Solution Manual

Chapter 1

1.1 For the two-dimensional flow field defined by the velocity components $v_x = \frac{1}{1+t}$, $v_y = 1$, $v_z = 0$, find the Lagrangian representation of the paths taken by the fluid particles.

Solution: Since the velocities all are dependent on time and independent of position, the path lines taken are all straight lines. The position in the y direction changes linearly in time, while the position in the z direction does not change. In the x direction the position changes as $\ln(1+t)$. Thus a particle which at time t_0 is at (X_0, Y_0, Z_0) will be at the position

$$(X, Y, Z) = (X_0, Y_0, Z_0) + \left(\ln \frac{1+t}{1+t_0}, t - t_0, 0\right).$$

1.2 Find the acceleration at point (1, 1, 1) for the velocity $\mathbf{v} = (yz + t,\ xz - t,\ xy)$.

Solution:

$$\mathbf{v} = (yz + t,\ xz - t,\ xy).$$

$$a_x = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}$$

$$= 1 + (yz + t) \cdot 0 + (xz - t) \cdot z + xy \cdot y = 1 + x\left(y^2 + z^2\right) - tz.$$

1.3 a. Find the relationship between velocity components in cylindrical polar coordinates in terms of components in Cartesian coordinates, as well as the inverse relations. Use Figure 1.4.1.

b. Find the relationships between velocity components in spherical polar coordinates in terms of components in Cartesian coordinates, as well as the inverse relations. Use Figure 1.4.3.

Solution:

a. Cylindrical polar coordinates:

$$v_r = v_x \cos\theta + v_y \sin\theta,\ \ v_\theta = -v_x \sin\theta + v_y \cos\theta,\ \ v_z = v_z.$$
$$v_x = v_r \cos\theta - v_\theta \sin\theta,\ \ v_y = v_r \sin\theta + v_\theta \cos\theta.$$

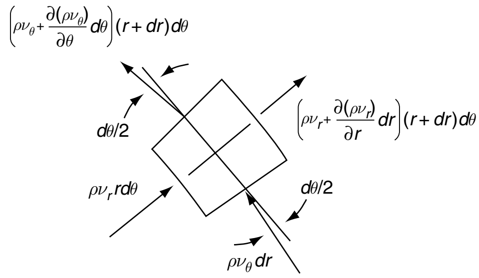
b Spherical polar coordinates:

$$v_R = \left(v_x \cos\theta + v_y \sin\theta\right)\sin\beta + v_z \cos\beta,\ \ v_\beta = \left(v_x \cos\theta + v_y \sin\theta\right)\cos\beta - v_z \sin\beta,$$
$$v_\theta = -v_x \sin\theta + v_y \cos\theta.$$
$$v_x = \left(v_R \sin\beta + v_\beta \cos\beta\right)\cos\theta - v_\theta \sin\theta,\ \ v_y = \left(v_R \sin\beta + v_\beta \cos\beta\right)\sin\theta + v_\theta \cos\theta,$$
$$v_z = v_R \cos\beta - v_\beta \sin\beta.$$

The spherical components are first found from the geometry of the appropriate figure, then the inverse result is found from solving the resulting set of algebraic equations.

1.4 Derive the continuity equation in cylindrical coordinates by examining a control volume bounded by the following: two cylinders perpendicular to the $x-y$ plane, the first of radius r, the second of radius $r+dr$; two planes perpendicular to the $x-y$ plane, the first making an angle θ with the x axis, the second an angle $\theta+d\theta$; two planes parallel to the $x-y$ plane, the first above it an amount z, the second an amount $z+dz$.

Solution:



The figure shows the rate at which mass enters and leaves the element through the sides. From the figure the net outflow through the 4 faces shown plus the two faces in the z direction is

$$\left[\rho v_r + \frac{\partial(\rho v_r)}{\partial r} dr\right](r+dr)d\theta dz - \rho v_r r d\theta dz$$
$$+\left[\rho v_\theta + \frac{\partial(\rho v_\theta)}{\partial\theta} d\theta\right] dr dz - \rho v_\theta r dr dz$$
$$+\left[\rho v_z + \frac{\partial(\rho v_z)}{\partial z} dz\right] r d\theta dr - \rho v_z r d\theta dr$$
$$= \left[\frac{\partial(\rho v_r)}{\partial r} + \frac{\rho v_r}{r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta} + \frac{\partial(\rho v_z)}{\partial z}\right] r d\theta dr dz$$

The change of mass in the imterior of the element is $\frac{\partial \rho}{\partial t} r dr d\theta dz$. Since the net change must be zero, dividing by the volume gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_r)}{\partial r} + \frac{\rho v_r}{r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0.$$

1.5 a. Find the stream function for the two-dimensional incompressible flow with velocity components $\mathbf{v} = (x^2 - 2xy\cos y^2, -2xy + \sin y^2,\ 0)$.

b. Find the discharge per unit between the points (1, π) and (0, 0).

Solution:

a. $v_y = -\frac{\partial \psi}{\partial x} = -2xy + \sin y^2,\ \therefore \psi = x^2 y - x\sin y^2 + f(y)$.

$$\frac{\partial \psi}{\partial y} = v_x = x^2 - 2xy\cos y^2 + \frac{df}{dy} = x^2 - 2xy\cos y^2.\ \ \therefore f = \text{constant}.$$

$$\psi(x, y) = x^2 y - x\sin y^2.$$

b. $Q = \psi(1, \pi) - \psi(0, 0) = 1 - \sin \pi^2$.

$$a_y = \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}$$

$$= -1 + (yz + t)\cdot z + (xz - t)\cdot 0 + xy\cdot x = -1 + y\left(x^2 + z^2\right) + tz.$$

$$a_x = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}$$

$$= 0 + (yz + t)\cdot y + (xz - t)\cdot x + xy\cdot 0 = z\left(x^2 + y^2\right) + t(y - x).$$

Therefore at (1,1,1), $a_x = 3 - t,\ a_y = 1 + t,\ a_z = 2$.

1.6 For the following flows, find the missing velocity component needed for the flow to satisfy the incompressible continuity equation.

a. $v_x = x^2 + y^2 + a^2,\ v_y = -xy - yz - xz,\ v_z = ?$

b. $v_x = \ln\left(y^2 + z^2\right),\ v_y = \sin\left(x^2 + z^2\right),\ v_z = ?$

c. $v_x = ?,\ v_y = \frac{y}{(x^2 + y^2 + z^2)^{3/2}},\ v_z = \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$.

Solution:

a. $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 2x - x - y = -\frac{\partial v_z}{\partial z},\ \therefore v_z = -z(x - y) + f(x, y)$.

b. $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 + 0 = -\frac{\partial v_z}{\partial z},\ \therefore v_z = f(x, y)$.

c. $\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{2}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3\left(y^2 + z^2\right)}{(x^2 + y^2 + z^2)^{5/2}} = -\frac{\partial v_x}{\partial x}$,

$$\therefore v_x = \frac{x}{(x^2 + y^2 + z^2)^{3/2}} + f(y, z).$$

1.7 For the flow field given by $\psi = A\ln(x^2+y^2)+yS$, find the discharge per unit width in the z direction between the points (1, 1, 0) and (−1, −1, 0)

Solution: The easy way to solve this is to say that the discharge is simply the difference between the values of the stream function at each point, thus

$$Q = A\ln(1^2+1^2)+S-A\ln(1^2+1^2)+S = 2S.$$

The student might want to check this by integration, giving

$$\begin{aligned} Q &= \int_{-1}^{1} -v_y(x,-1,0)dx + \int_{-1}^{1} v_x(1,y,0)dy \\ &= \int_{-1}^{1} -\left(-\frac{2Ax}{x^2+1}\right)dx + \int_{-1}^{1}\left(\frac{2Ay}{y^2+1}+S\right)dy \\ &= A\ \ln\left(x^2+1\right)\Big|_{-1}^{1} + A\ \ln\left(y^2+1\right)\Big|_{-1}^{1} + S\ y\Big|_{-1}^{1} = 2S. \end{aligned}$$

Here the path of integration is a horizontal line followed by a vertical line. Any other path will of course give the same value.

1.8 Find the stream function for the two-dimensional incompressible flow with a radial velocity (cylindrical polar coordinates) given by $v_r = \frac{A}{\sqrt{r}}\cos\theta$. Also find the missing velocity component.

Solution:

$$\begin{aligned} v_r &= -\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{A}{\sqrt{r}}\cos\theta,\ \therefore \psi = -A\sqrt{r}\sin\theta + f(r). \\ v_\theta &= \frac{\partial\psi}{\partial r} = -\frac{A}{2\sqrt{r}}\sin\theta + \frac{df}{dr}. \end{aligned}$$

1.9 Find the stream function for the two-dimensional incompressible flow with velocity components given by $v_r = U\left(1-\frac{a^2}{r^2}\right)\cos\theta,\ v_\theta = -U\left(1+\frac{a^2}{r^2}\right)\sin\theta$.

Solution:

$$\begin{aligned} v_r &= -\frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\left(1-\frac{a^2}{r^2}\right)\cos\theta,\ \therefore \psi = -rU\left(1-\frac{a^2}{r^2}\right)\sin\theta + f(r). \\ v_\theta &= \frac{\partial\psi}{\partial r} = -U\left(1+\frac{a^2}{r^2}\right)\sin\theta + \frac{df}{dr} = -U\left(1+\frac{a^2}{r^2}\right)\sin\theta.\ \therefore f = \text{constant}. \end{aligned}$$