2.1

$$a_i = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$a_i a_i = 6 \ \text{(scalar)}$$

(a) $$a_i a_j = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \ \text{(matrix)}$$

$$A_{ii} = A_{11} + A_{22} + A_{33} = 4 \ \text{(scalar)}$$

$$A_{ij} a_j = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \ \text{(vector)}$$

$$a_i = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$a_i a_i = 5 \ \text{(scalar)}$$

(b) $$a_i a_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix} \ \text{(matrix)}$$

$$A_{ii} = A_{11} + A_{22} + A_{33} = 2 \ \text{(scalar)}$$

$$A_{ij} a_j = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} \ \text{(vector)}$$

$$a_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$$a_i a_i = 6 \ \text{(scalar)}$$

(c) $$a_i a_j = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \ \text{(matrix)}$$

$$A_{ii} = A_{11} + A_{22} + A_{33} = 6 \ \text{(scalar)}$$

$$A_{ij} a_j = \begin{bmatrix} 3 \\ 8 \\ 6 \end{bmatrix} \ \text{(vector)}$$

2.2

$$A_{ij} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(a) $A_{(ij)} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) = \begin{bmatrix} 1 & 1/2 & 1 \\ 1/2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, six independent components

$A_{[ij]} = \frac{1}{2}\left(A_{ij} - A_{ji}\right) = \begin{bmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, three independent components

$$A_{ij} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(b) $A_{(ij)} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) = \begin{bmatrix} 1 & 3/2 & 1 \\ 3/2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, six independent components

$A_{[ij]} = \frac{1}{2}\left(A_{ij} - A_{ji}\right) = \begin{bmatrix} 0 & 1/2 & 0 \\ -1/2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, three independent components

$$A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

(c) $A_{(ij)} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) = \begin{bmatrix} 0 & 5/2 & 1 \\ 5/2 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$, six independent components

$A_{[ij]} = \frac{1}{2}\left(A_{ij} - A_{ji}\right) = \begin{bmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, three independent components

2.3.

$$A_{ij}B_{ij} = -A_{ji}B_{ji} = -A_{ij}B_{ij} \Rightarrow 2A_{ij}B_{ij} = 0 \Rightarrow A_{ij}B_{ij} = 0$$

2.4

$$\begin{aligned}
A_{ij}C_{ij} &= A_{ji}C_{ij} = A_{ij}C_{ji} \\
&= A_{ij}\left(C_{(ij)} + C_{[ij]}\right) = A_{ij}C_{(ij)} \\
B_{ij}C_{ij} &= -B_{ji}C_{ij} = -B_{ij}C_{ji} \\
&= B_{ij}\left(C_{(ij)} + C_{[ij]}\right) = B_{ij}C_{[ij]}
\end{aligned}$$

$$\det\left(B_{ij}\right) = \begin{vmatrix} 0 & B_{12} & B_{13} \\ -B_{12} & 0 & B_{23} \\ -B_{13} & -B_{23} & 0 \end{vmatrix} = -B_{12}B_{13}B_{23} + B_{13}B_{12}B_{23} = 0$$

2.5

$$\delta_{ij}a_{jk} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{21} + \delta_{13}a_{31} & \delta_{11}a_{12} + \delta_{12}a_{22} + \delta_{13}a_{32} & \delta_{11}a_{13} + \delta_{12}a_{23} + \delta_{13}a_{33} \\ . & . & . \\ . & . & . \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ik}$$

$$\delta_{ij}a_{kj} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{12} + \delta_{13}a_{13} & \delta_{11}a_{21} + \delta_{12}a_{22} + \delta_{13}a_{23} & \delta_{11}a_{31} + \delta_{12}a_{32} + \delta_{13}a_{33} \\ . & . & . \\ . & . & . \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = a_{ki}$$

2.6

$$\begin{aligned}
\det(A_{ij}) = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} &= \varepsilon_{123}A_{11}A_{22}A_{33} + \varepsilon_{231}A_{12}A_{23}A_{31} + \varepsilon_{312}A_{13}A_{21}A_{32} \\
&+ \varepsilon_{321}A_{13}A_{22}A_{31} + \varepsilon_{132}A_{11}A_{23}A_{32} + \varepsilon_{213}A_{12}A_{21}A_{33} \\
&= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} \\
&= A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31}) \\
&= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \text{ (using the usual first row expansion)}
\end{aligned}$$

Likewise the same for the form $\det[A_{ij}] = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$

2.7

$$\det[A_{ij}] = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21} \text{ , } \det[B_{ij}] = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = B_{11}B_{22} - B_{12}B_{21}$$

$$\Rightarrow \det[A_{ij}]\det[B_{ij}] = (A_{11}A_{22} - A_{12}A_{21})(B_{11}B_{22} - B_{12}B_{21})$$

$$= A_{11}A_{22}B_{11}B_{22} + A_{12}A_{21}B_{12}B_{21} - A_{11}A_{22}B_{12}B_{21} - A_{12}A_{21}B_{11}B_{22}$$

$$\det[\boldsymbol{AB}] = \begin{vmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{vmatrix}$$

$$= (A_{11}B_{11} + A_{12}B_{21})(A_{21}B_{12} + A_{22}B_{22}) - (A_{11}B_{12} + A_{12}B_{22})(A_{21}B_{11} + A_{22}B_{21})$$

$$= A_{11}A_{22}B_{11}B_{22} + A_{12}A_{21}B_{12}B_{21} - A_{11}A_{22}B_{12}B_{21} - A_{12}A_{21}B_{11}B_{22} = \det[A_{ij}]\det[B_{ij}]$$

Orthogonality Properties:

Given $A_{ik}A_{jk} = \delta_{ij} \Rightarrow \det\left(A_{ik}A_{jk}\right) = \det\delta_{ij} \Rightarrow \left(\det A\right)\left(\det A\right) = 1 \Rightarrow \det A = \pm 1$

2.8

Property (2.5.4) $\varepsilon_{ijk}\varepsilon_{pqr}\det[A_{mn}] = \begin{vmatrix} A_{ip} & A_{iq} & A_{ir} \\ A_{jp} & A_{jq} & A_{jr} \\ A_{kp} & A_{kq} & A_{kr} \end{vmatrix}$

For the case $A_{ij} = \delta_{ij} \Rightarrow \varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}$ (2.5.5)

From (2.5.5) with $p = i \Rightarrow \varepsilon_{ijk}\varepsilon_{iqr} = \begin{vmatrix} \delta_{ii} & \delta_{iq} & \delta_{ir} \\ \delta_{ji} & \delta_{jq} & \delta_{jr} \\ \delta_{ki} & \delta_{kq} & \delta_{kr} \end{vmatrix} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}$ (2.5.6)

From (2.5.6) with $q = j \Rightarrow \varepsilon_{ijk}\varepsilon_{ijr} = \delta_{jj}\delta_{kr} - \delta_{jr}\delta_{kj} = 3\delta_{kr} - \delta_{kr} = 2\delta_{kr}$ (2.5.7)

From (2.5.7) with $r = k \Rightarrow \varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{kk} = 6$ (2.5.8)

2.9

$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = 1$, $\boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = 1$, $\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1$

$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_1 = 0$, $\boldsymbol{e}_1 \cdot \boldsymbol{e}_3 = \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0$, $\boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = 1$, $\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1 \Rightarrow \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$

From (2.6.4) $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3 = -\boldsymbol{e}_2 \times \boldsymbol{e}_1$, $\boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1 = -\boldsymbol{e}_3 \times \boldsymbol{e}_2$, $\boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2 = -\boldsymbol{e}_3 \times \boldsymbol{e}_1$

$\Rightarrow \boldsymbol{e}_i \times \boldsymbol{e}_j = \varepsilon_{ijk}\boldsymbol{e}_k$

2.10

For 90° rotation about x_2-axis : $Q_{ij}=\begin{bmatrix}\cos 90^{\circ} & \cos 90^{\circ} & \cos 180^{\circ}\\ \cos 90^{\circ} & \cos 0^{\circ} & \cos 90^{\circ}\\ \cos 0^{\circ} & \cos 90^{\circ} & \cos 90^{\circ}\end{bmatrix}=\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$

Cases from Exercise 2.1:

$$\text{(a) } a_i' = Q_{ij}a_j = \begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix},\ |\boldsymbol{a}|=(a_ia_i)^{1/2}=\sqrt{6},\ |\boldsymbol{a}'|=(a_i'a_i')^{1/2}=\sqrt{6}$$

$$A_{ij}'=Q_{ip}Q_{jq}A_{pq}=\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 1 & 2 & 0\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}^T=\begin{bmatrix}1 & -2 & -1\\ 0 & 2 & 1\\ -1 & 0 & 1\end{bmatrix},\ A_{ii}=6,\ A_{ii}'=6$$

$$\text{(b) } a_i' = Q_{ij}a_j = \begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\ 2\\ 1\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ -1\end{bmatrix},\ |\boldsymbol{a}|=(a_ia_i)^{1/2}=\sqrt{6},\ |\boldsymbol{a}'|=(a_i'a_i')^{1/2}=\sqrt{6}$$

$$A_{ij}'=Q_{ip}Q_{jq}A_{pq}=\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ 1 & 0 & 0\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}^T=\begin{bmatrix}1 & -2 & -1\\ 0 & 0 & 1\\ -1 & 2 & 1\end{bmatrix},\ A_{ii}=2,\ A_{ii}'=2$$

$$\text{(c) } a_i' = Q_{ij}a_j = \begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}2\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}-1\\ 1\\ 2\end{bmatrix},\ |\boldsymbol{a}|=(a_ia_i)^{1/2}=\sqrt{6},\ |\boldsymbol{a}'|=(a_i'a_i')^{1/2}=\sqrt{6}$$

$$A_{ij}'=Q_{ip}Q_{jq}A_{pq}=\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 2 & 1\\ 3 & 2 & 0\\ 1 & 0 & 4\end{bmatrix}\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}^T=\begin{bmatrix}4 & 0 & -1\\ 0 & 2 & 3\\ -1 & 2 & 0\end{bmatrix},\ A_{ii}=6,\ A_{ii}'=6$$

2.11

$$\frac{d}{dt}\left(\boldsymbol{Q}\boldsymbol{Q}^T=\boldsymbol{I}\right)\Rightarrow \dot{\boldsymbol{Q}}\boldsymbol{Q}^T+\boldsymbol{Q}\dot{\boldsymbol{Q}}^T=0\Rightarrow$$

$$\dot{\boldsymbol{Q}}\boldsymbol{Q}^T=-\boldsymbol{Q}\dot{\boldsymbol{Q}}^T=-\left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T\right)^T$$

2.12

$$Q_{ij}=\begin{bmatrix}\cos(x_1',x_1) & \cos(x_1',x_2)\\ \cos(x_2',x_1) & \cos(x_2',x_2)\end{bmatrix}=\begin{bmatrix}\cos\theta & \cos(90^o-\theta)\\ \cos(90^o+\theta) & \cos\theta\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$b_i'=Q_{ij}b_j=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix}=\begin{bmatrix}b_1\cos\theta+b_2\sin\theta\\ -b_1\sin\theta+b_2\cos\theta\end{bmatrix}$$

$$A_{ij}'=Q_{ip}Q_{jq}A_{pq}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}^T$$

$$=\begin{bmatrix}A_{11}\cos^2\theta+(A_{12}+A_{21})\sin\theta\cos\theta+A_{22}\sin^2\theta & A_{12}\cos^2\theta-(A_{11}-A_{22})\sin\theta\cos\theta-A_{21}\sin^2\theta\\ A_{21}\cos^2\theta-(A_{11}-A_{22})\sin\theta\cos\theta-A_{12}\sin^2\theta & A_{11}\sin^2\theta-(A_{12}+A_{21})\sin\theta\cos\theta+A_{22}\cos^2\theta\end{bmatrix}$$

2.13

$$a_i=-\frac{1}{2}\varepsilon_{ijk}A_{jk}=-\frac{1}{2}\varepsilon_{ijk}\left(A_{(jk)}+A_{[jk]}\right)=-\frac{1}{2}\varepsilon_{ijk}A_{[jk]}\Rightarrow$$

$$a_i\varepsilon_{imn}=-\frac{1}{2}\varepsilon_{ijk}\varepsilon_{imn}A_{[jk]}=-\frac{1}{2}(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})A_{[jk]}=-\frac{1}{2}\left(A_{[mn]}-A_{[nm]}\right)=-A_{[mn]}$$

$$\therefore\ A_{[mn]}=-a_i\varepsilon_{imn}$$

2.14

Property 7 : Given $A_{ij}B_{ij}=c=$ scalar and B_{ij} is a second order tensor $\Rightarrow$

$A_{ij}'B_{ij}'=c'\Rightarrow A_{ij}'Q_{im}Q_{jn}B_{mn}=c=A_{mn}B_{mn}\Rightarrow A_{mn}=Q_{im}Q_{jn}A_{ij}'$ $\therefore$ $\boldsymbol{A}$ is a second order tensor

Property 8 : Given $A_{ij}b_j=c_i$ with b_j and c_i being first order tensors $\Rightarrow$

$A_{ij}'b_j'=c_i'\Rightarrow A_{ij}'Q_{jm}b_m=Q_{in}c_n=Q_{in}A_{nm}b_m\Rightarrow A_{ij}'Q_{jm}=Q_{in}A_{nm}\Rightarrow$

$Q_{km}\left(A_{ij}'Q_{jm}=Q_{in}A_{nm}\right)\Rightarrow A_{ij}'Q_{jm}Q_{km}=Q_{in}A_{nm}Q_{km}\Rightarrow A_{ij}'\delta_{jk}=A_{ik}'=Q_{in}Q_{km}A_{nm}$

$\therefore$ $\boldsymbol{A}$ is a second order tensor

2.15

$A_{ij}^{-1}A_{jm}=\dfrac{1}{2\det[\boldsymbol{A}]}\varepsilon_{ikl}\varepsilon_{jqr}A_{qk}A_{rl}A_{jm}=\dfrac{1}{2\det[\boldsymbol{A}]}\varepsilon_{ikl}\det[\boldsymbol{A}]\varepsilon_{mkl}$, where we have used (2.5.10)

$=\dfrac{1}{2}\varepsilon_{ikl}\varepsilon_{mkl}=\dfrac{1}{2}\varepsilon_{kli}\varepsilon_{klm}=\dfrac{1}{2}2\delta_{im}=\delta_{im}$, where we have used (2.5.7)

2.16

$I_1 = tr\boldsymbol{A} = I_A$

$I_2 = tr\boldsymbol{A^2} = A_{ij}A_{ji}$

From (2.11.3) $II_A = \dfrac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ij}) \Rightarrow A_{ij}A_{ij} = A_{ij}A_{ji} = A_{ii}A_{jj} - 2II_A$

$\therefore I_2 = I_A^2 - 2II_A$

$I_3 = tr\boldsymbol{A^3}$,

Using CT Theorem (2.13.1) $\Rightarrow \boldsymbol{A}^3 = I_A\boldsymbol{A^2} - II_A\boldsymbol{A} + III_A\boldsymbol{I}$

Taking trace operation gives $tr\boldsymbol{A^3} = I_A(I_A^2 - 2II_A) - II_AI_A + 3III_A$

$\therefore I_3 = I_A^3 - 3II_AI_A + 3III_A$

2.17

Given $A_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$I_A = A_{ii} = A_{11} + A_{22} + A_{33} = \lambda_1 + \lambda_2 + \lambda_3$

$II_A = \dfrac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ij}) = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}$

$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$

$III_A = \det[A_{ij}] = \lambda_1\lambda_2\lambda_3$

2.18

(a) $A_{ij}=\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}$

$I_A=A_{ii}=-4$, $II_A=\begin{vmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{vmatrix}+\begin{vmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{vmatrix}+\begin{vmatrix}A_{11} & A_{13}\\ A_{31} & A_{33}\end{vmatrix}=3$, $III_A=\det[A_{ij}]=0$

Characteristic Eqn. $-\lambda^3-4\lambda^2-3\lambda=0\Rightarrow\lambda(\lambda+1)(\lambda+3)=0,\ \lambda_1=0,\ \lambda_2=-1,\ \lambda_3=-3$

$\underline{\lambda_1=0}$, $(A_{ij}-\lambda\delta_{ij})n_j=0\Rightarrow$

$-2n_1^{(1)}+n_2^{(1)}=0$

$n_1^{(1)}-2n_2^{(1)}=0$

$\therefore\ n_1^{(1)}=n_2^{(1)}=0$ and thus $n_3^{(1)}=1\Rightarrow\boldsymbol{n}^{(1)}=\boldsymbol{e}_3$

$\underline{\lambda_2=-1}$, $(A_{ij}-\lambda\delta_{ij})n_j=0\Rightarrow$

$-n_1^{(2)}+n_2^{(2)}=0$

$n_1^{(2)}-n_2^{(2)}=0$

$n_3^{(2)}=0$

$\therefore\ n_1^{(1)}=n_2^{(1)}$ and thus $\Rightarrow\boldsymbol{n}^{(2)}=(\boldsymbol{e}_1+\boldsymbol{e}_2)/\sqrt{2}$

$\underline{\lambda_3=-3}$, $(A_{ij}-\lambda\delta_{ij})n_j=0\Rightarrow$

$n_1^{(3)}+n_2^{(3)}=0$

$n_1^{(3)}+n_2^{(3)}=0$

$3n_3^{(3)}=0$

$\therefore\ n_1^{(3)}=-n_2^{(3)}$ and thus $\Rightarrow\boldsymbol{n}^{(3)}=(\boldsymbol{e}_1-\boldsymbol{e}_2)/\sqrt{2}$

Rotation Matrix to Principal Axes: $Q_{ij}=\begin{bmatrix}0 & 0 & 1\\ 1/\sqrt{2} & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}\Rightarrow$

$$\boldsymbol{A}'=\boldsymbol{QAQ^T}=\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}$$

2.18

(b) $A_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$I_A = A_{ii} = 0 \ , \ II_A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = -4 \ , \ III_A = \det[A_{ij}] = 0$

Characteristic Eqn. $-\lambda^3 + 4\lambda = 0 \Rightarrow \lambda(\lambda^2 - 4) = 0, \ \lambda_1 = 0, \ \lambda_2 = 2, \ \lambda_3 = -2$

$\underline{\lambda_1 = 0}, \ (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$-n_1^{(1)} + n_2^{(1)} = 0$

$n_1^{(1)} - n_2^{(1)} = 0$

$2n_3^{(1)} = 0 \Rightarrow n_3^{(1)} = 0$

$\therefore \ n_1^{(1)} = n_2^{(1)}$ and normalizing $n_1^{(1)} = n_2^{(1)} = 1/\sqrt{2} \Rightarrow \boldsymbol{n}^{(1)} = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}$

$\underline{\lambda_2 = -2}, \ (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$n_1^{(2)} + n_2^{(2)} = 0$

$n_1^{(2)} + n_2^{(2)} = 0$

$4n_3^{(2)} = 0 \Rightarrow n_3^{(2)} = 0$

$\therefore \ n_1^{(2)} = -n_2^{(2)}$ and thus $\Rightarrow \boldsymbol{n}^{(3)} = (\boldsymbol{e}_1 - \boldsymbol{e}_2)/\sqrt{2}$

$\underline{\lambda_3 = 2}, \ (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$-3n_1^{(3)} + n_2^{(3)} = 0$

$n_1^{(3)} - 3n_2^{(3)} = 0$

$0 = 0 \Rightarrow$

$\therefore \ n_1^{(3)} = n_2^{(3)} = 0$ and thus $n_3^{(3)} = 1 \Rightarrow \boldsymbol{n}^{(3)} = \boldsymbol{e}_3$

Rotation Matrix to Principal Axes: $Q_{ij} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$

$$\boldsymbol{A}' = \boldsymbol{QAQ^T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2.18

(c) $A_{ij} = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$I_A = A_{ii} = 18$, $II_A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = 99$, $III_A = \det[A_{ij}] = 162$

Characteristic Eqn. $-\lambda^3 + 18\lambda^2 - 99\lambda + 162 = 0 \Rightarrow (\lambda - 9)(\lambda - 6)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 9,\ \lambda_2 = 6,\ \lambda_3 = 3$

$\underline{\lambda_1 = 9}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$-3n_1^{(1)} - 3n_2^{(1)} = 0$

$-3n_1^{(1)} - 3n_2^{(1)} = 0$

$-3n_3^{(1)} = 0 \Rightarrow n_3^{(1)} = 0$

$\therefore\ n_1^{(1)} = -n_2^{(1)}$ and normalizing $\boldsymbol{n}^{(1)} = (\boldsymbol{e}_1 - \boldsymbol{e}_2)/\sqrt{2}$

$\underline{\lambda_2 = 6}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$-3n_2^{(2)} = 0 \Rightarrow n_2^{(2)} = 0$

$-3n_1^{(2)} = 0 \Rightarrow n_1^{(2)} = 0$

$0 = 0$

$\therefore\ n_3^{(2)} = 1$ and thus $\Rightarrow \boldsymbol{n}^{(3)} = \boldsymbol{e}_3$

$\underline{\lambda_3 = 3}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$3n_1^{(3)} - 3n_2^{(3)} = 0$

$-3n_1^{(3)} + 3n_2^{(3)} = 0$

$3n_3^{(3)} = 0 \Rightarrow n_3^{(3)} = 0$

$\therefore\ n_1^{(3)} = n_2^{(3)}$ and normalizing $n_1^{(1)} = n_2^{(1)} = 1/\sqrt{2} \Rightarrow \boldsymbol{n}^{(3)} = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}$

Rotation Matrix to Principal Axes: $Q_{ij} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \Rightarrow$

$\boldsymbol{A}' = \boldsymbol{QAQ^T} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2.19

Mulitply basic equation $\boldsymbol{An} = \lambda\boldsymbol{n}$ by $\boldsymbol{A}^{-1}$ to get

$$\boldsymbol{A}^{-1}\boldsymbol{An} = \boldsymbol{A}^{-1}\lambda\boldsymbol{n} \Rightarrow \boldsymbol{n} = \lambda\boldsymbol{A}^{-1}\boldsymbol{n} \Rightarrow \boldsymbol{A}^{-1}\boldsymbol{n} = \frac{1}{\lambda}\boldsymbol{n}$$

$\therefore 1/\lambda$ are the principal values of $\boldsymbol{A}^{-1}$, and $\boldsymbol{n}$ are the principal directions

Using principal coordinates,

$$I_A = \lambda_1 + \lambda_2 + \lambda_3 \text{ , } II_A = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \text{ , } III_A = \lambda_1\lambda_2\lambda_3 \Rightarrow$$

$$II_A / III_A = \frac{\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1}{\lambda_1\lambda_2\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = I_{A^{-1}}$$

2.20

Using principal axes $A_{ij}b_ib_j = \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2$

If $\lambda_i \geq 0$, then clearly $A_{ij}b_ib_j \geq 0$

If $A_{ij}b_ib_j \geq 0 \Rightarrow \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 \geq 0$

Since vector $\boldsymbol{b}$ is arbitrary, choose $\boldsymbol{b} = (1,0,0) \Rightarrow \lambda_1 \geq 0$

Likewise for other choices $\boldsymbol{b} = (0,1,0) \Rightarrow \lambda_2 \geq 0$, $\boldsymbol{b} = (0,0,1) \Rightarrow \lambda_3 \geq 0$

2.21

(a) $A_{ij} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow A_{mm} = 9$

$$\tilde{A}_{ij} = \frac{1}{3} A_{mm} \delta_{ij} = 3\delta_{ij} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\hat{A}_{ij} = A_{ij} - \tilde{A}_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(b) $A_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow A_{mm} = 3$

$$\tilde{A}_{ij} = \frac{1}{3} A_{mm} \delta_{ij} = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{A}_{ij} = A_{ij} - \tilde{A}_{ij} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

(c) $A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix} \Rightarrow A_{mm} = 6$

$$\tilde{A}_{ij} = \frac{1}{3} A_{mm} \delta_{ij} = 2\delta_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\hat{A}_{ij} = A_{ij} - \tilde{A}_{ij} = \begin{bmatrix} -2 & 2 & 1 \\ 3 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

2.22

From (2.13.2) $\Rightarrow \boldsymbol{A}^3 = I_A\boldsymbol{A}^2 - II_A\boldsymbol{A} + III_A\boldsymbol{I}$

Multiply by $\boldsymbol{A} \Rightarrow \boldsymbol{A}^4 = I_A\boldsymbol{A}^3 - II_A\boldsymbol{A}^2 + III_A\boldsymbol{A}$; substituting in for $\boldsymbol{A}^3$ from (2.13.2) $\Rightarrow$

$\boldsymbol{A}^4 = I_A(I_A\boldsymbol{A}^2 - II_A\boldsymbol{A} + III_A\boldsymbol{I}) - II_A\boldsymbol{A}^2 + III_A\boldsymbol{A} = (I_A^2 - II_A)\boldsymbol{A}^2 + (III_A - I_A II_A)\boldsymbol{A} + I_A III_A\boldsymbol{I}$

Next multiplying by $\boldsymbol{A}$ again $\Rightarrow$

$\boldsymbol{A}^5 = (I_A^2 - II_A)\boldsymbol{A}^3 + (III_A - I_A II_A)\boldsymbol{A}^2 + I_A III_A\boldsymbol{A}$; substituting in for $\boldsymbol{A}^3$ from (2.13.2) $\Rightarrow$

$\boldsymbol{A}^5 = (I_A^2 - II_A)(I_A\boldsymbol{A}^2 - II_A\boldsymbol{A} + III_A\boldsymbol{I}) + (III_A - I_A II_A)\boldsymbol{A}^2 + I_A III_A\boldsymbol{A}$

$\boldsymbol{A}^5 = (I_A^3 - 2I_A II_A + III_A)\boldsymbol{A}^2 + (I_A III_A - I_A^2 II_A + II_A^2)\boldsymbol{A} + (I_A^2 III_A - II_A III_A)\boldsymbol{I}$

2.23

$\boldsymbol{A} = \boldsymbol{RU} = \boldsymbol{VR} \Rightarrow$

$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{RU} = \boldsymbol{U}^T\boldsymbol{IU} = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}^2$

$\boldsymbol{AA}^T = \boldsymbol{VRR}^T\boldsymbol{V}^T = \boldsymbol{VI}\,\boldsymbol{V}^T = \boldsymbol{V}\,\boldsymbol{V}^T = \boldsymbol{V}\,\boldsymbol{V} = \boldsymbol{V}^2$

2.24

(a) $\boldsymbol{u} = x_1\boldsymbol{e_1} + x_1x_2\boldsymbol{e_2} + 2x_1x_2x_3\boldsymbol{e_3}$

$\nabla \cdot \boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 1 + x_1 + 2x_1x_2$

$$\nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1x_2 & 2x_1x_2x_3 \end{vmatrix} = 2x_1x_3\boldsymbol{e_1} - 2x_2x_3\boldsymbol{e_2} + x_2\boldsymbol{e_3}$$

$\nabla^2\boldsymbol{u} = \nabla^2 u_1\boldsymbol{e_1} + \nabla^2 u_2\boldsymbol{e_2} + \nabla^2 u_3\boldsymbol{e_3} = 0\boldsymbol{e_1} + 0\boldsymbol{e_2} + 0\boldsymbol{e_3} = 0$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2x_3 & 2x_1x_3 & 2x_1x_2 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 1 + x_1 + 2x_1x_2$$

(b) $\boldsymbol{u} = x_1^2\boldsymbol{e_1} + 2x_1x_2\boldsymbol{e_2} + x_3^3\boldsymbol{e_3}$

$\nabla \cdot \boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 2x_1 + 2x_1 + 3x_3^2$

$$\nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1x_2 & x_3^3 \end{vmatrix} = 0\boldsymbol{e_1} - 0\boldsymbol{e_2} + 2x_2\boldsymbol{e_3}$$

$\nabla^2\boldsymbol{u} = \nabla^2 u_1\boldsymbol{e_1} + \nabla^2 u_2\boldsymbol{e_2} + \nabla^2 u_3\boldsymbol{e_3} = 2\boldsymbol{e_1} + 0\boldsymbol{e_2} + 6x_3\boldsymbol{e_3} = 0$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 4x_1 + 3x_3^2$$

(c) $\boldsymbol{u} = x_2^2\boldsymbol{e_1} + 2x_2x_3\boldsymbol{e_2} + 4x_1^2\boldsymbol{e_3}$

$\nabla \cdot \boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 0 + 2x_3 + 0$

$$\nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{vmatrix} = -2x_2\boldsymbol{e_1} - 8x_1\boldsymbol{e_2} - 2x_2\boldsymbol{e_3}$$

$\nabla^2\boldsymbol{u} = \nabla^2 u_1\boldsymbol{e_1} + \nabla^2 u_2\boldsymbol{e_2} + \nabla^2 u_3\boldsymbol{e_3} = 2\boldsymbol{e_1} + 0\boldsymbol{e_2} + 8\boldsymbol{e_3}$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 2x_3$$

2.25

(a)

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k} = \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi = \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi$$

$$\nabla\cdot(\phi\boldsymbol{u}) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k}u_k = \nabla\phi\cdot\boldsymbol{u} + \phi(\nabla\cdot\boldsymbol{u})$$

(b)

$$\boldsymbol{\nabla}\times(\phi\boldsymbol{u}) = \varepsilon_{ijk}(\phi u_k)_{,j} = \varepsilon_{ijk}(\phi u_{k,j} + \phi_{,j}u_k) = \varepsilon_{ijk}\phi_{,j}u_k + \phi\varepsilon_{ijk}u_k = \boldsymbol{\nabla}\phi\times\boldsymbol{u} + \phi(\boldsymbol{\nabla}\times\boldsymbol{u})$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = (\varepsilon_{ijk}u_j v_k)_{,i} = \varepsilon_{ijk}(u_j v_{k,i} + u_{j,i}v_k) = v_k\varepsilon_{ijk}u_{j,i} + u_j\varepsilon_{ijk}v_{k,i} = \boldsymbol{v}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}) - \boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0 \text{ because of symmetry and antisymmetry in } jk$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = (\phi_{,k})_{,k} = \phi_{,kk} = \nabla^2\phi$$

(c)

$$\boldsymbol{\nabla}\cdot(\nabla\times\boldsymbol{u}) = (\varepsilon_{ijk}u_{k,j})_{,i} = \varepsilon_{ijk}u_{k,ji} = 0, \text{ because of symmetry and antisymmetry in } ij$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = \varepsilon_{mni}(\varepsilon_{ijk}u_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk}u_{k,jn} = (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})u_{k,jn} = u_{n,nm} - u_{m,nn}$$

$$= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u}$$

$$\boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = \varepsilon_{ijk}u_j(\varepsilon_{kmn}u_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn}u_j u_{n,m} = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_j u_{n,m} = u_n u_{n,i} - u_m u_{i,m}$$

$$= \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\cdot\nabla\boldsymbol{u}$$

2.26

Cylindrical coordinates: $\xi^1 = r,\ \xi^2 = \theta,\ \xi^3 = z$

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2 \Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = 1$$

$$\hat{\boldsymbol{e}}_r = \cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{e}_2,\ \hat{\boldsymbol{e}}_\theta = -\sin\theta\boldsymbol{e}_1 + \cos\theta\boldsymbol{e}_2,\ \hat{\boldsymbol{e}}_z = \boldsymbol{e}_3$$

$$\frac{\partial\hat{\boldsymbol{e}}_r}{\partial\theta} = \hat{\boldsymbol{e}}_\theta,\ \frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial\theta} = -\hat{\boldsymbol{e}}_r,\ \frac{\partial\hat{\boldsymbol{e}}_r}{\partial r} = \frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial r} = \frac{\partial\hat{\boldsymbol{e}}_z}{\partial r} = \frac{\partial\hat{\boldsymbol{e}}_z}{\partial\theta} = \frac{\partial\hat{\boldsymbol{e}}_z}{\partial z} = 0$$

$$\boldsymbol{\nabla} = \hat{\boldsymbol{e}}_r\frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}$$

$$\boldsymbol{\nabla}f = \hat{\boldsymbol{e}}_r\frac{\partial f}{\partial r} + \hat{\boldsymbol{e}}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{\boldsymbol{e}}_z\frac{\partial f}{\partial z}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\boldsymbol{e}}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right)\hat{\boldsymbol{e}}_z$$