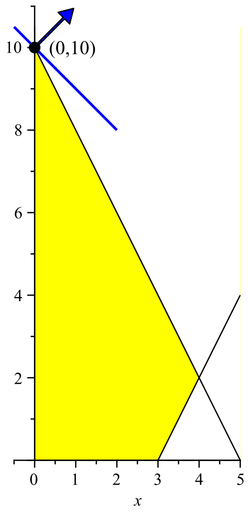
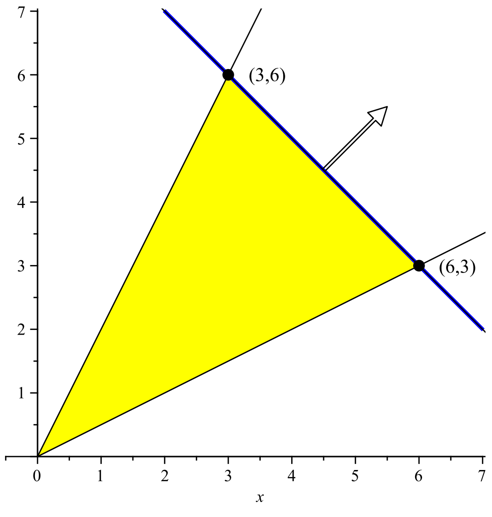
# Chapter 1

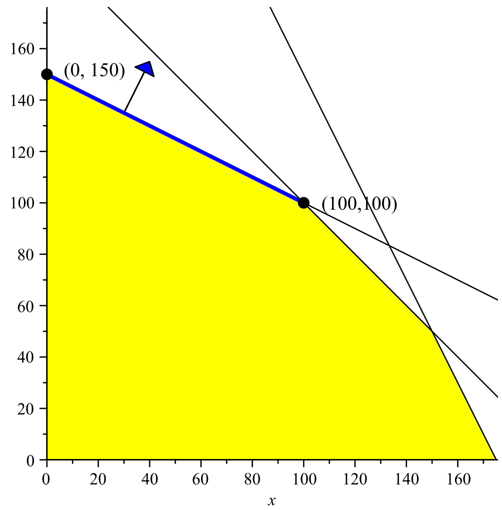
1.1. (a) The optimal solution occurs at $(0, 10)$ with optimal value 20.



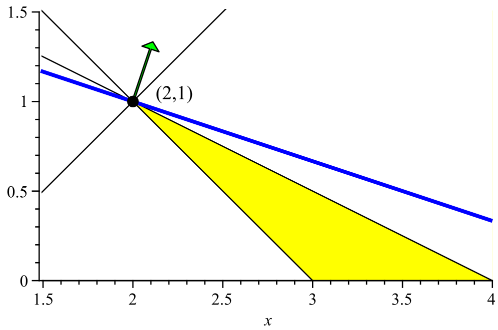

(b) The optimal solution occurs on the line segment joining $(3, 6)$ and $(6, 3)$ with optimal value 9.



(c) The optimal solutions on the line segment joining occurs at $(0,150)$ and $(100,100)$ with optimal value 7500.




(d) The optimal solution occurs at $(2,1)$ with optimal value 7.




(e) The feasible region is empty, and hence there is no optimal solution.

1.2. (a) The optimal range for coefficient $c_x$ of $x$ is $c_x \leq 4$. The range for coefficient $c_y$ of $y$ is $c_y >= 1.5$.

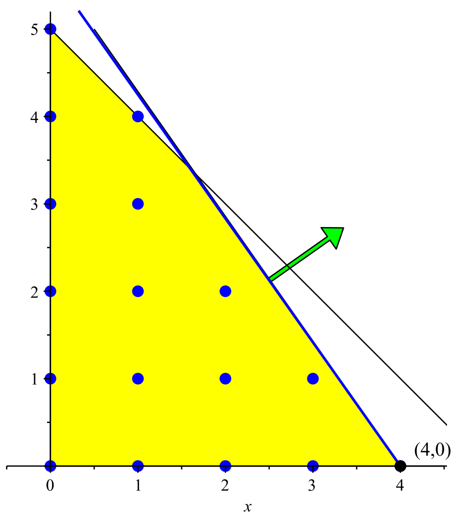
(b) For optimal solution $(6,3)$ to remain optimal, the coefficient $c_x$ of $x$ must be so that the slope of the objective line remains negatives, which indicates that $-c_x \leq -1$ or that $c_x \geq 1$, while the coefficient $c_y$ of $y$ must be in the interval $-2 \leq c_y \leq 1$. For the optimal solution $(3,6)$ to remain optimal, the coefficient $c_x$ must remain in the interval $-2 \leq c_x \leq 1$ while the coefficient $c_y$ of $y$ must satisfy $c_y \geq 1$, since it must remain positive and the slope of the objective line must be no more than $-1$.

(c) For the solution $(0, 150)$ to remain optimal, the coefficient $c_1$ of $x_1$ must satisfy $c_1 \geq 25$, and the coefficient $c_2$ must satisfy $c_2 \geq 50$. For the solution $(100, 100)$ to remain optimal, we must have $25 \leq c_1 \leq 50$ and $25 \leq c_2 \leq 50$.

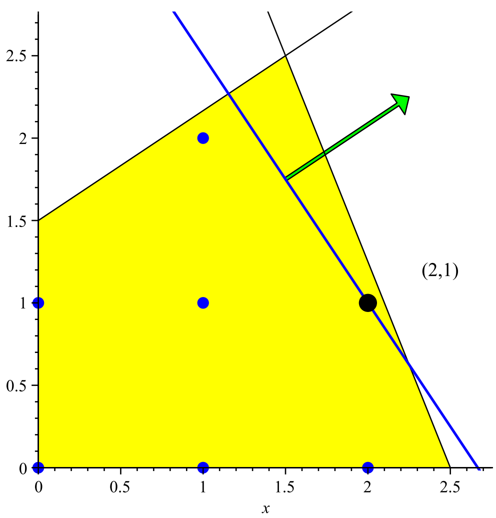
(d) For $(2, 1)$ to remain optimal the coefficient $c_1$ of $x_1$ must satisfy $-3 \leq c_1 \leq \frac{3}{2}$ and the coefficient $c_2$ of $x_2$ must satisfy $c_2 \geq 2$.

(e) The feasible region is empty, and hence there is no optimal solution.

1.3. The optimal solution occurs at $(4, 0)$ with optimal value 68.




1.4. The optimal solution occurs at $(2, 1)$ with optimal value 8.



# Chapter 2

Note: Many of these solutions are available electronically in a variety of modeling languages. Many of the solutions given here are presented in the modeling language Mosel (Xpress-MP) as an example. In addition, we often provide a general model (without the data values) as opposed to a specific instance.

2.1. Let $UT, UC$ denote the number of unfinished tables and chairs made, $FT, FC$ be the number of finished tables and chairs made, and $W$ be the amount of wood purchased. Our model is then

$$
\begin{aligned}
\max \quad & 80UT + 120FT + 30UC + 55FC - 2W \\
\text{s.t.} \quad & \\
& 4UT + 2UC + 12FT + 8FC \leq 2500 && \text{(Hours)} \\
& W = 25(UT + FT) + 10(UC + FC) && \text{(Wood)} \\
& W \leq 10000 \\
& UC + FC \geq 2(UT + FT) \\
& UC + FC \geq 450 \\
& UT + FT \geq 200 \\
& UT, UC, FT, FT, W \geq 0.
\end{aligned}
$$

The optimal solution is $UT = 130$, $FT = 90$, $UC = 450$, $FC = 0$, and $W = 10000$ with value 14700.

2.2. Below summarizes the changes in optimal solution and profit for each amount of labor.

| Labor Hours | Solution | Profit |
|---|---|---|
| 290 | $(9.35484, 0, 9.35484)$ | 935.484 |
| 300 | $(9.67742, 0, 9.67742)$ | 967.742 |
| 310 | $(10, 0, 10)$ | 1000 |
| 320 | $(5.92593, 3.7037, 9.62963)$ | 1000 |

As labor increases to 310 hours, we do not make any High Security doors in the optimal profit because their labor usage is too high. Once we have enough labor, we begin to produce these doors.

2.3. If we let $x_j$ be the number of packages of thickness $j$ and $Labor_m$ to be the amount of time (in minutes) used on machine $m$, then our model is as follows.

$$\begin{aligned}
\max \quad & \sum_{j \in Thickness} (Revenue_j - MatCost_j)\, x_j) - \sum_{m \in M} LaborCost_m Labor_m \\
\text{s.t.} \quad & \\
& \sum_{j \in Thickness} MachTime_{jm} x(j) \leq Labor_m && m \in M \\
& Labor_m \leq AvailTime && m \in M \\
& x_j \leq MaxDemand_j \\
& x_j, Labor_m \geq 0.
\end{aligned}$$

The optimal solution is to produce 0 units 0.5mm, 106.25 units of 1mm, 250 units of 2mm, and 200 units of 5mm This uses 2331.25 minutes on Machine 1, 3600 minutes on Machine 2, and 3006.25 units on Machine 3, resulting in an Optimal Profit of \$34275.

2.4. Let $A, B, C$ denote the number of stained bookshelves and $UA, UB, UC$ the number of unstained bookshelves to produce. Our model is as follows.

$$\begin{aligned}
\max \quad & 60A + 40B + 75C + 30UA + 20UB + 40UC \\
\text{s.t.} \quad & \\
& (A + UA) + 0.5(B + UB) + 2(C + UC) \leq 200 && \text{(cutting)} \\
& 4(A + UA) + 3(B + UB) + 6(C + UC) \leq 700 && \text{(assembly)} \\
& 7A + 5B + 8C \leq 550 && \text{(staining)} \\
& UA + UB + UC \leq 50 && \text{(Unstained Max)} \\
& B + UB \geq 20 \\
& \quad A, B, C, UA, UB, UC \geq 0
\end{aligned}$$

(a) The optimal solution is to $A = 30$, $B = 20$,$C = 30$, and $UC = 50$, for a profit of \$6850.

(b) There are only 40 hours available in Assembly. The others use all available hours.

(c) When the profit margin is 50 or 55, the optimal solution changes to $A = 0$, $B = 50$,$C = 37.5$, and $UC = 50$. When the profit margin is 65, the optimal solution remains at $A = 30$, $B = 20$,$C = 30$, and $UC = 50$. When the profit margin is 70, the solution become $A = 64.2857$, $B = 20$,$C = 0$, and $UC = 50$

(d) When the profit margin is 20 and 25, the solution remains as it was for 30: $A = 30$, $B = 20$,$C = 30$, and $UC = 50$. When the profit margin becomes 35, the solution changes to $A = 0$, $B = 20$,$C = 56.25$, $UA = 22.5$, and $UC = 27.5$. When the profit margin is 40, the solution becomes $A = 0$, $B = 20$,$C = 56.25$, and $UA = 50$.

(e) When cutting hours are 150, the optimal solution is $A = 64.2857$, $B = 20$,$C = 0$, $UA = 24.2857$, and $UC = 27.7143$ for a profit of \$6414.29. When cutting hours are 175, the optimal solution is $A = 63.3333$, $B = 20$,$C = 0.83333$, $UA = 0$, and $UC = 50$ for a profit of \$6662.50. When cutting hours are 225 and 250, the optimal solution is $A = 0$, $B = 20$,$C = 56.25$, $UA = 0$, and $UC = 50$ for a profit of \$7018.75.

(f) When assembly hours are 650, the optimal solution is $A = 38$, $B = 20$,$C = 23$, and $UC = 50$, for a profit of \$6805. When assembly hours are 675, 700, 725, and 750, the optimal solution is $A = 30$, $B = 20$,$C = 30$, and $UC = 50$, for a profit of \$6850.

2.5. Let $PoleM$, $BackboardM$, and $RimM$ be the number of items manufactured and $PoleP$, and $RimP$ be the amounts produced. Our model is then

$$\begin{array}{lllll}
\min & 60PoleM + 80BackboardM + 30RimM + 95PoleP + 45RimP & & & \\
\text{s.t.} & & & & \\
& 2PoleM + 2.5BackboardM + \; RimM & \leq 2000 & \text{(Dept. A)} \\
& 0.5PoleM + \; BackboardM + 2RimM & \leq 900 & \text{(Dept. B)} \\
& PoleM + \; 2BackboardM + \; RimM & \leq 1500 & \text{(Dept. C)} \\
& PoleM + PoleP & \geq 500 & \text{(Pole)} \\
& BackboardM & \geq 500 & \text{(Backboard)} \\
& RimM + RimP & \geq 500 & \text{(Rim)} \\
& PoleM, BackboardM, RimM, PoleP, RimP \geq 0. & & 
\end{array}$$

(a) The optimal solution is $PoleM = 375$, $BackboardM = 500$, $PoleP = 125$, and $RimP = 500$, with total cost \$96875.

(b) Department B has 212.50 hours unused and Department C has 125 unused hours.

(c) When manufacturing costs on poles is 50, 55, or 60, the optimal solution is $PoleM = 375$, $BackboardM = 500$, $PoleP = 125$, and $RimP = 500$. When manufacturing costs on poles is 65 or 70, the optimal solution is $PoleM = 290$, $BackboardM = 500$, , $RimM = 170$, $PoleP = 210$, and $RimP = 330$.

(d) When purchasing costs on rims are 40 or 45, the optimal solution is $PoleM = 375$, $BackboardM = 500$, $PoleP = 125$, and $RimP = 500$. When purchasing costs on rims are 50, 55, or 60, the optimal solution is $PoleM = 290$, $BackboardM = 500$, , $RimM = 170$, $PoleP = 210$, and $RimP = 330$.

(e) When Department C hours are 1350, the solution is $PoleM = 350$, $BackboardM = 500$, $PoleP = 150$, and $RimP = 500$. When hours are 1375 and above, the solution remains as it originally was.

2.6. If we let $Emp8hr(j)$ and $Emp12hr(j)$ denote the number of 8-hour and 12-hour workers starting their shift in time slot $j$, then our model (in Mosel) would be as follows.

---

```
forall(i in 1..2) do
  PERIOD12(i):= sum(j in maxlist(i-2, 1)..i) Emp8hr(j) +
                sum(j in minlist(i+5, 7)..6) Emp8hr(j) +
                Emp12hr(1)
                >= minpeople(i)

  Emp8hr(i) is_integer
  Emp12hr(i) is_integer
end-do

PERIOD3 := sum(j in 2..3) Emp8hr(j) +Emp12hr(1) + Emp12hr(3) >= minpeople(3)

Emp8hr(3) is_integer
Emp12hr(3) is_integer

forall(i in 4..5) do
  PERIOD45(i):= sum(j in maxlist(i-2, 1)..i) Emp8hr(j) + Emp12hr(3) >= minpeople(i)
```

```
    Emp8hr(i) is_integer
    Emp12hr(i) is_integer
  end-do

  PERIOD6 := sum(j in 5..6) Emp8hr(j) >= minpeople(6)

  Emp8hr(6) is_integer
  Emp12hr(6) is_integer

  Emp12hr(2) = 0
  Emp12hr(4) = 0
  Emp12hr(5) = 0
  Emp12hr(6) = 0

  TotalCost := sum(i in periods) (cost8hr*Emp8hr(i) + cost12hr*Emp12hr(i))

  minimize(TotalCost)
```

An optimal schedule uses only 8-hour shift and costs $11200. The number of people starting at each shift is given below.

| Period | 8-hr Starts |
|---|---|
| 12AM-4AM | 2 |
| 4AM-8AM | 8 |
| 8AM-12PM | 8 |
| 12PM-4PM | 5 |
| 4PM-8PM | 6 |
| 8PM-12AM | 6 |

2.7. We now need variables Emp8hr(d, i) and Emp12hr(d,i) for the number of 8-hr and 12-hr employees who start their work weeks on day $d$, period $i$, and a constraint for each day and each period.

```
forall(d in Days) do
  forall(i in 1..2) do
    P12(d,i):=
        sum(dd in maxlist(d-4,1)..d, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
        sum(dd in maxlist(d-4,1)..d, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
        sum(dd in minlist(d+3,8)..7, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
        sum(dd in minlist(d+3,8)..7, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
        sum(dd in maxlist(d-2,1)..d) Emp12hr(dd, 1)  +
        sum(dd in minlist(d+5,8)..7) Emp12hr(dd, 1)
        >= minpeople(i)

    Emp8hr(d, i) is_integer
    Emp12hr(d, i) is_integer
  end-do
```

```
  P3(d) := sum(dd in maxlist(d-4,1)..d, j in 2..3) Emp8hr(dd, j) +
           sum(dd in minlist(d+3,8)..7, j in 2..3) Emp8hr(dd, j) +
           sum(dd in maxlist(d-2,1)..d) (Emp12hr(dd, 2) + Emp12hr(dd,3))  +
           sum(dd in minlist(d+5,8)..7) (Emp12hr(dd, 2) + Emp12hr(dd,3))
           >= minpeople(3)

  Emp8hr(d,3) is_integer
  Emp12hr(d,3) is_integer

  forall(i in 4..5) do
    P45(d,i):=
      sum(dd in maxlist(d-4,1)..d, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
      sum(dd in maxlist(d-4,1)..d, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
      sum(dd in minlist(d+3,8)..7, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
      sum(dd in minlist(d+3,8)..7, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
      sum(dd in maxlist(d-2,1)..d) Emp12hr(dd, 3)  +
      sum(dd in minlist(d+5,8)..7) Emp12hr(dd, 3)
      >= minpeople(i)

    Emp8hr(d,i) is_integer
    Emp12hr(d,i) is_integer
  end-do

  P6(d) := sum(dd in maxlist(d-4,1)..d, j in 5..6) Emp8hr(dd, j) +
           sum(dd in minlist(d+3,8)..7, j in 5..6) Emp8hr(dd, j)
           >= minpeople(6)

  Emp8hr(d,6) is_integer
  Emp12hr(d,6) is_integer

  Emp12hr(d,2) = 0
  Emp12hr(d,4) = 0
  Emp12hr(d,5) = 0
  Emp12hr(d,6) = 0
end-do

TotalCost := sum(d in Days, i in periods) (cost8hr*Emp8hr(d,i) +
                                           cost12hr*Emp12hr(d,i))

minimize(TotalCost)
```

Solving this model may take a long time (this is due to the symmetry of the days), but after a few seconds the following solution is found, again using only 8-hour shifts:

| Period/Day | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 12AM-4AM | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| 4AM-8AM | 2 | 1 | 2 | 1 | 2 | 2 | 2 |
| 8AM-12PM | 3 | 0 | 0 | 4 | 1 | 4 | 1 |
| 12PM-4PM | 1 | 2 | 2 | 0 | 0 | 0 | 0 |
| 4PM-8PM | 0 | 2 | 2 | 0 | 2 | 1 | 1 |
| 8PM-12AM | 2 | 1 | 0 | 3 | 0 | 2 | 2 |

The cost for this solution is \$80,000.

2.8. We now need to constrain over a 2-week time frame. This affects our Emp12hr variables, which now need an additional index: Emp12hr(w,d,i) for week $w \in \{1, 2\}$. Our model is now

```
! week 1 first
forall(d in Days) do
  forall(i in 1..2) do
    P112(d, i):=
      sum(dd in maxlist(d-4,1)..d, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
      sum(dd in maxlist(d-4,1)..d, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
      sum(dd in minlist(d+3,8)..7, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
      sum(dd in minlist(d+3,8)..7, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
      sum(dd in maxlist(d-2,1)..d) Emp12hr(1,dd, 1)  +
      sum(dd in minlist(d+5,8)..7) Emp12hr(2,dd, 1)
      >= minpeople(i)

    Emp8hr(d, i) is_integer
    Emp12hr(1, d, i) is_integer
  end-do

  P13(d) :=
      sum(dd in maxlist(d-4,1)..d, j in 2..3) Emp8hr(dd, j) +
      sum(dd in minlist(d+3,8)..7, j in 2..3) Emp8hr(dd, j) +
      sum(dd in maxlist(d-2,1)..d) (Emp12hr(1, dd, 2) + Emp12hr(1, dd,3)) +
      sum(dd in minlist(d+5,8)..7) (Emp12hr(2, dd, 2) + Emp12hr(2, dd,3))
      >= minpeople(3)

  Emp8hr(d,3) is_integer
  Emp12hr(1, d,3) is_integer

  forall(i in 4..5) do
    P145(d,i):=
      sum(dd in maxlist(d-4,1)..d, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
      sum(dd in maxlist(d-4,1)..d, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
      sum(dd in minlist(d+3,8)..7, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
      sum(dd in minlist(d+3,8)..7, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
      sum(dd in maxlist(d-2,1)..d) Emp12hr(1, dd, 3)  +
      sum(dd in minlist(d+5,8)..7) Emp12hr(2, dd, 3)
      >= minpeople(i)
```

```
    Emp8hr(d,i) is_integer
    Emp12hr(1, d,i) is_integer
  end-do
  P16(1, d) :=
      sum(dd in maxlist(d-4,1)..d, j in 5..6) Emp8hr(dd, j) +
      sum(dd in minlist(d+3,8)..7, j in 5..6) Emp8hr(dd, j)
      >= minpeople(6)

  Emp8hr(d,6) is_integer
  Emp12hr(1, d,6) is_integer

  Emp12hr(1,d,2) = 0
  Emp12hr(1,d,4) = 0
  Emp12hr(1,d,5) = 0
  Emp12hr(1,d,6) = 0
end-do

! week 2 next
forall(d in Days) do
  forall(i in 1..2) do
    P212(d, i):=
      sum(dd in maxlist(d-4,1)..d, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
      sum(dd in maxlist(d-4,1)..d, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
      sum(dd in minlist(d+3,8)..7, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
      sum(dd in minlist(d+3,8)..7, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
      sum(dd in maxlist(d-2,1)..d) Emp12hr(2,dd, 1)  +
      sum(dd in minlist(d+5,8)..7) Emp12hr(1,dd, 1)
      >= minpeople(i)

    Emp12hr(2, d, i) is_integer
  end-do

  P3(2, d) :=
      sum(dd in maxlist(d-4,1)..d, j in 2..3) Emp8hr(dd, j) +
      sum(dd in minlist(d+3,8)..7, j in 2..3) Emp8hr(dd, j) +
      sum(dd in maxlist(d-2,1)..d) (Emp12hr(2, dd, 2) + Emp12hr(1, dd,3)) +
      sum(dd in minlist(d+5,8)..7) (Emp12hr(1, dd, 2) + Emp12hr(2, dd,3))
      >= minpeople(3)

  Emp12hr(2, d,3) is_integer

  forall(i in 4..5) do
    P245(d,i):=
      sum(dd in maxlist(d-4,1)..d, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
      sum(dd in maxlist(d-4,1)..d, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
      sum(dd in minlist(d+3,8)..7, j in maxlist(i-2, 1)..i) Emp8hr(dd,j) +
      sum(dd in minlist(d+3,8)..7, j in minlist(i+5, 7)..6) Emp8hr(dd,j) +
```

```
      sum(dd in maxlist(d-2,1)..d) Emp12hr(2, dd, 3)  +
      sum(dd in minlist(d+5,8)..7) Emp12hr(1, dd, 3)
      >= minpeople(i)

    Emp12hr(2, d,i) is_integer
  end-do

  P26(1, d) :=
      sum(dd in maxlist(d-4,1)..d, j in 5..6) Emp8hr(dd, j) +
      sum(dd in minlist(d+3,8)..7, j in 5..6) Emp8hr(dd, j)
      >= minpeople(6)

  Emp12hr(2, d,6) is_integer

  Emp12hr(2,d,2) = 0
  Emp12hr(2,d,4) = 0
  Emp12hr(2,d,5) = 0
  Emp12hr(2,d,6) = 0
end-do

TotalCost :=
      sum(d in Days, i in periods) 2*5*cost8hr*Emp8hr(d,i) +
      sum(w in Week, d in Days, i in periods) 7*cost12hr*Emp12hr(w,d,i)

minimize(TotalCost)
```

Again, no 12-hr shifts are used, and so our optimal schedule is the same as in Exercise 2.7, with 2-week cost of $160,000.

2.9. Let $P(i, j)$ denote the amount of product $i$ processed into product $j$, $Sold(i)$ indicate the amount of Product $i$ sold, $Total(i)$ denote the amount of Product $i$ available, and $Raw$ denote the amount of raw material purchased. Our model is then

$$\max \quad 10Sold(1) + 20Sold(2) + 30Sold(3) - ((25+1)Raw + P(1,2) + 2P(1,3) + 6P(2,3))$$

s.t.

$$2Raw + 2P(1,2) + 3P(1,3) + P(2,3) \leq 25000 \qquad \text{(Labor)}$$
$$Total(1) = 3Raw$$
$$Total(1) = Sold(1) + P(1,2) + P(1,3)$$
$$Total(2) = Raw + \tfrac{2}{3}P(1,2)$$
$$Total(2) = Sold(2) + P(2,3)$$
$$Sold(3) = \tfrac{1}{2}P(1,3) + \tfrac{3}{4}P(2,3)$$
$$Sold(1) \leq 5000$$
$$Sold(2) \leq 4000$$
$$Sold(3) \leq 3000$$

All variables nonnegative.