1 Essentials of Logic and Set Theory

Review exercises for Chapter 1

3. Consider the Venn diagram for three sets depicted in Fig. SM1.R.3. Let n_k denote the number of students in the set marked (k), for $k = 1, 2, \ldots, 8$. Suppose the sets A, B, and C refer to those who study English, French, and Spanish, respectively. Since 10 students take all three languages, $n_7 = 10$. There are 15 who take French and Spanish, so $15 = n_2 + n_7$, and thus $n_2 = 5$. Furthermore, $32 = n_3 + n_7$, so $n_3 = 22$. Also, $110 = n_1 + n_7$, so $n_1 = 100$. The rest of the information implies that $52 = n_2 + n_3 + n_6 + n_7$, so $n_6 = 52 - 5 - 22 - 10 = 15$. Moreover, $220 = n_1 + n_2 + n_5 + n_7$, so $n_5 = 220 - 100 - 5 - 10 = 105$. Finally, $780 = n_1 + n_3 + n_4 + n_7$, so $n_4 = 780 - 100 - 22 - 10 = 648$. The answers are therefore:

   (a) $n_1 = 100$,

   (b) $n_3 + n_4 = 648 + 22 = 670$,

   (c) $1000 - \sum_{i=1}^{7} n_i = 1000 - 905 = 95$.

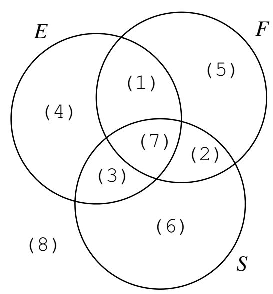

Figure SM1.R.3

4. (a) $\Rightarrow$ is true; $\Leftarrow$ is false, because $x = y = 1$ also solves $x + y = 2$.

   (b) $\Rightarrow$ is false, because $x^2 = 16$ also has the solution $x = -4$; $\Leftarrow$ true, because if $x = 4$, then $x^2 = 16$.

   (c) $\Rightarrow$ is true, because $(x-3)^2 \geq 0$; $\Leftarrow$ false because with $y > -2$ and $x = 3$, one has $(x-3)^2(y+2) = 0$.

   (d) $\Rightarrow$ and $\Leftarrow$ are both true, since the equation $x^3 = 8$ has the solution $x = 2$ and no others.[1]

5. For (a) and (b) see the solutions in the book. For (c), note that when $n = 1$, the inequality is obviously correct.[2] As the induction hypothesis when n equals the arbitrary natural number k, suppose that $(1+x)^k \geq 1 + kx$. Because $1 + x \geq 0$, we then have

$$(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x) = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x.$$

   where the last inequality holds because $k > 0$. Thus, the induction hypothesis holds for $n = k + 1$. Therefore, by induction, Bernoulli's inequality is true for all natural numbers n.

[1] In the terminology of Section 6.3, function $f(x) = x^3$ is strictly increasing. See Fig. 4.3.7 and Exercise 6.3.3.
[2] And for $n = 2$, it is correct by part (b).

2 Algebra

2.3 Rules of algebra

4. (a) $(2t-1)(t^2-2t+1) = 2t(t^2-2t+1)-(t^2-2t+1) = 2t^3-4t^2+2t-t^2+2t-1 = 2t^3-5t^2+4t-1$.

(b) $(a+1)^2+(a-1)^2-2(a+1)(a-1) = (a^2+2a+1)+(a^2-2a+1)-2(a^2-1) = 4$.[3]

(c) $(x+y+z)^2 = (x+y+z)(x+y+z) = x(x+y+z)+y(x+y+z)+z(x+y+z) = (x^2+xy+xz)+(yx+y^2+yz)+(zx+zy+z^2) = x^2+y^2+z^2+2xy+2xz+2yz$.

(d) Put $a = x+y+z$ and $b = x-y-z$. Then

$$(x+y+z)^2-(x-y-z)^2 = a^2-b^2 = (a+b)(a-b) = 2x(2y+2z) = 4x(y+z)$$

2.4 Fractions

5. (a) $\dfrac{1}{x-2}-\dfrac{1}{x+2} = \dfrac{x+2}{(x-2)(x+2)}-\dfrac{x-2}{(x+2)(x-2)} = \dfrac{x+2-x+2}{(x-2)(x+2)} = \dfrac{4}{x^2-4}$.

(b) Since $4x+2 = 2(2x+1)$ and $4x^2-1 = (2x+1)(2x-1)$, the lowest common denominator, LCD, is $2(2x+1)(2x-1)$. Then,

$$\frac{6x+25}{4x+2}-\frac{6x^2+x-2}{4x^2-1} = \frac{(6x+25)(2x-1)-2(6x^2+x-2)}{2(2x+1)(2x-1)} = \frac{42x-21}{2(2x+1)(2x-1)} = \frac{21}{2(2x+1)}.$$

(c) $\dfrac{18b^2}{a^2-9b^2}-\dfrac{a}{a+3b}+2 = \dfrac{18b^2-a(a-3b)+2(a^2-9b^2)}{(a+3b)(a-3b)} = \dfrac{a(a+3b)}{(a+3b)(a-3b)} = \dfrac{a}{a-3b}$.

(d) $\dfrac{1}{8ab}-\dfrac{1}{8b(a+2)} = \dfrac{(a+2)-a}{8ab(a+2)} = \dfrac{2}{8ab(a+2)} = \dfrac{1}{4ab(a+2)}$.

(e) $\dfrac{2t-t^2}{t+2}\cdot\left(\dfrac{5t}{t-2}-\dfrac{2t}{t-2}\right) = \dfrac{t(2-t)}{t+2}\cdot\dfrac{3t}{t-2} = \dfrac{-t(t-2)}{t+2}\cdot\dfrac{3t}{t-2} = \dfrac{-3t^2}{t+2}$.

(f) Note that $\dfrac{a\left(1-\frac{1}{2a}\right)}{0.25} = \dfrac{a-\frac{1}{2}}{\frac{1}{4}} = 4a-2$, so

$$2-\frac{a\left(1-\frac{1}{2a}\right)}{0.25} = 2-(4a-2) = 4-4a = 4(1-a).$$

6. (a) $\dfrac{2}{x}+\dfrac{1}{x+1}-3 = \dfrac{2(x+1)+x-3x(x+1)}{x(x+1)} = \dfrac{2-3x^2}{x(x+1)}$

(b) $\dfrac{t}{2t+1}-\dfrac{t}{2t-1} = \dfrac{t(2t-1)-t(2t+1)}{(2t+1)(2t-1)} = \dfrac{-2t}{4t^2-1}$

(c) $\dfrac{3x}{x+2}-\dfrac{4x}{2-x}-\dfrac{2x-1}{(x-2)(x+2)} = \dfrac{3x(x-2)+4x(x+2)-(2x-1)}{(x-2)(x+2)} = \dfrac{7x^2+1}{x^2-4}$

(d) The expression equals $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{xy}} = \dfrac{\left(\dfrac{1}{x}+\dfrac{1}{y}\right)xy}{\dfrac{1}{xy}\cdot xy} = \dfrac{y+x}{1} = x+y$.

[3] Alternatively, apply the quadratic identity $x^2+y^2-2xy = (x-y)^2$ with $x = a+1$ and $y = a-1$ to obtain $(a+1)^2+(a-1)^2-2(a+1)(a-1) = [(a+1)-(a-1)]^2 = 2^2 = 4$.

(e) The expression equals $\dfrac{\dfrac{1}{x^2}-\dfrac{1}{y^2}}{\dfrac{1}{x^2}+\dfrac{1}{y^2}} = \dfrac{\left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right)\cdot x^2y^2}{\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}\right)\cdot x^2y^2} = \dfrac{y^2-x^2}{y^2+x^2}$.

(f) To clear the fractions within both the numerator and denominator, multiply both by xy to get

$$\frac{a(y-x)}{a(y+x)} = \frac{y-x}{y+x}$$

8. (a) $\dfrac{1}{4}-\dfrac{1}{5}=\dfrac{5}{20}-\dfrac{4}{20}=\dfrac{1}{20}$, so $\left(\dfrac{1}{4}-\dfrac{1}{5}\right)^{-2}=\left(\dfrac{1}{20}\right)^{-2}=20^2=400$.

(b) $n-\dfrac{n}{1-\frac{1}{n}} = n-\dfrac{n\cdot n}{\left(1-\frac{1}{n}\right)\cdot n} = n-\dfrac{n^2}{n-1} = \dfrac{n(n-1)-n^2}{n-1} = -\dfrac{n}{n-1}$.

(c) Let $u=x^{p-q}$. Then $\dfrac{1}{1+x^{p-q}}+\dfrac{1}{1+x^{q-p}} = \dfrac{1}{1+u}+\dfrac{1}{1+1/u} = \dfrac{1}{1+u}+\dfrac{u}{1+u} = 1$.

(d) Using $x^2-1=(x+1)(x-1)$, one has

$$\frac{\left(\dfrac{1}{x-1}+\dfrac{1}{x^2-1}\right)(x^2-1)}{\left(x-\dfrac{2}{x+1}\right)(x^2-1)} = \frac{(x+1)+1}{x(x^2-1)-2(x-1)} = \frac{x+2}{(x-1)[x(x+1)-2]},$$

which reduces to

$$\frac{x+2}{(x-1)(x^2+x-2)} = \frac{x+2}{(x-1)[(x+2)(x-1)]} = \frac{1}{(x-1)^2}.$$

(e) Since

$$\frac{1}{(x+h)^2}-\frac{1}{x^2} = \frac{x^2-(x+h)^2}{x^2(x+h)^2} = \frac{-2xh-h^2}{x^2(x+h)^2},$$

it follows that

$$\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} = \frac{-2x-h}{x^2(x+h)^2}.$$

(f) Multiplying both numerator and denominator by $x^2-1=(x+1)(x-1)$ yields $\dfrac{10x^2}{5x(x-1)}$, which reduces to $\dfrac{2x}{x-1}$.

2.5 Fractional powers

5. The answers for each respective part that are given in the book emerge after multiplying both numerator and denominator by the following terms:

(a) $\sqrt{7}-\sqrt{5}$	(b) $\sqrt{5}-\sqrt{3}$	(c) $\sqrt{3}+2$
(d) $x\sqrt{y}-y\sqrt{x}$	(e) $\sqrt{x+h}+\sqrt{x}$	(f) $1-\sqrt{x+1}$

11. (a) $(2^x)^2=2^{2x}$, which equals 2^{x^2} if and only if $2x=x^2$, or if and only if $x=0$ or $x=2$.

(b) Correct because $a^{p-q} = a^p/a^q$.

(c) Correct because $a^{-p} = 1/a^p$.

(d) $5^{1/x} = 1/5^x = 5^{-x}$ if and only if $1/x = -x$ or $-x^2 = 1$, so there is no real x that satisfies the equation.

(e) Put $u = a^x$ and $v = a^y$, which reduces the equation to $uv = u + v$, or $0 = uv - u - v = (u-1)(v-1) - 1$. This is true only for special values of u and v and so for special values of x and y. In particular, the equation is false when $x = y = 1$.

(f) Putting $u = \sqrt{x}$ and $v = \sqrt{y}$ reduces the equation to $2^u \cdot 2^v = 2^{uv}$, which holds if and only if $uv = u + v$, as in (e) above.

2.6 Inequalities

3. (a) This inequality has the same solutions as

$$\frac{3x+1}{2x+4} - 2 > 0, \quad \text{or} \quad \frac{3x+1-2(2x+4)}{2x+4} > 0, \quad \text{or} \quad \frac{-x-7}{2x+4} > 0.$$

A sign diagram reveals that the inequality is satisfied for $-7 < x < -2$. A serious error is to multiply the inequality by $2x+4$, without checking the sign of $2x+4$. If $2x+4 < 0$, mulitiplying by this number will reverse the inequality sign.[4]

(b) The inequality is equivalent to $120/n \leq 0.75 = 3/4$, or $(480 - 3n)/4n \leq 0$. A sign diagram reveals that the inequality is satisfied for $n < 0$ and for $n \geq 160$.[5]

(c) This is easy: $g(g-2) \leq 0$ and so $0 \leq g \leq 2$.

(d) Note that $p^2 - 4p + 4 = (p-2)^2$, so the inequality reduces to $\dfrac{(p-2)+3}{(p-2)^2} = \dfrac{p+1}{(p-2)^2} \geq 0$. The fraction makes no sense if $p = 2$. The conclusion that $p \geq -1$ and $p \neq 2$ follows.

(e) The inequality is equivalent to

$$\frac{-n-2}{n+4} - 2 > 0 \iff \frac{-n-2-2n-8}{n+4} > 0 \iff \frac{-3n-10}{n+4} > 0 \iff -4 < n < -\frac{10}{3}$$

(f) $x^4 - x^2 = x^2(x^2-1) < 0 \iff x \neq 0$ and $x^2 < 1 \iff -1 < x < 0$ or $0 < x < 1$

6. (a) It is easy to see by means of a sign diagram that $x(x+3) < 0$ precisely when x lies in the open interval $(-3, 0)$. Therefore we have $\Rightarrow$, but not $\Leftarrow$: for example, if $x = 10$, then $x(x+3) = 130$.

(b) $x^2 < 9 \iff -3 < x < 3$, so $x^2 < 9 \Rightarrow x < 3$. If $x = -5$, for instance, we have $x < 3$ but $x^2 > 9$, hence we cannot have $\Leftarrow$ here.

(c) If $x > 0$, then $x^2 > 0$, but $x^2 > 0$ also when $x < 0$. So, we have $\Leftarrow$ but not $\Rightarrow$.

(d) As $y^2 \geq 0$ for any real number y, we have that $x > 0$ whenever $x > y^2$. But $x = 1 > 0$ does not imply that $x > y^2$ for $y \geq 1$. That is, we have $\Rightarrow$ but not $\Leftarrow$.

[4] It could be instructive to test the inequality for some values of x. For example, for $x = 0$ it is not true. What about $x = -5$?

[5] Note that for $n = 0$ the inequality makes no sense. For $n = 160$, we have equality.