

## Chapter 1 Solutions

### Section 1.2

1. (a) Since  $(-a) + a = 0$ , uniqueness of the additive inverse of  $(-a)$  implies that  $-(-a) = a$ .
- (b)  $[(ab) + (-a)b] = [a + (-a)]b = 0 \cdot b = 0$ , so uniqueness of the additive inverse implies  $-(ab) = (-a)b$ . A similar argument works for the second equality.
- (c) By (b) and (a),  $(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$ .
- (d) By (b),  $(-1)a = 1(-a) = -a$ .
- (e) By commutativity and associativity of multiplication,

$$(a/b)(bc) = a(b^{-1}b)c = ac = c(d^{-1}d)a = (c/d)(ad),$$

hence the first equality follows from 1.2.1(h). For the second equality, by commutativity and associativity of multiplication and 1.2.1(i),

$$(a/b)(c/d) = (ab^{-1})(cd^{-1}) = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1} = (ac)/(bd).$$

- (f) Using commutativity and associativity of multiplication, the distributive law, and 1.2.1(i),

$$\begin{aligned} a/b + c/d &= ab^{-1}(dd^{-1}) + cd^{-1}(bb^{-1}) = ad(b^{-1}d^{-1}) + bc(b^{-1}d^{-1}) \\ &= ad(bd)^{-1} + bc(bd)^{-1}(ad + bc)/(bd). \end{aligned}$$

2. Let  $r = m/n$  and  $s = p/q$  where  $m, n, p, q \in \mathbb{N}$  and  $nq \neq 0$ . By Exercise 1,  $r \pm s = (mq \pm pn)/(nq)$  and  $rs = (mp)/(nq)$ , which are rational. Since  $1/s = (pq^{-1})^{-1} = p^{-1}q = q/p$ ,  $r/s$  is the product of rational numbers hence is rational.
3. If  $s := r/x \in \mathbb{Q}$ , then, by Exercise 2,  $x = r/s \in \mathbb{Q}$ , a contradiction. Therefore,  $r/x \in \mathbb{I}$ . The remaining parts have similar proofs.
4. (a) By commutativity and associativity of multiplication and the distributive law,

$$\begin{aligned} (x - y) \sum_{j=1}^n x^{n-j} y^{j-1} &= \sum_{j=1}^n x^{n-j+1} y^{j-1} - \sum_{j=1}^n x^{n-j} y^j \\ &= \sum_{j=0}^{n-1} x^{n-j} y^j - \sum_{j=1}^n x^{n-j} y^j \\ &= x^n - y^n. \end{aligned}$$

(b) Replace  $y$  in part (a) by  $-y$ .

(c) Replace  $x$  and  $y$  in part (a) by  $x^{-1}$  and  $y^{-1}$ , respectively.

5. The left side of (a) is  $\frac{n-1}{n} \frac{n-2}{n} \cdots \frac{1}{n} = \frac{n!}{n^n}$ . For (b),

$$\begin{aligned}(2n)! &= [2n(2n-2)(2n-4) \cdots 4 \cdot 2] [(2n-1)(2n-3) \cdots 3 \cdot 1] \\ &= 2^n [n(n-1)(n-2) \cdots 2 \cdot 1] [(2n-1)(2n-3) \cdots 3 \cdot 1].\end{aligned}$$

$$\begin{aligned}6. \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} \\ &= \frac{kn! + (n-k+1)n!}{(n-k+1)!k!} \\ &= \binom{n+1}{k}\end{aligned}$$

7. Let  $a_n$  denote the difference of the two sides of the equation in (a). Combining fractions in the resulting summation leads to

$$a_n = \sum_{k=0}^n \frac{n-2k}{(n+2)(k+1)(n-k+1)}.$$

Making the index change  $j = n - k$  results in

$$a_n = \sum_{j=0}^n \frac{2j-n}{(n+2)(j+1)(n-j+1)} = -a_n.$$

Therefore,  $a_n = 0$ . Part (b) is proved similarly.

8.  $f(k) = k^3 - (k-1)^3 = 3k^2 - 3k + 1$ .

### Section 1.3

- (a) If  $a > 0$  and  $b < 0$ , then  $-(ab) = a(-b) > 0$  hence  $ab < 0$ .  
(b) If  $a > 0$  and  $1/a < 0$ , then  $1 = a(1/a) < 0$ . The converse is similar.  
(c) Follows from  $a/b - c/d = (ad - bc)/bd$ .
- Multiply the given inequalities by  $x$ , using (d) of 1.3.2.
- Part (a) follows from a double application of 1.3.2(d). Part (b) follows from (a) by noting that  $-y < -x$  and  $0 < -b < -a$ . Part (c) follows from (a).
- If  $0 < x < y$ , then multiplying the inequality by  $1/(xy)$  and using (d) of 1.3.2 shows that  $1/y < 1/x$ . If  $x < y < 0$ , then  $0 < -y < -x$  hence, by the first part,  $1/(-x) < 1/(-y)$  so  $1/x > 1/y$ .

5. If  $-1 < x < y$  or  $x < y < -1$ , then  $(y + 1)(x + 1) > 0$  hence

$$\frac{y}{y+1} - \frac{x}{x+1} = \frac{y-x}{(y+1)(x+1)} > 0.$$

If  $x < -1 < y$ , then  $(y + 1)(x + 1) < 0$  and the inequality is reversed.

6. (a) By Exercise 1.2.4,  $y^n - x^n = (y - x) \sum_{j=1}^n y^{n-j} x^{j-1}$ . Each term of the sum is positive and less than  $y^{n-j} y^{j-1} = y^{n-1}$ . Since there are  $n$  terms, part (a) follows.

(b) The inequality is equivalent to

$$n(n+1)xy + ny + nx + x + 1 < n(n+1)xy + ny + nx + y + 1,$$

which reduces to  $x < y$ .

7. The given inequality implies that  $mx > nx - n$  and  $m < n$ . Therefore,  $n > (n - m)x > x$ .

8.  $a = ta + (1 - t)a < tb + (1 - t)b = b$ .

9. If the inequality holds, take  $x = y = 1$  to get  $a \geq -2$  and  $x = 1, y = -1$  to get  $a \leq 2$ . Conversely, suppose that  $0 \leq a \leq 2$ . The inequality then holds trivially if  $xy \geq 0$ , and if  $xy < 0$  then  $x^2 + y^2 + axy = (x + y)^2 + (2 - a)(-xy) \geq 0$ . A similar argument works for the case  $-2 \leq a \leq 0$ .

10. If  $a > b$  then  $x := (a - b)/2 > 0$  and  $a > b + x$ , contradicting the hypothesis.

11. Note that  $b > 0$ . Suppose  $a > b$ . Then  $x := (1 + a/b)/2 > 1$  and  $bx = (a + b)/2 < a$ , contradicting the hypothesis.

12. The inequality is equivalent to  $a < x^2 + x$  for all  $x > 0$ . Assume  $a > 0$ . If  $a \geq 1$  then  $x = 1/2$  violates the condition. If  $0 < a < 1$ , then  $x := a/4 < 1$  so  $a > x + x > x^2 + x$ , again, violating the condition. Therefore,  $a \leq 0$ .

13. (a) Follows from  $0 \leq (x - y)^2 = x^2 - 2xy + y^2$ .

(b)  $0 \leq (x - y)^2 + (y - z)^2 + (z - x)^2 = 2(x^2 + y^2 + z^2) - 2(xy + yz + xz)$ .

(c) By expansion, the inequality is equivalent to  $2xyzw \leq (yz)^2 + (xw)^2$ , which follows from (a).

(d) Follows from (a).

14. Expand  $(x - a)^2 \geq 0$  and divide by  $x$ .

15. (a) Write  $x - y = (x - z) + (z - y)$  and apply the triangle inequality.

(b)  $|x - L| < \varepsilon$  iff  $-\varepsilon < x - L < \varepsilon$ .

16. (a) Let  $S = \{x_1, \dots, x_n\}$ , where  $x_1 < \dots < x_n$ . Then  $\min\{S\} = x_1$  and  $\max\{-S\} = -x_1$ . Part (b) is proved in a similar manner.  
 (c) Let  $x = \max(S \cup T)$  and assume without loss of generality that  $x \in S$ . Then  $x = \max S$  and  $t \leq x$  for all  $t \in T$  hence  $\max T \leq x$ . Therefore,  $x = \max\{\max S, \max T\}$ . Part (d) is proved similarly.
17. (a) For the equalities, consider the cases  $x \geq 0$  and  $x \leq 0$ .  
 (b) Follows from (a).  
 (c) Add and subtract the equations  $x = y - z$  and  $|x| = y + z$ .  
 (d) Use (b) and the triangle inequality.  
 (e)  $(x - y)^- = \max\{y - x, 0\} \leq y$ .
18. If  $a \leq x \leq b$ , then  $x \leq |b|$  and  $-x \leq -a \leq |a|$ , hence  $|x| \leq \max\{|a|, |b|\}$ .
19. Consider cases  $x \geq y$  and  $x \leq y$ .
20. Set  $x := \max\{a, b\}$ . By Exercises 16 and 19,  $x = \frac{1}{2}(a + b + |a - b|)$  and  $\max\{a, b, c\} = \max\{x, c\} = \frac{1}{2}(x + c + |x - c|)$ . Substituting the expression for  $x$  gives the formula for  $\max\{a, b, c\}$ . The corresponding formula for  $\min\{a, b, c\}$  may be found similarly or may be derived from (a).
21. Assume without loss of generality that  $S_1 = S \setminus \{a_1, \dots, a_k\}$ , so  $\min S_1 = a_{k+1}$ . Each of the remaining sets  $S_j$  contains at least one of  $a_1, \dots, a_k$  hence  $\min S_j \leq a_k < a_{k+1}$ , verifying the assertion.

## Section 1.4

- $x \in -A \Rightarrow -x \in A \Rightarrow -x \leq \sup A \Rightarrow x \geq -\sup A$ . Therefore,  $-\sup A$  is a lower bound for  $-A$  hence  $-\sup A \leq \inf(-A)$ . Similarly,  $a \in A \Rightarrow -a \in -A \Rightarrow -a \geq \inf(-A) \Rightarrow a \leq -\inf(-A)$ , so  $-\inf(-A)$  is an upper bound for  $A$  hence  $\sup A \leq -\inf(-A)$  or  $-\sup A \geq \inf(-A)$
- (a)  $\sup = 12$ ,  $\inf = -12$ . (b)  $\sup = 1$ ,  $\inf = -1$ .  
 (c)  $\sup = 3/2$ ,  $\inf = -3/2$ . (d)  $\sup = 0$ ,  $\inf = -2$ .
- (a)  $\sup = 3$ ,  $\inf = 2$ , (b)  $\sup = 3$ ,  $\inf = -2$ .  
 (c)  $\sup = 10/3$ ,  $\inf = 3$ . (d)  $\sup = \frac{3 + \sqrt{5}}{2}$ ,  $\inf = -\infty$ .  
 (e)  $\sup = +\infty$ ,  $\inf = -\infty$ . (f)  $\sup = 2$ ,  $\inf = 3/2$ .  
 (g)  $\sup = \frac{3 + \sqrt{2}}{2}$ ,  $\inf = \frac{3 - \sqrt{2}}{2}$ . (h)  $\sup = 3$ ,  $\inf = 0$ .  
 (i)  $\sup = \frac{1}{2} + \frac{\sqrt{2}}{4}$ ,  $\inf = \frac{1}{2} - \frac{\sqrt{2}}{4}$ . (j)  $\sup = \frac{1}{2} + \frac{\sqrt{6}}{4}$ ,  $\inf = -1/8$ .  
 (k)  $\sup = 4$ ,  $\inf = -2$ . (l)  $\sup = 2$ ,  $\inf = -2$ .  
 (m)  $\sup = 4/3$ ,  $\inf = -1$ . (n)  $\sup = 3/2$ ,  $\inf = -5/4$ .

4. If  $B$  is bounded above then any upper bound of  $B$  is an upper bound of  $A$  hence  $\sup A \leq \sup B$ . The inequality still holds if  $B$  is unbounded above. A similar argument establishes the other inequality.
5. Let  $x, y \in A$ . Then  $\pm(x-y) \leq \sup A - \inf A$  hence  $|x-y| \leq \sup A - \inf A$ . Since  $|x| - |y| \leq |x-y|$ ,  $|x| - |y| \leq \sup A - \inf A$  so  $|x| \leq \sup A - \inf A + |y|$ . Since  $x$  was arbitrary, we have  $\sup |A| \leq \sup A - \inf A + |y|$  hence  $\sup |A| - \sup A + \inf A \leq |y|$ . Since  $y$  was arbitrary it follows that  $\sup |A| - \sup A + \inf A \leq \inf |A|$ .
6. (a)  $a \in A$  and  $b \in B \Rightarrow a + b \leq \sup A + \sup B \Rightarrow \sup(A + B) \leq \sup A + \sup B$ . The infimum case is similar.
- (b) Since  $x > 0$ ,  $xa \leq x \sup A$  for all  $a \in A$ , hence  $\sup(xA) \leq x \sup A$ . Replacing  $x$  by  $1/x$  proves the inequality in the other direction.
- (c) For any  $a \in A$  and  $b \in B$ ,  $ab \geq \inf A \inf B$ , so  $\inf AB \geq \inf A \inf B$ . If  $\inf A = 0$ , choose a sequence  $a_n$  in  $A$  with  $a_n \rightarrow 0$ . Fix any  $b \in B$ . Then  $\inf AB \leq a_n b \rightarrow 0$  so  $\inf AB \leq \inf A \inf B$  in this case. Now suppose  $\inf A \neq 0$ . Then  $ab \geq \inf AB \Rightarrow a \leq b^{-1} \inf AB \Rightarrow \inf A \geq b^{-1} \inf AB \Rightarrow b \inf A \geq \inf AB \Rightarrow b \geq [\inf A]^{-1} \inf AB \Rightarrow \inf B \leq [\inf A]^{-1} \inf AB \Rightarrow \inf A \inf B \geq \inf AB$ .
- (d)  $a \in A \Rightarrow a^r \leq (\sup A)^r \Rightarrow \sup A^r \leq (\sup A)^r$ . Also,  $a = (a^r)^{1/r} \leq (\sup A^r)^{1/r}$  hence  $\sup A \leq (\sup A^r)^{1/r}$ .
- (e)  $a \in A \Rightarrow \inf A \leq a \Rightarrow 1/\inf A \geq 1/a \Rightarrow 1/\inf A \geq \sup A^{-1}$ . Also,  $1/a \leq \sup A^{-1} \Rightarrow a \geq 1/(\sup A^{-1}) \Rightarrow \inf A \geq 1/(\sup A^{-1})$ , or  $1/(\inf A) \leq \sup A^{-1}$ .
7. Let  $r$  denote the infimum. By the approximation property for suprema, there exists  $x \in A$  such that  $\sup A - r < x \leq \sup A$ . Suppose  $x < \sup A$ . Choose  $y \in A$  such that  $x < y \leq \sup A$ . Then  $y - x < r$ , a contradiction. Therefore,  $\sup A = x \in A$ .
8. For all  $x, y \in A$ ,  $x < y + r$  hence  $\sup A \leq y + r$  or  $\sup A - r \leq y$ . Therefore,  $\sup A - r \leq \inf A$  or  $\sup A - \inf A \leq r$ .
9. Let  $a < b$  and let  $r \in (a - \sqrt{2}, b - \sqrt{2})$  be rational. Then  $r + \sqrt{2} \in (a, b)$  is irrational.
10. If  $r_1 < \dots < r_n$  are rationals in  $(a, b)$  then there exists a rational in  $(r_n, b)$ . Therefore, the number of rationals in  $(a, b)$  must be infinite. A similar argument applies to irrationals.
11. Choose  $n \in \mathbb{N}$  such that  $n(b - a) > 1$  and let  $m = \lfloor 2^n a \rfloor + 1$ . Then  $2^n a < m \leq 2^n a + 1 < 2^n b$ , the last inequality because  $2^n > n$ . Therefore,  $a < m/2^n < b$ .

12. (a) If  $n := \lfloor x \rfloor = \lfloor -x \rfloor$ , then  $x - 1 < n \leq x$  and  $-x - 1 < n \leq -x$ . Adding these inequalities gives  $-2 < 2n \leq 0$  so  $n = 0$ . The converse is trivial.
- (b) If  $n := \lfloor x \rfloor = -\lfloor -x \rfloor$ , then  $x - 1 < n \leq x$  and  $x \leq n < x + 1$ . This is possible only if  $x = n$ . The converse is trivial.
- (c) By definition  $-x - 1 < \lfloor -x \rfloor \leq -x$ .
- (d) Adding  $m - x - 1 < \lfloor m - x \rfloor \leq m - x$  to  $x - 1 < \lfloor x \rfloor \leq x$  gives  $m - 2 < \lfloor x \rfloor + \lfloor m - x \rfloor \leq m$ .
13. (a) Let  $s = \sum_{j=0}^n x_j$  and  $t = \sum_{j=0}^n \lfloor x_j \rfloor$ . Then  $s - 1 < \lfloor s \rfloor \leq s$  and  $s - (n + 1) < t \leq s$ . Adding the first inequality to  $-s \leq -t < n + 1 - s$  gives  $-1 < \lfloor s \rfloor - t < n + 1$ , hence  $0 \leq \lfloor s \rfloor - t \leq n$ .
- (b) By (a),  $\lfloor s \rfloor - t = k$  for some  $k = 0, 1, \dots, n$ . By definition of  $\lfloor s \rfloor$ ,  $s - 1 < k + t \leq s$ .
14. Let  $x := (b^m)^{1/n}$  and  $y := (b^{1/n})^m$ . By definition,  $x$  is the unique positive solution of  $x^n = b^m$ . Since  $y^n = \left[ (b^{1/n})^m \right]^n = \left[ (b^{1/n})^n \right]^m = b^m$ ,  $x = y$ .
15. Use Exercise 1.2.4 with  $x = a^{1/n}$  and  $y = b^{1/n}$ .
16. Use Exercise 15.
17. Let  $\ell \leq x \leq u$  for all  $x \in A$ . By the Archimedean principle, there exist positive integers  $m$  and  $n$  such that  $-m < \ell \leq u < n$ . Set  $N = \max\{m, n\}$ .
18. This follows from 1.4.11.
19. Let  $x = a + b\sqrt{2}$  and  $y = c + d\sqrt{2}$ ,  $a, b, c, d \in \mathbb{Q}$ . Then, for example,

$$xy = (ac + 2bd) + (bc + ad)\sqrt{2} \in \mathbb{Q}(\sqrt{2}) \quad \text{and}$$

$$1/y = (c - d\sqrt{2})(c^2 + 2d^2) \in \mathbb{Q}(\sqrt{2}).$$

The set  $\{x \in \mathbb{Q}(\sqrt{2}) : x^2 < \sqrt{3}\}$  is bounded above but has no least upper bound in  $\mathbb{Q}(\sqrt{2})$  hence  $\mathbb{Q}(\sqrt{2})$  is not complete.

20. For any  $a \in \mathbb{N}$ , if  $r := \sqrt{n+a} + \sqrt{n} \in \mathbb{Q}$ , then squaring both sides of  $\sqrt{n+a} = r - \sqrt{n}$  shows that  $\sqrt{n} \in \mathbb{Q}$  and hence that  $n = j^2$  for some  $j \in \mathbb{N}$  (1.4.11). Then  $\sqrt{n+a} \in \mathbb{Q}$  hence  $n+a = k^2$  for some  $k \in \mathbb{N}$ . Therefore,  $a = k^2 - j^2 = (k-j)(k+j)$ . If  $a = 11$ , then  $k-j = 1$  and  $j+k = 11$  so  $n = 25$ . If  $a = 21$ , then either  $k-j = 1$  and  $j+k = 21$  or  $k-j = 3$  and  $j+k = 7$ . The first choice leads to  $j = 10$  and  $n = 100$ , and the second to  $j = 2$  and  $n = 4$ .

21. Let  $r = (\sqrt{n+1})(\sqrt{n+p+1})^{-1}$ . If  $n = (p-1)^2/4$ , then  $n+p = (p+1)^2/4$ , hence  $r \in \mathbb{Q}$ . Conversely, let  $r \in \mathbb{Q}$ . Since

$$r^2(n+p) = 2(r-1)\sqrt{n} + n + (1-r)^2,$$

$\sqrt{n}$  is rational and hence  $n$  is a perfect square, say  $n = m^2$ ,  $m \in \mathbb{N}$  (1.4.11). Since

$$\sqrt{n+p} = r^{-1}(\sqrt{n} + 1) - 1 = r^{-1}(m+1) - 1,$$

$\sqrt{n+p}$  is rational hence  $n+p = k^2$  for some  $k \in \mathbb{N}$ . Therefore  $p = k^2 - m^2 = (k-m)(k+m)$ . Since  $p$  is prime,  $k-m = 1$  and  $k+m = p$ . Thus  $m = (p-1)/2$ , hence  $n = (p-1)^2/4$ .

### Section 1.5

1. Let  $P(n)$  be the assertion that  $a < x_n < x_{n+1} < b$ . Since  $x_1 - a < 1$ ,  $x_1 - a < \sqrt{x_1 - a} < 1$  hence  $x_1 = a + (x_1 - a) < a + \sqrt{x_1 - a} = x_2 < b$ . Therefore,  $P(1)$  holds. Assume  $P(n)$  holds. Then

$$0 < \sqrt{x_n - a} < \sqrt{x_{n+1} - a} < 1$$

so  $a < a + \sqrt{x_n - a} < a + \sqrt{x_{n+1} - a} < a + 1$ , which is  $P(n+1)$ . A similar argument proves the other inequality.

2. Let  $P(n)$  be the statement that a set with  $n$  members has a largest and a smallest element. Clearly  $P(1)$  and  $P(2)$  are true. Let  $n \geq 2$  and assume that  $P(n)$  holds. If  $S$  is a set with  $n+1$  members then removing a member  $a$  from  $S$  produces a set  $T$  with  $n$  members. Let  $m$  be the smallest and  $M$  the largest element of  $T$ . Then  $\min\{m, a\}$  is the smallest and  $\max\{M, a\}$  the largest element of  $S$ . Therefore  $P(n+1)$  holds.
3. Let  $f(n)$  denote the sum on the left side of the equation and  $g(n)$  the sum on the right. Then  $f(1) = 1/2 = g(1)$ . Now let  $n \geq 1$ . Then

$$f(n+1) - f(n) = \sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$g(n+1) - g(n) = \sum_{k=n+2}^{2n+2} \frac{1}{k} - \sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}.$$

Since the right sides are equal,  $f(n) = g(n) \Rightarrow f(n+1) = g(n+1)$ .

4. Let  $S(n)$  denote the sum on the left side of the equation and  $g(n)$  the expression on the right. In each part, one easily checks that  $S(1) = g(1)$ . Now let  $n > 1$  and assume that  $S(n-1) = g(n-1)$ . Then the last term of the sum  $S(n)$  is  $S(n) - S(n-1) = S(n) - g(n-1)$ . This shows that the induction step  $S(n) = g(n)$  holds iff the last term of the sum  $S(n)$  is  $g(n) - g(n-1)$ . For example,

$$(a) n = \frac{n(n+1)}{2} - \frac{(n-1)n}{2},$$

$$(c) n^3 = \frac{n^2}{4}[(n+1)^2 - (n-1)^2],$$

$$(f) \frac{1}{\sqrt{n} + \sqrt{n-1}} = \sqrt{n} - \sqrt{n-1}.$$

$$5. \frac{25}{3}n^3 - \frac{15}{2}n^2 + \frac{1}{6}n.$$

$$6. (a) \sum_{k=1}^{999} k + \sum_{k=1}^{999} k^2 = \frac{999 \cdot 1000}{2} + \frac{999 \cdot 1000 \cdot 1999}{6} = 333,333,000.$$

$$(b) \sum_{k=1}^{500} (4k^2 - 1) = 4 \frac{500 \cdot 501 \cdot 1001}{6} - 500 = 167,166,500.$$

$$(c) \sum_{k=1}^{251} (4k-3)(4k-1) = 16 \frac{251 \cdot 252 \cdot 503}{6} - 16 \frac{251 \cdot 252}{2} + 3 \cdot 251.$$

$$= 85,348,785$$

7. For  $n \geq 1$ , let  $Q(n)$  be the statement  $P(n-1+n_0)$ . Then  $Q(1) = P(n_0)$  is true. Assume  $Q(n) = P(n-1+n_0)$  is true. Then  $Q(n+1) = P(n+n_0)$  is true. By mathematical induction,  $Q(n) = P(n-1+n_0)$  is true for all  $n \geq 1$ , that is,  $P(n)$  is true for every  $n \geq n_0$ .

8. In each case, let  $f(n)$  be the left side of the inequality and  $g(n)$  the right side, and let  $P(n) : f(n) < g(n)$ . Let  $n_0$  be the base value of  $n$  for which  $P(n)$  is true. It is straightforward to check that in each case  $f(n_0) < g(n_0)$ . Assume  $P(n)$  holds for some  $n \geq n_0$ , so that  $f(n)/g(n) < 1$ . Then

$$(a) \frac{f(n+1)}{g(n+1)} = \frac{2n+3}{2^{n+1}} = \frac{f(n)}{2g(n)} + \frac{1}{2^n} < 1.$$

$$(b) \frac{f(n+1)}{g(n+1)} = \frac{n^2+2n+1}{2^{n+1}} = \frac{f(n)}{2g(n)} + \frac{2n+1}{2^{n+1}} < 1$$

$$(c) \frac{f(n+1)}{g(n+1)} = \frac{2^{n+1}}{(n+1)!} = \frac{2}{n+1} \frac{f(n)}{g(n)} < 1.$$

$$(d) \frac{f(n+1)}{g(n+1)} = \frac{3^{n+1}}{(n+1)!} = \frac{3}{n+1} \frac{f(n)}{g(n)} < 1.$$

$$(e) \frac{f(n+1)}{g(n+1)} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} = \frac{f(n)}{g(n)} \frac{2}{(1+1/n)^n} < 1.$$

$$(f) \frac{f(n+1)}{g(n+1)} = \frac{8^{n+1}(n+1)!}{(2n+2)!} = \frac{f(n)}{g(n)} \frac{4}{2n+1} < 1.$$

9. Check that  $6 < \ln(6!)$ . For the induction step, use  $(n+1)! = (n+1)n!$ .

10. The inequality clearly holds for  $n = 0$ . Suppose  $(1 + x)^n \geq 1 + nx$  for some  $n \geq 0$ . Then  $(1 + x)^{n+1} = (1 + x)^n(1 + x) \geq (1 + nx)(1 + x) = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x$ .
11. For  $n \geq 1$ , let  $Q(n)$  be the statement that  $P(k - 1 + n_0)$  is true for  $k = 1, \dots, n$ . Then  $Q(1) = P(n_0)$  is true. Assume  $Q(n)$  is true, so  $P(k - 1 + n_0)$  is true for  $k = 1, \dots, n$ , equivalently,  $P(j)$  is true for  $n_0 \leq j \leq n - 1 + n_0$ . By hypothesis,  $P(n + n_0)$  is true hence  $P(j)$  is true for  $n_0 \leq j \leq n + n_0$ . Thus  $P(k - 1 + n_0)$  is true for  $k = 1, \dots, n + 1$ , that is,  $Q(n + 1)$  is true. By mathematical induction,  $Q(n)$  is true for every  $n \geq 1$  hence  $P(n)$  is true for every  $n \geq n_0$ .
12. Obvious for  $n = 2$ . Let  $n > 2$  and suppose the prime factorization holds for all integers  $m$  with  $2 \leq m \leq n$ . If  $n + 1$  is prime, we're done. Otherwise  $n + 1 = mk$  where  $2 \leq m, k < n$ . By hypothesis,  $m$  and  $k$  have prime factorizations hence so does the product.
13. Let  $g_n$  denote the expression on the right in the assertion. One checks directly that  $g_0 = g_1 = 1$ . Let  $n \geq 2$  and assume that  $f_j = g_j$  for all  $2 \leq j \leq n$ . Then

$$\begin{aligned} g_{n+1} - f_{n+1} &= g_{n+1} - f_n - f_{n-1} = g_{n+1} - g_n - g_{n-1} \\ &= \frac{1}{\sqrt{5}}(a^{n+2} - a^{n+1} - a^n) + \frac{1}{\sqrt{5}}(b^{n+2} - b^{n+1} - b^n) \\ &= \frac{a^n}{\sqrt{5}}(a^2 - a - 1) + \frac{b^n}{\sqrt{5}}(b^2 - b - 1) = 0. \end{aligned}$$

14. Let  $b_n$  denote the right side of the equation. One checks directly that  $b_n = a_n$  for  $n = 0, 1$ . Let  $n \geq 2$  and assume that  $b_j = a_j$  for  $2 \leq j \leq n$ . We show that  $b_{n+1} = a_{n+1}$  or, equivalently,  $2b_{n+1} = b_n + b_{n-1}$ :

$$\begin{aligned} b_n + b_{n-1} &= \left[ \frac{(-1)^n}{3 \cdot 2^{n-1}} + \frac{(-1)^{n-1}}{3 \cdot 2^{n-2}} \right] (a_0 - a_1) + \frac{2}{3}(a_0 + 2a_1) \\ &= \frac{(-1)^{n-1}(a_0 - a_1)}{3 \cdot 2^{n-2}} \left[ \frac{-1}{2} + 1 \right] + \frac{2}{3}(a_0 + 2a_1) \\ &= \frac{2(-1)^{n+1}(a_0 - a_1)}{3 \cdot 2^n} + \frac{2}{3}(a_0 + 2a_1) \\ &= 2b_{n+1}. \end{aligned}$$

15. The set of all nonnegative integers of the form  $m - qn$ ,  $q \in \mathbb{Z}$ , is nonempty (Archimedean principle) hence has a smallest member  $r = m - qn$  (well ordering principle). If  $r \geq n$ , then  $0 \leq r - n = m - (q + 1)n < r$ , contradicting the minimal property of  $r$ . Therefore,  $m = qn + r$  has the required form. If also  $m = q'n + r'$ ,  $q' \in \mathbb{Z}$ ,  $r' \in \{0, \dots, n - 1\}$ , then  $|q - q'|n = |r - r'| < n$  hence  $q' = q$  and  $r' = r$ .

16. Clearly,  $n = 1$  has a decimal representation. Assume all integers  $q \leq n$  have decimal representations. By the division algorithm,  $n + 1 = 10q + d$ ,  $d \in \{0, 1, \dots, 9\}$ . Since  $q \leq n$ ,  $q$  has a decimal representation, say  $q = d_p d_{p-1} \dots d_0$ . Then

$$n + 1 = \sum_{k=0}^p d_k 10^{k+1} + d = d_p d_{p-1} \dots d_0 d.$$

Therefore, by induction, all positive integers have decimal representations. To see that the representation is unique, suppose that

$$n = \sum_{k=0}^p d_k 10^k = \sum_{k=0}^q e_k 10^k, \quad d_j, e_j \in \{0, 1, \dots, 9\}.$$

Then

$$e_0 - d_0 = \sum_{k=1}^p d_k 10^k - \sum_{k=1}^q e_k 10^k,$$

which is divisible by 10. Therefore  $e_0 = d_0$  and  $\sum_{k=1}^p d_k 10^k = \sum_{k=1}^q e_k 10^k$ . Arguing similarly, we see that  $e_1 = d_1$ . Continuing in this manner, eventually  $p = q$  and  $e_j = d_j$ ,  $0 \leq j \leq p$ .

## Section 1.6

$$1. \quad \mathbf{x} = \mathbf{c} - \frac{\mathbf{d} \cdot \mathbf{e} - (\mathbf{b} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})}{1 - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{d})} \mathbf{a}, \quad \mathbf{y} = \mathbf{e} - \frac{\mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{d} \cdot \mathbf{e})}{1 - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{d})} \mathbf{d}.$$

2. By 1.6.3,

$$\|\mathbf{x} + \mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2 + 2(\mathbf{x} \cdot \mathbf{y}) \quad \text{and} \quad \|\mathbf{x} - \mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2 - 2(\mathbf{x} \cdot \mathbf{y}).$$

Adding and subtracting gives (a) and (b).

(c) By the triangle inequality,

$$\|\mathbf{x}\|_2 = \|\mathbf{x} - \mathbf{y} + \mathbf{y}\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 + \|\mathbf{y}\|_2$$

hence  $\|\mathbf{x}\|_2 - \|\mathbf{y}\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2$ . Similarly,  $\|\mathbf{y}\|_2 - \|\mathbf{x}\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2$ .

(d) Use induction.

$$3. \quad \text{By 1.6.3, } \|\mathbf{x}_1 + \mathbf{x}_2 + \dots + \mathbf{x}_k\|_2^2 = \sum_{i,j=1}^n \mathbf{x}_i \cdot \mathbf{x}_j = \sum_{j=1}^k \mathbf{x}_j \cdot \mathbf{x}_j.$$

4. For the triangle inequality, we have

$$\|\mathbf{x} + \mathbf{y}\|_1 = \sum_{j=0}^n |x_j + y_j| \leq \sum_{j=0}^k |x_j| + |y_j| = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$$