the equations

$$\begin{cases} 2t_1+3=15-7t_2, \\ 3t_1+3=t_2-2, \text{ and} \\ 2t_1+1=3t_2-7. \end{cases}$$

Eliminate t_1 by subtracting the third equation from the first to get $t_2 = 2$. Substitute back into any of the equations to get $t_1 = -1$. Using either set of equations, you'll find that the point of intersection is $(1, 0, -1)$.

43. The way the problem is phrased tips us off that something is going on. Let's handle this the same way we did in Exercise 42.

$$\begin{cases} 2t_1+1=3t_2+1, \\ -3t_1=t_2+5, \text{ and} \\ t_1-1=7-t_2. \end{cases}$$

Adding the last two equations eliminates t_2 and gives us $t_1 = 13/2$. This corresponds to the point (14, $-39/2$, 11/2). Substituting this value of t_1 into the third equation gives us $t_2 = 3/2$, while substituting this into the first equation gives us $t_2 = 13/3$. This inconsistency tells us that the second line doesn't pass through the point $(14, -39/2, 11/2)$.

44. **(a)** The distance is $\sqrt{(3t-5+2)^2+(1-t-1)^2+(4t+7-5)^2}=\sqrt{26t^2-2t+13}$.

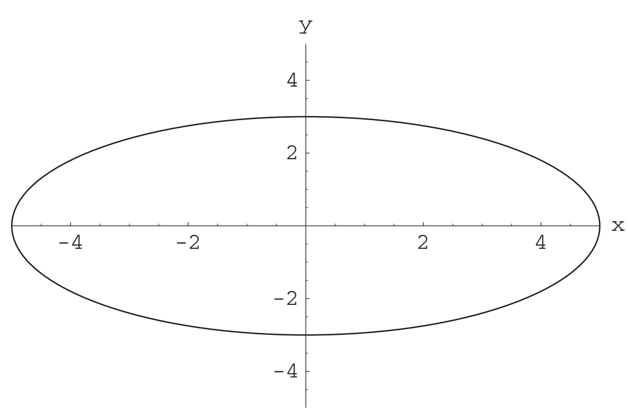
(b) Using a standard first year calculus trick, the distance is minimized when the square of the distance is minimized. So we find $D = 26t^2 - 2t + 13$ is minimized (at the vertex of the parabola) when $t = 1/26$. Substitute back into our answer for (a) to find that the minimal distance is $\sqrt{337/26}$.

45. **(a)** As in Example 2, this is the equation of a circle of radius 2 centered at the origin. The difference is that you are traveling around it three times as fast. This means that if t varied between 0 and 2π that the circle would be traced three times.

(b) This is just like part (a) except the radius of the circle is 5.

(c) This is just like part (b) except the x and y coordinates have been switched. This is the same as reflecting the circle about the line $y = x$ and so this is also a circle of radius 5. If you care, the circle in (b) was drawn starting at the point (5, 0) counterclockwise while this circle is drawn starting at (0, 5) clockwise.

(d) This is an ellipse with major axis along the x-axis intersecting it at $(\pm5, 0)$ and minor axis along the y-axis intersecting it at $(0, \pm3)$: $\frac{x^2}{25}+\frac{y^2}{9}=1$.



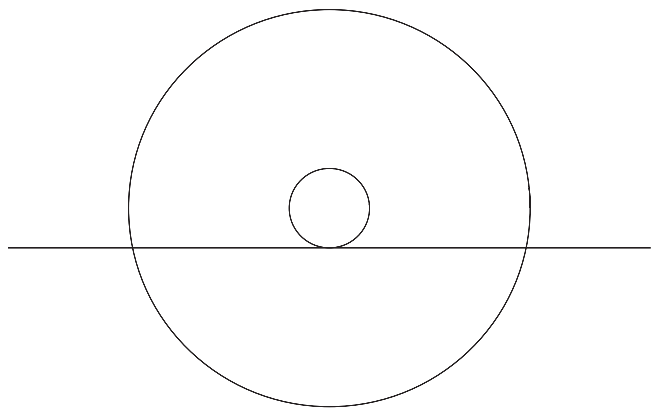
46. The discussion in the text of the cycloid looked at the path traced by a point on the circumference of a circle of radius a as it is rolled without slipping on the x-axis. The vector from the origin to our point P was split into two pieces: $\overrightarrow{OA}$ (the vector from the origin to the center of the circle) and $\overrightarrow{AP}$ (the vector from the center of the circle to P). This split remains the same in our problem.

The center of the circle is always a above the x-axis, and after the wheel has rolled through a central angle of t radians the x coordinate is just *at*. So $\overrightarrow{OA} = (at, a)$. This does not change in our problem.

The vector $\overrightarrow{AP}$ was calculated to be $(-a\sin t, -a\cos t)$. The direction of the vector is still correct but the length is not. If we are b units from the center then $\overrightarrow{AP} = -b(\sin t, \cos t)$.

We conclude then that the parametric equations are $x = at - b\sin t, y = a - b\cos t$. When $a = b$ this is the case of the cycloid described in the text; when $a > b$ we have the curtate cycloid; and when $a < b$ we have the prolate cycloid.

For a picture of how to generate one consider the diagram:



Here the inner circle is rolling along the ground and the prolate cycloid is the path traced by a point on the outer circle. There is a classic toy with a plastic wheel that runs along a handheld track, but your students are too young for that. You could pretend that the big circle is the end of a round roast and the little circle is the end of a skewer. In a regular rotisserie the roast would just rotate on the skewer, but we could imagine rolling the skewer along the edges of the grill. The motion of a point on the outside of the roast would be a prolate cycloid.

47. You are to picture that the circular dispenser stays still so Egbert has to unwind around the dispenser. The direction is $(\cos\theta, \sin\theta)$. The length is the radius of the circle a, plus the amount of tape that's been unwound. The tape that's been unwound is the distance around the circumference of the circle. This is $a\theta$ where θ is again in radians. The equation is therefore $(x, y) = a(1 + \theta)(\cos\theta, \sin\theta)$.

1.3 The Dot Product

Exercises 1–16 are just straightforward calculations. For 1–6 use Definition 3.1 and formula (1). For 7–11 use formula (4). For 12–16 use formula (5).

1. $(1, 5)\cdot(-2, 3) = 1(-2) + 5(3) = 13, \quad \|(1, 5)\| = \sqrt{1^2 + 5^2} = \sqrt{26},$
$\|(-2, 3)\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$

2. $(4, -1)\cdot(1/2, 2) = 4(1/2) - 1(2) = 0, \quad \|(4, -1)\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$
$\|(1/2, 2)\| = \sqrt{(1/2)^2 + 2^2} = \sqrt{17}/2.$

3. $(-1, 0, 7)\cdot(2, 4, -6) = -1(2) + 0(4) + 7(-6) = -44, \quad \|(-1, 0, 7)\| = \sqrt{(-1)^2 + 0^2 + 7^2} = \sqrt{50} = 5\sqrt{2}$, and $\|(2, 4, -6)\| = \sqrt{2^2 + 4^2 + (-6)^2} = \sqrt{56} = 2\sqrt{14}.$

4. $(2, 1, 0)\cdot(1, -2, 3) = 2(1) + 1(-2) + 0(3) = 0, \quad \|(2, 1, 0)\| = \sqrt{2^2 + 1} = \sqrt{5}$, and $\|(1, -2, 3)\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}.$

5. $(4\mathbf{i} - 3\mathbf{j} + \mathbf{k})\cdot(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 4(1) + -3(1) + 1(1) = 2, \quad \|4\mathbf{i} - 3\mathbf{j} + \mathbf{k}\| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$, and $\|\mathbf{i} + \mathbf{j} + \mathbf{k}\| = \sqrt{1 + 1 + 1} = \sqrt{3}.$

6. $(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\cdot(-3\mathbf{j} + 2\mathbf{k}) = 2(-3) - 1(2) = -8, \quad \|\mathbf{i} + 2\mathbf{j} - \mathbf{k}\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$, and $\| - 3\mathbf{j} + 2\mathbf{k}\| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}.$

7. $\theta = \cos^{-1}\left(\dfrac{(\sqrt{3}\mathbf{i} + \mathbf{j})\cdot(-\sqrt{3}\mathbf{i} + \mathbf{j})}{\|(\sqrt{3}\mathbf{i} + \mathbf{j})\|\;\| - \sqrt{3}\mathbf{i} + \mathbf{j}\|}\right) = \cos^{-1}\left(\dfrac{-3 + 1}{(2)(2)}\right) = \cos^{-1}\left(\dfrac{-1}{2}\right) = \dfrac{2\pi}{3}.$

8. $\theta = \cos^{-1}\left(\frac{(-1,2)\cdot(3,1)}{\|(-1,2)\|\,\|(3,1)\|}\right) = \cos^{-1}\left(\frac{-3+2}{\sqrt{5}\sqrt{10}}\right) = \cos^{-1}\left(-\frac{1}{5\sqrt{2}}\right).$

9. $\theta = \cos^{-1}\left(\frac{(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\|\mathbf{i}+\mathbf{j}\|\,\|\mathbf{i}+\mathbf{j}+\mathbf{k}\|}\right) = \cos^{-1}\left(\frac{1+1}{\sqrt{2}\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right).$

10. $\theta = \cos^{-1}\left(\frac{(\mathbf{i}+\mathbf{j}-\mathbf{k})\cdot(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})}{\|\mathbf{i}+\mathbf{j}-\mathbf{k}\|\,\|-\mathbf{i}+2\mathbf{j}+2\mathbf{k}\|}\right) = \cos^{-1}\left(\frac{-1+2-2}{(\sqrt{3})(\sqrt{3})}\right) = \cos^{-1}\left(\frac{-1}{3\sqrt{3}}\right).$

11. $\theta = \cos^{-1}\left(\frac{(1,-2,3)\cdot(3,-6,-5)}{\|(1,-2,3)\|\,\|(3,-6,-5)\|}\right) = \cos^{-1}\left(\frac{3+12-15}{\sqrt{14}\sqrt{70}}\right) = \cos^{-1}(0) = \frac{\pi}{2}.$

Note: The answers to 12 and 13 are the same. You may want to assign both exercises and ask your students why this should be true. You might then want to ask what would happen if vector **a** *was the same but vector* **b** *was divided by* $\sqrt{2}$.

12. $\text{proj}_{\mathbf{i}+\mathbf{j}}(2\mathbf{i}+3\mathbf{j}-\mathbf{k}) = \left(\frac{(\mathbf{i}+\mathbf{j})\cdot(2\mathbf{i}+3\mathbf{j}-\mathbf{k})}{(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})}\right)(\mathbf{i}+\mathbf{j}) = \frac{2+3}{1+1}(1,1,0) = \left(\frac{5}{2},\frac{5}{2},0\right).$

13. $\text{proj}_{\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}}(2\mathbf{i}+3\mathbf{j}-\mathbf{k}) = \left(\frac{\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right)\cdot(2\mathbf{i}+3\mathbf{j}-\mathbf{k})}{\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right)\cdot\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right)}\right)\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}(2+3)}{\frac{1+1}{2}}\frac{(1,1,0)}{\sqrt{2}} = \left(\frac{5}{2},\frac{5}{2},0\right).$

14. $\text{proj}_{5\mathbf{k}}(\mathbf{i}-\mathbf{j}+2\mathbf{k}) = \left(\frac{(5\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+2\mathbf{k})}{(5\mathbf{k})\cdot(5\mathbf{k})}\right)(5\mathbf{k}) = \frac{10}{25}(5\mathbf{k}) = 2\mathbf{k}.$

15. $\text{proj}_{-3\mathbf{k}}(\mathbf{i}-\mathbf{j}+2\mathbf{k}) = \left(\frac{(-3\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+2\mathbf{k})}{(-3\mathbf{k})\cdot(-3\mathbf{k})}\right)(-3\mathbf{k}) = \frac{-6}{9}(-3\mathbf{k}) = 2\mathbf{k}.$

16. $\text{proj}_{\mathbf{i}+\mathbf{j}+2\mathbf{k}}(2\mathbf{i}-4\mathbf{j}+\mathbf{k}) = \left(\frac{(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}-4\mathbf{j}+\mathbf{k})}{(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+2\mathbf{k})}\right)(\mathbf{i}+\mathbf{j}+2\mathbf{k}) = \frac{2-4+2}{1+1+4}(1,1,2) = 0.$

17. We just divide the vector by its length: $\frac{2\mathbf{i}-\mathbf{j}+\mathbf{k}}{||2\mathbf{i}-\mathbf{j}+\mathbf{k}||} = \frac{1}{\sqrt{6}}(2,-1,1).$

18. Here we take the negative of the vector divided by its length: $\frac{\mathbf{i}-2\mathbf{k}}{\|\mathbf{i}-2\mathbf{k}\|} = \frac{1}{\sqrt{5}}(1,0,-2).$

19. Same idea as Exercise 17, but multiply by 3: $\frac{3(\mathbf{i}+\mathbf{j}-\mathbf{k})}{\|\mathbf{i}+\mathbf{j}-\mathbf{k}\|} = \frac{3}{\sqrt{3}}(1,1,-1) = \sqrt{3}(1,1,-1).$

20. There are a whole plane full of perpendicular vectors. The easiest three to find are when we set the coefficients of the coordinate vectors equal to zero in turn: $\mathbf{i}+\mathbf{j}$, $\mathbf{j}+\mathbf{k}$, and $-\mathbf{i}+\mathbf{k}$.

21. We have two cases to consider.
If either of the projections is zero: $\text{proj}_{\mathbf{a}}\mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a}\cdot\mathbf{b} = 0 \Leftrightarrow \text{proj}_{\mathbf{b}}\mathbf{a} = \mathbf{0}$.
If neither of the projections is zero, then the directions must be the same. This means that **a** must be a multiple of **b**. Let $\mathbf{a} = c\mathbf{b}$, then on the one hand

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{c\mathbf{b}}\mathbf{b} = \frac{c\mathbf{b}\cdot\mathbf{b}}{c\mathbf{b}\cdot c\mathbf{b}}c\mathbf{b} = \mathbf{b}.$$

On the other hand

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \text{proj}_{\mathbf{b}}c\mathbf{b} = \frac{\mathbf{b}\cdot c\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = c\mathbf{b}.$$

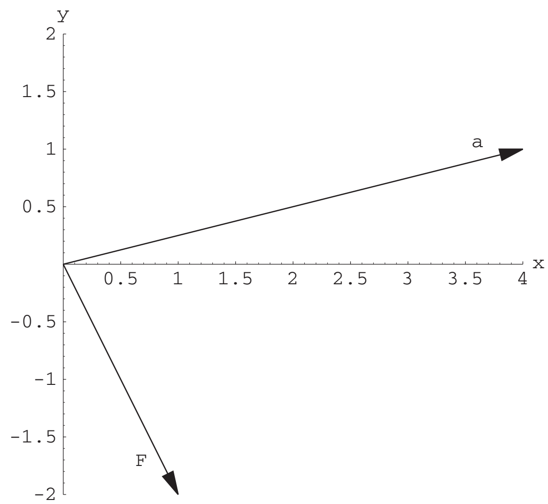
These are equal only when $c = 1$.

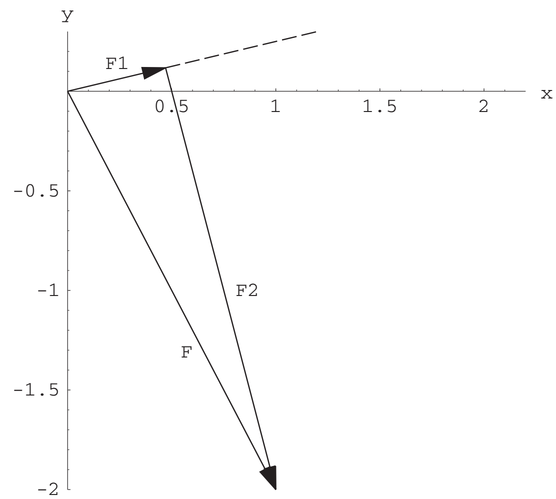
In other words, $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{a}$ when $\mathbf{a}\cdot\mathbf{b} = 0$ or when $\mathbf{a} = \mathbf{b}$.

22. Property 2: $\mathbf{a}\cdot\mathbf{b} = (a_1,a_2,a_3)\cdot(b_1,b_2,b_3) = a_1b_1+a_2b_2+a_3b_3 = b_1a_1+b_2a_2+b_3a_3 = \mathbf{b}\cdot\mathbf{a}$.
Property 3: $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = (a_1,a_2,a_3)\cdot((b_1,b_2,b_3)+(c_1,c_2,c_3)) = (a_1,a_2,a_3)\cdot(b_1+c_1,b_2+c_2,b_3+c_3) = a_1(b_1+c_1)+a_2(b_2+c_2)+a_3(b_3+c_3) = (a_1b_1+a_2b_2+a_3b_3)+(a_1c_1+a_2c_2+a_3c_3) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$.
Property 4: $(k\mathbf{a})\cdot b = (k(a_1,a_2,a_3))\cdot(b_1,b_2,b_3) = (ka_1,ka_2,ka_3)\cdot(b_1,b_2,b_3) = ka_1b_1+ka_2b_2+ka_3b_3$(*for the* 1^{st} *equality*) $= k(a_1b_1+a_2b_2+a_3b_3) = k(\mathbf{a}\cdot\mathbf{b})$. (*for the* 2^{nd} *equality*) $= a_1\,kb_1+a_2\,kb_2+a_3\,kb_3 = (a_1,a_2,a_3)\cdot(kb_1,kb_2,kb_3) = \mathbf{a}\cdot(k\mathbf{b})$.

23. We have $\|k\mathbf{a}\| = \sqrt{k\mathbf{a}\cdot k\mathbf{a}} = \sqrt{k^2(\mathbf{a}\cdot\mathbf{a})} = \sqrt{k^2}\sqrt{\mathbf{a}\cdot\mathbf{a}} = |k|\,\|\mathbf{a}\|$.

24. The following diagrams might be helpful:





To find $\mathbf{F}_1$, the component of $\mathbf{F}$ in the direction of $\mathbf{a}$, we project $\mathbf{F}$ onto $\mathbf{a}$:

$$\mathbf{F}_1 = \text{proj}_{\mathbf{a}}\mathbf{F} = \left(\frac{(\mathbf{i} - 2\mathbf{j}) \cdot (4\mathbf{i} + \mathbf{j})}{(4\mathbf{i} + \mathbf{j}) \cdot (4\mathbf{i} + \mathbf{j})}\right)(4\mathbf{i} + \mathbf{j}) = \frac{2}{17}(4, 1).$$

To find $\mathbf{F}_2$, the component of $\mathbf{F}$ in the direction perpendicular to $\mathbf{a}$, we can just subtract $\mathbf{F}_1$ from $\mathbf{F}$:

$$\mathbf{F}_2 = (1, -2) - \frac{2}{17}(4, 1) = \left(\frac{9}{17}, \frac{-36}{17}\right) = \frac{9}{17}(1, -4).$$

Note that $\mathbf{F}_1$ is a multiple of $\mathbf{a}$ so that $\mathbf{F}_1$ does point in the direction of $\mathbf{a}$ and that $\mathbf{F}_2 \cdot \mathbf{a} = 0$ so $\mathbf{F}_2$ is perpendicular to $\mathbf{a}$.

25. (a) The work done by the force is given to be the product of the length of the displacement ($\|\overrightarrow{PQ}\|$) and the component of force in the direction of the displacement ($\pm\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|$ or in the case pictured in the text, $\|\mathbf{F}\| \cos\theta$). That is,

$$\text{Work} = \|\overrightarrow{PQ}\|\,\|\mathbf{F}\| \cos\theta = \mathbf{F} \cdot \overrightarrow{PQ}$$

using Theorem 3.3.

(b) The displacement vector is $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and so, using part (a), we have

$$\text{Work} = \mathbf{F} \cdot \overrightarrow{PQ} = (\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 1 + 5 - 4 = 2.$$

26. The amount of work is

$$\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| \cos 20^\circ = 60 \cdot 12 \cdot \cos 20^\circ \approx 676.6 \text{ ft-lb}.$$

27. To move the bananas, one must exert an *upward* force of 500 lb. Such a force makes an angle of 60° with the ramp, and it is the ramp that gives the direction of displacement. Thus the amount of work done is

$$\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| \cos 60^\circ = 500 \cdot 40 \cdot \tfrac{1}{2} = 10,000 \text{ ft-lb}.$$

28. Note that **i**, **j**, and **k** each point along the positive x-, y-, and z-axes. Therefore, we may use Theorem 3.3 to calculate that

$$\cos\alpha = \frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot \mathbf{i}}{\|\mathbf{i} + 2\mathbf{j} - \mathbf{k}\|(1)} = \frac{1}{\sqrt{6}};$$

$$\cos\beta = \frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot \mathbf{j}}{\|\mathbf{i} + 2\mathbf{j} - \mathbf{k}\|(1)} = \frac{2}{\sqrt{6}};$$

$$\cos\gamma = \frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot \mathbf{k}}{\|\mathbf{i} + 2\mathbf{j} - \mathbf{k}\|(1)} = -\frac{1}{\sqrt{6}}.$$

29. As in the previous problem, we use $\mathbf{a} = 3\mathbf{i} + 4\mathbf{k}$ to find that

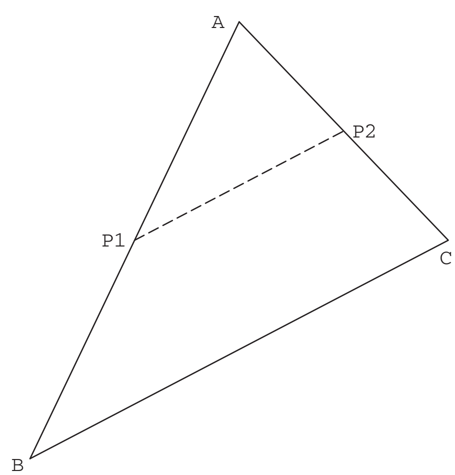
$$\cos\alpha = \frac{(3\mathbf{i}+4\mathbf{k})\cdot\mathbf{i}}{\|3\mathbf{i}+4\mathbf{k}\|(1)} = \frac{3}{5};$$

$$\cos\beta = \frac{(3\mathbf{i}+4\mathbf{k})\cdot\mathbf{j}}{\|3\mathbf{i}+4\mathbf{k}\|(1)} = 0;$$

$$\cos\gamma = \frac{(3\mathbf{i}+4\mathbf{k})\cdot\mathbf{k}}{\|3\mathbf{i}+4\mathbf{k}\|(1)} = \frac{4}{5}.$$

30. You could either use the three right triangles determined by the vector **a** and the three coordinate axes, or you could use Theorem 3.3. By that theorem, $\cos\alpha = \dfrac{\mathbf{a}\cdot\mathbf{i}}{\|\mathbf{a}\|\,\|\mathbf{i}\|} = \dfrac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}$. Similarly, $\cos\beta = \dfrac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}$ and $\cos\gamma = \dfrac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}$.

31. Consider the figure:

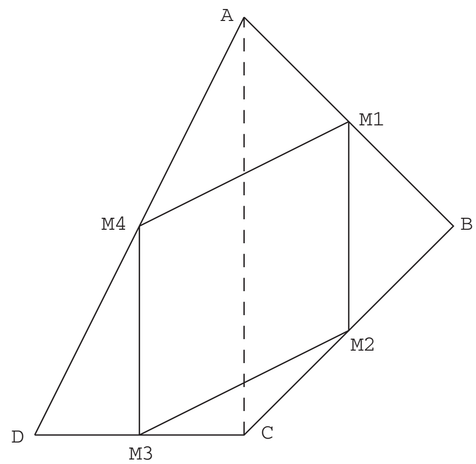


If P_1 is the point on $\overline{AB}$ located r times the distance from A to B, then the vector $\overrightarrow{AP_1} = r\overrightarrow{AB}$. Similarly, since P_2 is the point on $\overline{AC}$ located r times the distance from A to C, then the vector $\overrightarrow{AP_2} = r\overrightarrow{AC}$. So now we can look at the line segment $\overline{P_1P_2}$ using vectors.

$$\overrightarrow{P_1P_2} = \overrightarrow{AP_2} - \overrightarrow{AP_1} = r\overrightarrow{AC} - r\overrightarrow{AB} = r(\overrightarrow{AC} - \overrightarrow{AB}) = r\overrightarrow{BC}.$$

The two conclusions now follow. Because $\overrightarrow{P_1P_2}$ is a scalar multiple of $\overrightarrow{BC}$, they are parallel. Also the positive scalar r pulls out of the norm so $\|\overrightarrow{P_1P_2}\| = \|r\overrightarrow{BC}\| = r\|\overrightarrow{BC}\|$.

32. This now follows immediately from Exercise 31 or Example 6 from the text. Consider first the triangle *ABC*.



If M_1 is the midpoint of $\overline{AB}$ and M_2 is the midpoint of $\overline{BC}$, we've just shown that $\overline{M_1M_2}$ is parallel to $\overline{AC}$ and has half its length. Similarly, consider triangle *DAC* where M_3 is the midpoint of $\overline{CD}$ and M_4 is the midpoint of $\overline{DA}$. We see that $\overline{M_3M_4}$ is parallel to *AC* and has half its length. The first conclusion is that $\overline{M_1M_2}$ and $\overline{M_3M_4}$ have the same length and are parallel. Repeat this process for triangles *ABD* and *CBD* to conclude that $\overline{M_1M_4}$ and $\overline{M_2M_3}$ have the same length and are parallel. We conclude that $M_1M_2M_3M_4$ is a parallelogram. *For kicks—have your students draw the figure for ABCD a non-convex quadrilateral. The argument and the conclusion still hold even though one of the "diagonals" is not inside of the quadrilateral*.

33. In the diagram in the text, the diagonal running from the bottom left to the top right is $\mathbf{a}+\mathbf{b}$ and the diagonal running from the bottom right to the top left is $\mathbf{b}-\mathbf{a}$.

$$\begin{aligned}
\|\mathbf{a}+\mathbf{b}\| &= \|-\mathbf{a}+\mathbf{b}\| && \Leftrightarrow \\
\sqrt{(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})} &= \sqrt{(-\mathbf{a}+\mathbf{b})\cdot(-\mathbf{a}+\mathbf{b})} && \Leftrightarrow \\
\sqrt{\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+2\mathbf{a}\cdot\mathbf{b}} &= \sqrt{(-1)^2\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}} && \Leftrightarrow \\
\mathbf{a}\cdot\mathbf{b} &= 0
\end{aligned}$$

Since neither **a** nor **b** is zero, they must be orthogonal.

34. Using the same set up as that in Exercise 33, we note first that

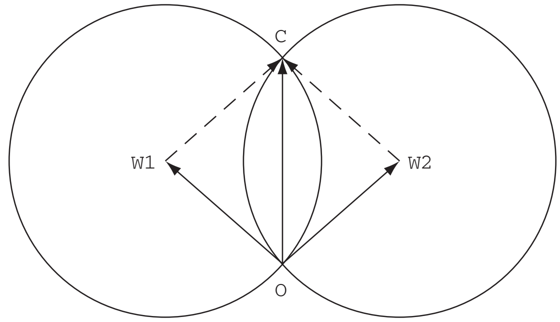
$$(\mathbf{a}+\mathbf{b})\cdot(-\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot(-\mathbf{a})+\mathbf{b}\cdot(-\mathbf{a})+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b} = -\|\mathbf{a}\|^2+\|\mathbf{b}\|^2.$$

It follows immediately that

$$(\mathbf{a}+\mathbf{b})\cdot(-\mathbf{a}+\mathbf{b}) = 0 \Leftrightarrow \|\mathbf{a}\| = \|\mathbf{b}\|.$$

In other words that the diagonals of the parallelogram are perpendicular if and only if the parallelogram is a rhombus.

35. (a) Let's start with the two circles with centers at W_1 and W_2. Assume that in addition to their intersection at point O that they also intersect at point C as shown below.



The polygon OW_1CW_2 is a parallelogram. In fact, because all sides are equal, it is a rhombus. We can, therefore, write the vector $\mathbf{c} = \overrightarrow{OC} = \overrightarrow{OW}_1 + \overrightarrow{OW}_2 = \mathbf{w}_1 + \mathbf{w}_2$. Similarly, we can write $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_3$ and $\mathbf{a} = \mathbf{w}_2 + \mathbf{w}_3$.

(b) Let's use the results of part (a) together with the hint. We need to show that the distance from each of the points A, B, and C to P is r. Let's show, for example, that $\|\overrightarrow{CP}\|$ is r:

$$\|\overrightarrow{CP}\| = \|\overrightarrow{OP} - \overrightarrow{OC}\| = \|(\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3) - (\mathbf{w}_1 + \mathbf{w}_2)\| = \|\mathbf{w}_3\| = r.$$

The arguments for the other two points are analogous.

(c) What we really need to show is that each of the lines passing through O and one of the points A, B, or C is perpendicular to the line containing the two other points. Using vectors we will show that $\overrightarrow{OA} \perp \overrightarrow{BC}$, $\overrightarrow{OB} \perp \overrightarrow{AC}$, and $\overrightarrow{OC} \perp \overrightarrow{AB}$ by showing their dot products are 0. It's enough to show this for one of them: $\overrightarrow{OA} \cdot \overrightarrow{BC} = (\mathbf{w}_2 + \mathbf{w}_3) \cdot ((\mathbf{w}_1 + \mathbf{w}_2) - (\mathbf{w}_1 + \mathbf{w}_3)) = (\mathbf{w}_2 + \mathbf{w}_3) \cdot (\mathbf{w}_2 - \mathbf{w}_3) = \mathbf{w}_2 \cdot \mathbf{w}_2 + \mathbf{w}_3 \cdot \mathbf{w}_2 - \mathbf{w}_2 \cdot \mathbf{w}_3 - \mathbf{w}_3 \cdot \mathbf{w}_3 = r^2 - r^2 = 0$.

36. (a) This follows immediately from Exercise 34 if you notice that the vectors are the diagonals of the rhombus with two sides $\|\mathbf{b}\|\mathbf{a}$ and $\|\mathbf{a}\|\mathbf{b}$.

Or we can proceed with the calculation: $(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|) \cdot (\|\mathbf{b}\|\mathbf{a} - \|\mathbf{a}\|\mathbf{b})$. The only bit of good news here is that the cross terms clearly cancel each other out and we're left with: $\|\mathbf{b}\|^2(\mathbf{a} \cdot \mathbf{a}) - \|\mathbf{a}\|^2(\mathbf{b} \cdot \mathbf{b}) = \|\mathbf{b}\|^2\|\mathbf{a}\|^2 - \|\mathbf{a}\|^2\|\mathbf{b}\|^2 = 0$.

(b) As in (a), the slicker way is to recall (or reprove geometrically) that the diagonals of a rhombus bisect the vertex angles. Then note that $(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})$ is the diagonal of the rhombus with sides $\|\mathbf{b}\|\mathbf{a}$ and $\|\mathbf{a}\|\mathbf{b}$ and so bisects the angle between them which is the same as the angle between **a** and **b**.

Another way is to let θ_1 be the angle between **a** and $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$, and let θ_2 be the angle between **b** and $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$. Then

$$\cos^{-1}\theta_1 = \frac{\mathbf{a} \cdot (\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})}{(\|\mathbf{a}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\|\mathbf{a}\|^2\|\mathbf{b}\| + \|\mathbf{a}\|\mathbf{a} \cdot \mathbf{b}}{(\|\mathbf{a}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\|\mathbf{a}\|\ \|\mathbf{b}\| + \mathbf{a} \cdot \mathbf{b}}{\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|}.$$

Also

$$\cos^{-1}\theta_2 = \frac{\mathbf{b} \cdot (\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})}{(\|\mathbf{b}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\|\mathbf{b}\|\mathbf{b} \cdot \mathbf{a} + \|\mathbf{b}\|^2\|\mathbf{a}\|}{(\|\mathbf{b}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\mathbf{b} \cdot \mathbf{a} + \|\mathbf{a}\|\ \|\mathbf{b}\|}{\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|}.$$

So $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$ bisects the angle between the vectors **a** and **b**.

1.4 The Cross Product

For Exercises 1–4 use Definition 4.2.

1. $(2)(3) - (4)(1) = 2$.

2. $(0)(6) - (5)(-1) = 5$.

3. $(1)(2)(3) + (3)(7)(-1) + (5)(0)(0) - (5)(2)(-1) - (1)(7)(0) - (3)(0)(3) = -5$.

4. $(-2)(6)(2) + (0)(-1)(4) + (1/2)(3)(-8) - (1/2)(6)(4) - (-2)(-1)(-8) - (0)(3)(2) = -32$.

Note: In Exercises 5–7, the difference between using (2) and (3) really amounts to changing the coefficient of **j** *from* $(a_3b_1 - a_1b_3)$ *in formula (2) to* $-(a_1b_3 - a_3b_1)$ *in formula (3). The details are only provided in Exercise 5.*

5. First we'll use formula (2):

$$\begin{aligned}(1,3,-2) \times (-1,5,7) &= [(3)(7) - (-2)(5)]\mathbf{i} + [(-2)(-1) - (1)(7)]\mathbf{j} + [(1)(5) - (3)(-1)]\mathbf{k} \\ &= 31\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} = (31, -5, 8).\end{aligned}$$

If instead we use formula (3), we get:

$$\begin{aligned}(1,3,-2) \times (-1,5,7) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ -1 & 5 & 7 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -2 \\ 5 & 7 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ -1 & 7 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix}\mathbf{k} \\ &= 31\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} = (31, -5, 8).\end{aligned}$$

6. Just using formula (3):

$$(3\mathbf{i}-2\mathbf{j}+\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-2&1\\1&1&1\end{vmatrix}=\begin{vmatrix}-2&1\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&1\\1&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&-2\\1&1\end{vmatrix}\mathbf{k}$$
$$=-3\mathbf{i}-2\mathbf{j}+5\mathbf{k}=(-3,-2,5).$$

7. Note that these two vectors form a basis for the xy-plane so the cross product will be a vector parallel to (0, 0, 1). Again, just using formula (3):

$$(\mathbf{i}+\mathbf{j})\times(-3\mathbf{i}+2\mathbf{j})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\-3&2&0\end{vmatrix}=\begin{vmatrix}1&0\\2&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\-3&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&1\\-3&2\end{vmatrix}\mathbf{k}=5\mathbf{k}=(0,0,5).$$

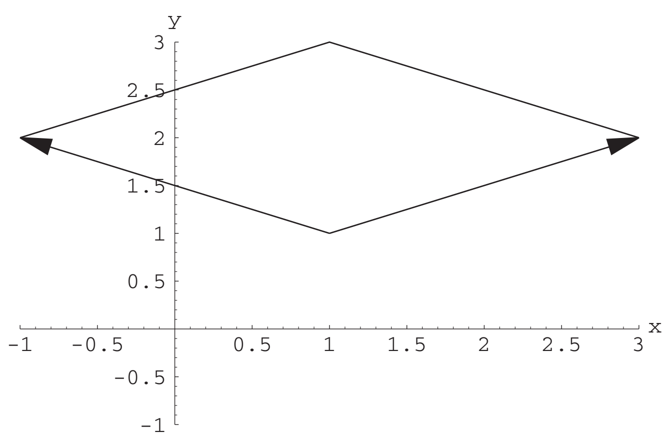
8. By (1) $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=-\mathbf{c}\times(\mathbf{a}+\mathbf{b})$.
By (2), this $=-\mathbf{c}\times\mathbf{a}+-\mathbf{c}\times\mathbf{b}$.
By (1), this $=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$.

9. $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=(\mathbf{a}\times\mathbf{a})+(\mathbf{b}\times\mathbf{a})-(\mathbf{a}\times\mathbf{b})-(\mathbf{b}\times\mathbf{b})$. The cross product of a vector with itself is $\mathbf{0}$ and also $(\mathbf{b}\times\mathbf{a})=-(\mathbf{a}\times\mathbf{b})$, so

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=-2(\mathbf{a}\times\mathbf{b}).$$

You may wish to have your students consider what this means about the relationship between the cross product of the sides of a parallelogram and the cross product of its diagonals. In any case, we are given that $\mathbf{a}\times\mathbf{b}=(3,-7,-2)$, so $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=(-6,14,4)$.

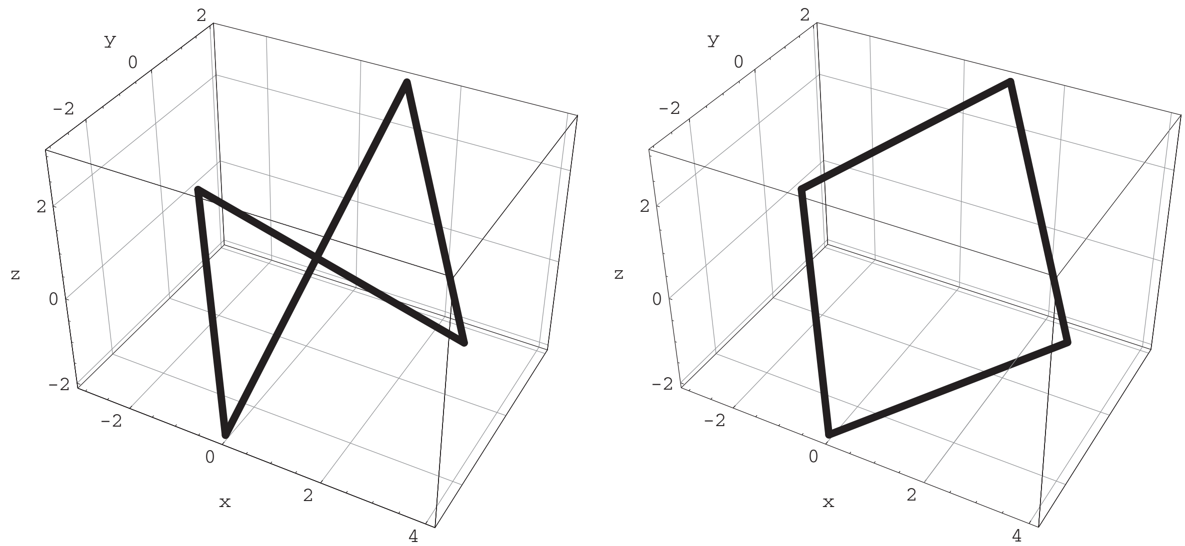
10. If you plot the points you'll see that they are given in a counterclockwise order of the vertices of a parallelogram. To find the area we will view the sides from (1, 1) to (3, 2) and from (1, 1) to $(-1, 2)$ as vectors by calculating the displacement vectors: $(3,2)-(1,1)$ and $(-1,2)-(1,1)$. We then embed the problem in $\mathbf{R}^3$ and take a cross product. The length of this cross product is the area of the parallelogram.



$$(3-1,2-1,0)\times(-1-1,2-1,0)=(2,1,0)\times(-2,1,0)=4\mathbf{k}=(0,0,4).$$

So the area is $\|(0,0,4)\|=4$.

11. This is tricky, as the points are not given in order. The figure on the left shows the sides connected in the order that the points are given.



As the figure on the right shows, if you take the first side to be the side that joins the points (1, 2, 3) and $(4, -2, 1)$ then the next side is the side that joins $(4, -2, 1)$ and $(0, -3, -2)$. We will again calculate the length of the cross product of the displacement vectors. So the area of the parallelogram will be the length of

$$(0-4, -3-(-2), -2-1) \times (1-4, 2-(-2), 3-1) = (-4, -1, -3) \times (-3, 4, 2) = (10, 17, -19).$$

The length of $(10, 17, -19)$ is $\sqrt{10^2 + 17^2 + (-19)^2} = \sqrt{750} = 5\sqrt{30}$.

12. The cross product will give us the right direction; if we then divide this result by its length we will get a unit vector:

$$\frac{(2,1,-3) \times (1,0,1)}{\|(2,1,-3) \times (1,0,1)\|} = \frac{(1,-5,-1)}{\|(1,-5,-1)\|} = \frac{1}{\sqrt{27}}(1,-5,-1).$$

13. For $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ to be zero either

- One or more of the three vectors is $\mathbf{0}$,
- $(\mathbf{a} \times \mathbf{b}) = \mathbf{0}$ which would happen if $\mathbf{a} = k\mathbf{b}$ for some real k, or
- $\mathbf{c}$ is in the plane determined by $\mathbf{a}$ and $\mathbf{b}$.

For Exercises 14–17 we'll just take half of the length of the cross product. Unlike Exercises 10 and 11, in Exercises 16 and 17 we don't have to worry about the ordering of the points. In a triangle, whichever order we choose we are traveling either clockwise or counterclockwise. Just choose any of the vertices as the base for the cross product. Our choices may differ, but the solution won't.

14. $(1/2)\|(1,1,0) \times (2,-1,0)\| = (1/2)\|(0,0,-3)\| = 3/2$.

15. $(1/2)\|(1,-2,6) \times (4,3,-1)\| = (1/2)\|(-16,25,11)\| = \sqrt{1002}/2$.

16. $(1/2)\|(-1-1, 2-1, 0) \times (-2-1, -1-1, 0)\| = (1/2)\|(-2,1,0) \times (-3,-2,0)\| =$
$(1/2)\|(0,0,7)\| = 7/2$.

17. $(1/2)\|(0-1, 2, 3-1) \times (-1-1, 5, -2-1)\| = (1/2)\|(-1,2,2) \times (-2,5,-3)\| =$
$(1/2)\|(-16,-7,-1)\| = \sqrt{306}/2 = 3\sqrt{34}/2$.

The triple scalar product is used in Exercises 18 and 19 and the equivalent determinant form mentioned in the text is proved in Exercise 20.

Some people write this product as $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ *instead of* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. *Exercise 28 shows that these are equivalent.*

18. Here we are given the vectors so we can just use the triple scalar product:

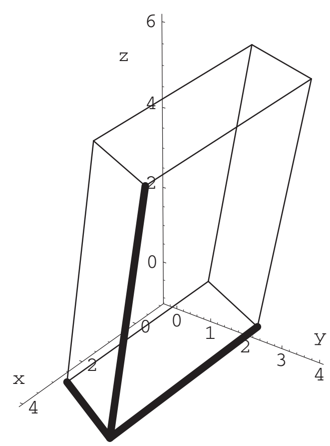
$$
\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= ((3\mathbf{i}-\mathbf{j})\times(-2\mathbf{i}+\mathbf{k}))\cdot(\mathbf{i}-2\mathbf{j}+4\mathbf{k}) = \begin{vmatrix} 3 & -1 & 0 \\ -2 & 0 & 1 \\ 1 & -2 & 4 \end{vmatrix} \\
&= 3\begin{vmatrix} 0 & 1 \\ -2 & 4 \end{vmatrix} - (-1)\begin{vmatrix} -2 & 1 \\ 1 & 4 \end{vmatrix} + 0\begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix} = 3(2) + (-9) = -3.
\end{aligned}
$$

Volume $= |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = 3$.

19. You need to figure out a useful ordering of the vertices. You can either plot them by hand or use a computer package to help or you can make some observations about them. First look at the z coordinates. Two points have $z=-1$ and two have $z=0$. These form your bottom face. Of the remaining points two have $z=5$—these will match up with the bottom points with $z=-1$, and two have $z=6$—these will match up with the bottom points with $z=0$. The parallelepiped is shown below.

We'll use the highlighted edges as our three vectors **a**, **b**, and **c**. You could have based the calculation at any vertex. I have chosen $(4,2,-1)$. The three vectors are:

$$
\begin{aligned}
\mathbf{a} &= (0,3,0)-(4,2,-1) = (-4,1,1) \\
\mathbf{b} &= (4,3,5)-(4,2,-1) = (0,1,6) \\
\mathbf{c} &= (3,0,-1)-(4,2,-1) = (-1,-2,0)
\end{aligned}
$$



We can now calculate

$$
\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= ((-4,1,1)\times(0,1,6))\cdot(-1,-2,0) = \begin{vmatrix} -4 & 1 & 1 \\ 0 & 1 & 6 \\ -1 & -2 & 0 \end{vmatrix} \\
&= -4\begin{vmatrix} 1 & 6 \\ -2 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 6 \\ -1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} = -4(12) - (6) + (1) = -53.
\end{aligned}
$$

Finally, Volume $= |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = 53$.

Note: The proofs of Exercises 20 and 28 are easier if you remember that if matrix A is just matrix B with any two rows interchanged then the determinant of A is the negative of the determinant of B. If you don't use this fact (which is explored in exercises later in this chapter), you can prove this with a long computation. That is why the author of the text suggests that a computer algebra system could be helpful—and this would be a great place to use it in a class demonstration.

20. This is not as bad as it might first appear.

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \left(\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\right)\cdot(c_1, c_2, c_3)$$

$$= \left(\mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)\cdot(c_1, c_2, c_3)$$

$$= c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

21. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ by Exercise 20. Similarly, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$ by Exercise 20.
Expand these determinants to see that they are equal.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = b_1(a_3c_2 - a_2c_3) - b_2(a_3c_1 - a_1c_3) + b_3(a_2c_1 - a_1c_2)$$

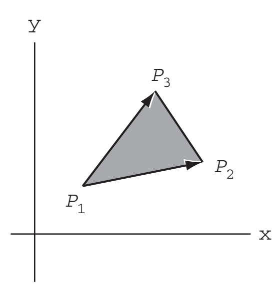
22. The value of $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$ is the volume of the parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. But so is $|\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})|$, so the quantities must be equal.

23. (a) We have

$$\text{Area} = \tfrac{1}{2}\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\|$$
$$= \tfrac{1}{2}\|(x_2 - x_1, y_2 - y_1, 0)\times(x_3 - x_1, y_3 - y_1, 0)\|$$

Now $\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix}$

$$= [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]\mathbf{k}$$



Hence the area is $\frac{1}{2}|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|$. On the other hand

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \frac{1}{2}\left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right).$$

Expanding and taking absolute value, we obtain

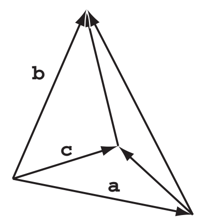
$$\tfrac{1}{2}|x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1|.$$

From here, its easy to see that this agrees with the formula above.

(b) We compute the absolute value of $\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -4 \\ 2 & 3 & -4 \end{vmatrix} = \frac{1}{2}(-8-8+3-4+12+4) = \frac{1}{2}(-1) = -\frac{1}{2}$.

Thus the area is $\left|-\frac{1}{2}\right| = \frac{1}{2}$.

24. Surface area $= \frac{1}{2}(\|\mathbf{a}\times\mathbf{b}\| + \|\mathbf{b}\times\mathbf{c}\| + \|\mathbf{a}\times\mathbf{c}\| + \|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\|)$



25. We assume that **a**, **b**, and **c** are non-zero vectors in $\mathbf{R}^3$.

(a) The cross product $\mathbf{a}\times\mathbf{b}$ is orthogonal to both **a** and **b**.

(b) Scale the cross product to a unit vector by dividing by the length and then multiply by 2 to get $2\left(\frac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|}\right)$.

(c) $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$.

(d) Here we divide vector **a** by its length and multiply it by the length of **b** to get $\left(\frac{\|\mathbf{b}\|}{\|\mathbf{a}\|}\right)\mathbf{a}$.

(e) The cross product of two vectors is orthogonal to each: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

(f) A vector perpendicular to $\mathbf{a}\times\mathbf{b}$ will be back in the plane determined by **a** and **b**, so our answer is $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

26. *I love this problem—students tend to go ahead and calculate without thinking through what they're doing first. This would make a great quiz at the beginning of class.*

(a) Vector: The cross product of the vectors **a** and **b** is a vector so you can take its cross product with vector **c**.

(b) Nonsense: The dot product of the vectors **a** and **b** is a scalar so you can't dot it with a vector.

(c) Nonsense: The dot products result in scalars and you can't find the cross product of two scalars.

(d) Scalar: The cross product of the vectors **a** and **b** is a vector so you can take its dot product with vector **c**.

(e) Nonsense: The cross product of the vectors **a** and **b** is a vector so you can take its cross product with vector that is the result of the cross product of **c** and **d**.

(f) Vector: The dot product results in a scalar that is then multiplied by vector **d**. We can evaluate the cross product of vector **a** with this result.

(g) Scalar: We are taking the dot product of two vectors.

(h) Vector: You are subtracting two vectors.

Note: You can have your students use a computer algebra system for these as suggested in the text. I've included worked out solutions for those as old fashioned as I am.

27. Exercise 25(f) shows us that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is in the plane determined by **a** and **b** and so we expect the solution to be of the form $k_1\mathbf{a} + k_2\mathbf{b}$ for scalars k_1 and k_2.

Using formula (3) from the text for $\mathbf{a}\times\mathbf{b}$:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \left(-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_3 - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_2\right)\mathbf{i} - \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_3 - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_1\right)\mathbf{j} \\ &\quad + \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_1\right)\mathbf{k}\end{aligned}$$

Look first at the coefficient of **i**: $-a_1b_3c_3 + a_3b_1c_3 - a_1b_2c_2 + a_2b_1c_2$. If we add and subtract $a_1b_1c_1$ and regroup we have: $b_1(a_1c_1 + a_2c_2 + a_3c_3) - a_1(b_1c_1 + b_2c_2 + b_3c_3) = b_1(\mathbf{a}\cdot\mathbf{c}) - a_1(\mathbf{b}\cdot\mathbf{c})$. Similarly for the coefficient of **j**. Expand then add and subtract $a_2b_2b_3$ and regroup to get $b_2(\mathbf{a}\cdot\mathbf{c}) - a_2(\mathbf{b}\cdot\mathbf{c})$. Finally for the coefficient of **k**, expand then add and subtract $a_3b_3c_3$ and regroup to obtain $b_3(\mathbf{a}\cdot\mathbf{c}) - a_3(\mathbf{b}\cdot\mathbf{c})$. This shows that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$.

Now here's a version of Exercise 27 worked on *Mathematica*. First you enter the following to define the vectors **a**, **b**, and **c**.

$$a = \{a1, a2, a3\}$$
$$b = \{b1, b2, b3\}$$
$$c = \{c1, c2, c3\}$$

The reply from *Mathematica* is an echo of your input for **c**. Let's begin by calculating the cross product. You can either select the cross product operator from the typesetting palette or you can type the escape key followed by "cross" followed by the escape key. *Mathematica* should reformat this key sequence as $\times$ and you should be able to enter

$$(a \times b) \times c.$$

Mathematica will respond with the calculated cross product

$$\{a2b1c2 - a1b2c2 + a3b1c3 - a1b3c3,$$
$$-a2b1c1 + a1b2c1 + a3b2c3 - a2b3c3,$$
$$-a3b1c1 + a1b3c1 - a3b2c2 + a2b3c2\}.$$

Now you can check the other expression. Use a period for the dot in the dot product.

$$(a.c)b - (b.c)\ a$$

Mathematica will immediately respond

$$\{b1(a1c1 + a2c2 + a3c3) - a1(b1c1 + b2c2 + b3c3),$$
$$b2(a1c1 + a2c2 + a3c3) - a2(b1c1 + b2c2 + b3c3),$$
$$b3(a1c1 + a2c2 + a3c3) - a3(b1c1 + b2c2 + b3c3)\}$$

This certainly looks different from the previous expression. Before giving up hope, note that this one has been factored and the earlier one has not. You can expand this by using the command

$$\text{Expand}[(a.c)b - (b.c)a]$$

or use *Mathematica*'s command % to refer to the previous entry and just type

$$\text{Expand}[\%].$$

This still might not look familiar. So take a look at

$$(a \times b) \times c - [(a.c)b - (b.c)a].$$

If this *still* isn't what you are looking for, simplify it with the command

$$\text{Simplify}[\%]$$

and *Mathematica* will respond

$$\{0, 0, 0\}.$$

28. The exercise asks us to show that six quantities are equal.

The most important pair is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$. Because of the commutative property of the dot product $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ and so we are showing that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$
$$= (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

The determinants of the 3 by 3 matrices above are equal because we had to interchange two rows twice to get from one to the other. This fact has not yet been presented in the text. This would be an excellent time to use a computer algebra system to show the two determinants are equal. Of course, you could use *Mathematica* or some other such system to do the entire problem.

To show that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$ we use a similar approach:

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} &= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}).\end{aligned}$$

So we've established that the first three triple scalars are equal.

We get the rest almost for free by noticing that three pairs of equations are trivial:

$$\begin{aligned}&\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}),\\ &\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}), \text{ and}\\ &\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}).\end{aligned}$$

Each of the above pairs are equal by the anticommutativity property of the cross product. If you prefer the matrix approach, this also follows from the fact that interchanging two rows changes the sign of the determinant.

29. By Exercise 28, $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b}))$.
By anticommutativity, $\mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b})) = -\mathbf{c}\cdot((\mathbf{a}\times\mathbf{b})\times\mathbf{d})$.
By Exercise 27, $-\mathbf{c}\cdot((\mathbf{a}\times\mathbf{b})\times\mathbf{d}) = -\mathbf{c}\cdot((\mathbf{a}\cdot\mathbf{d})\mathbf{b} - (\mathbf{b}\cdot\mathbf{d})\mathbf{a}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$.

30. Apply the results of Exercise 27 to each of the three components:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{b}\times\mathbf{c})\times\mathbf{a} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b} &= [(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}] + [(\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}] \\ &\quad + [(\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}] = \mathbf{0}.\end{aligned}$$

(For example, the $(\mathbf{a}\cdot\mathbf{c})\mathbf{b}$ cancels with the $(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$ because of the commutative property for the dot product.)

31. If your students are using a computer algebra system, they may not notice that this is *exactly* the same problem as Exercise 27. Just replace **c** with $(\mathbf{c}\times\mathbf{d})$ on both sides of the equation in Exercise 27 to obtain the result here.

32. First apply Exercise 29 to the dot product to get

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}) = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})][\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})] - [\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})][\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})].$$

You can either observe that two of these quantities must be 0, or you can apply Exercise 28 to see $\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{a}) = 0$. Exercise 28 also shows that $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. The result follows.

33. We did this above in Exercise 29.

34. The amount of torque is the product of the length of the "wrench" and the component of the force perpendicular to the "wrench". In this case, the wrench is the door—so the length is four feet. The 20 lb force is applied perpendicular to the plane of the doorway and the door is open 45°. So from the text, the amount of torque $= \|\mathbf{a}\|\|\mathbf{F}\|\sin\theta = (4)(20)(\sqrt{2}/2) = 40\sqrt{2}$ ft-lb.

35. **(a)** Here the length of **a** is 1 foot, the force $\mathbf{F} = 40$ pounds and angle $\theta = 120$ degrees. So

$$\mathbf{Torque} = (1)(40)\sin 120^\circ = 40\left(\frac{\sqrt{3}}{2}\right) = 20\sqrt{3} \text{ foot-pounds.}$$

(b) Here all that has changed is that $\|\mathbf{a}\|$ is 1.5 feet, so

$$\mathbf{Torque} = (3/2)(40)\sin 120^\circ = 60\left(\frac{\sqrt{3}}{2}\right) = 30\sqrt{3} \text{ foot-pounds.}$$

36. $\mathbf{a} = 2$ in but torque is measured in foot-pounds so $\|\mathbf{a}\| = (1/6)$ ft.

$$\textbf{Torque} = \mathbf{a} \times \mathbf{F} = \left(\frac{1}{6}, 0, 0\right) \times (0, 15, 0) = \left(0, 0, \frac{5}{2}\right).$$

So Egbert is using 5/2 foot-pounds straight up.

37. From the figure

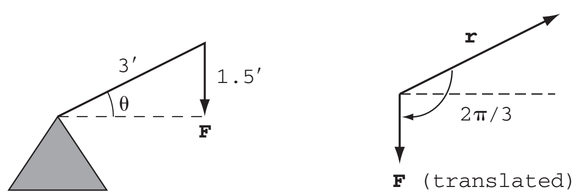
$$\sin\theta = \frac{1.5}{3} = \frac{1}{2}$$
$$\Rightarrow \theta = \pi/6.$$

This is the angle the seesaw makes with horizontal. The angle we want is

$$\pi/6 + \pi/2 = 2\pi/3.$$

Since $\|\mathbf{r}\| = 3$ and $\|\mathbf{F}\| = 50$, the amount of torque is

$$\|\mathbf{T}\| = \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\,\|\mathbf{F}\| \sin\frac{2\pi}{3}$$
$$= 3 \cdot 50 \cdot \frac{\sqrt{3}}{2} = 75\sqrt{3} \text{ ft-lb}$$



38. (a) The linear velocity is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ so that

$$\|\mathbf{v}\| = \|\boldsymbol{\omega}\|\,\|\mathbf{r}\| \sin\theta.$$

We have that the angular speed is $\frac{2\pi \text{ radians}}{24 \text{ hrs}} = \frac{\pi}{12}$ radians/hr (this is $\|\boldsymbol{\omega}\|$.) Also $\|\mathbf{r}\| = 3960$, so at 45° North latitude, $\|\mathbf{v}\| = \frac{\pi}{12} \cdot 3960 \cdot \sin 45^\circ = \frac{330\pi}{\sqrt{2}} \approx 733.08$ mph.

(b) Here the only change is that $\theta = 90^\circ$. Thus $\|\mathbf{v}\| = \frac{\pi}{2} \cdot 3960 \cdot \sin 90^\circ = 330\pi \approx 1036.73$ mph.

39. Archie's actual experience isn't important in solving this problem; he could have ridden closer to the center. Since we are only interested in comparing Archie's experience with Annie's, it turns out that their difference would be the same so long as the difference in their distance from the center remained at 2 inches. The difference in speed is $(331/3)(2\pi)(6) - (331/3)(2\pi)(4) = (331/3)(2\pi)(2) = 4\pi(331/3) = 1331/3\pi = 400\pi/3$ in/ min.

40. (a) $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = (0, 0, 12) \times (2, -1, 3) = (12, 24, 0) = 12\mathbf{i} + 24\mathbf{j}$.

(b) The height of the point doesn't change so we can view this as if it were a problem in $\mathbf{R}^2$. When $x = 2$ and $y = -1$, we can find the central angle by taking $\tan^{-1}(-1/2)$. In one second the angle has moved 12 radians so the new point is

$$(\sqrt{5}\cos(\tan^{-1}(-1/2) + 12), \sqrt{5}\sin(\tan^{-1}(-1/2) + 12), 3) \approx (1.15, -1.92, 3).$$