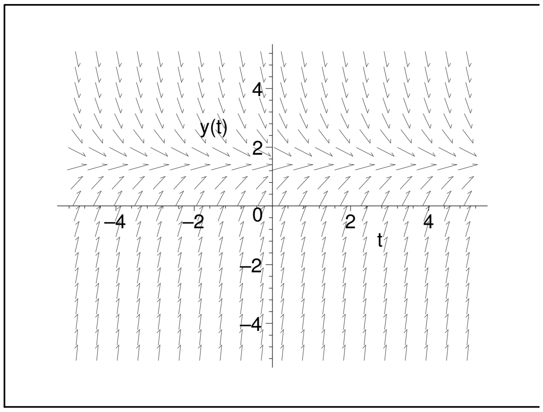
Chapter 1

Introduction

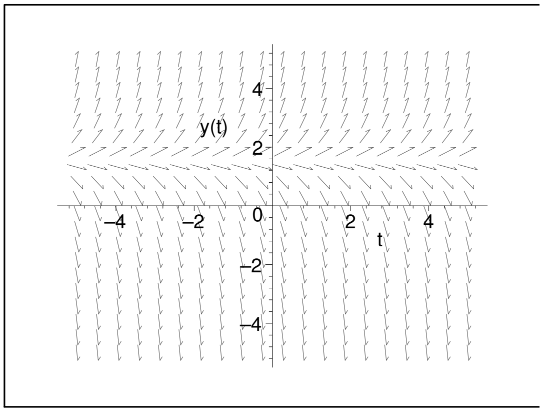
1.1 Mathematical Models, Solutions, and Direction Fields

1.



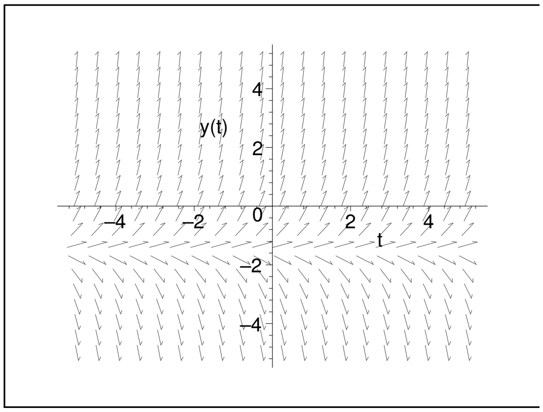
For $y > 3/2$, the slopes are negative, and, therefore the solutions decrease. For $y < 3/2$, the slopes are positive, and, therefore, the solutions increase. As a result, $y \to 3/2$ as $t \to \infty$

2.



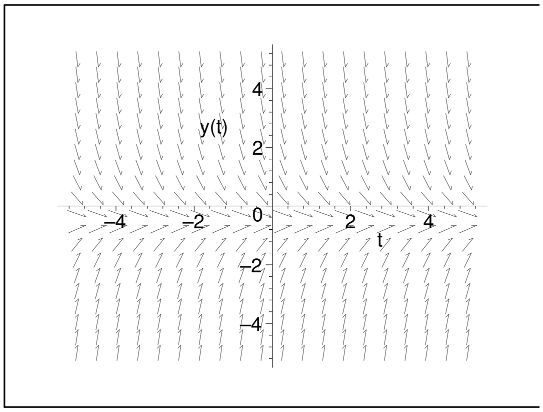
For $y > 3/2$, the slopes are positive, therefore the solutions increase. For $y < 3/2$, the slopes are negative, therefore, the solutions decrease. As a result, y diverges from $3/2$ as $t \to \infty$ if $y(0) \neq 3/2$.

3.



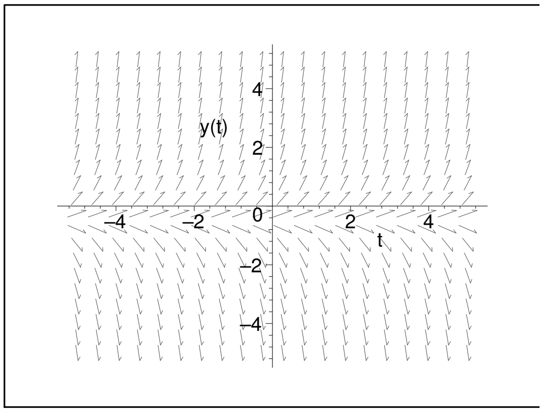
For $y > -3/2$, the slopes are positive, and, therefore the solutions increase. For $y < -3/2$, the slopes are negative, and, therefore, the solutions decrease. As a result, y diverges from the equilibrium $-3/2$ as $t \to \infty$

4.



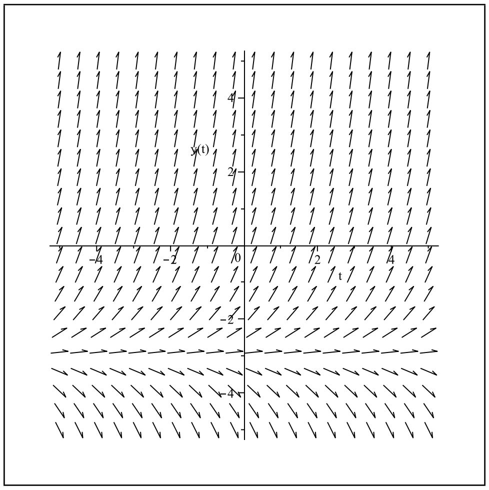
For $y > -1/2$, the slopes are negative, therefore the solutions decrease. For $y < -1/2$, the slopes are positive, therefore, the solutions increase. As a result, $y \to -1/2$ as $t \to \infty$.

5.



For $y > -1/2$, the slopes are positive, and, therefore the solutions increase. For $y < -1/2$, the slopes are negative, and, therefore, the solutions decrease. As a result, y diverges from the equilibrium $-1/2$ as $t \to \infty$

6.



For $y > -3$, the slopes are positive, therefore the solutions increase. For $y < -3$, the slopes are negative, therefore, the solutions decrease. As a result, y diverges from -3 as $t \to \infty$.

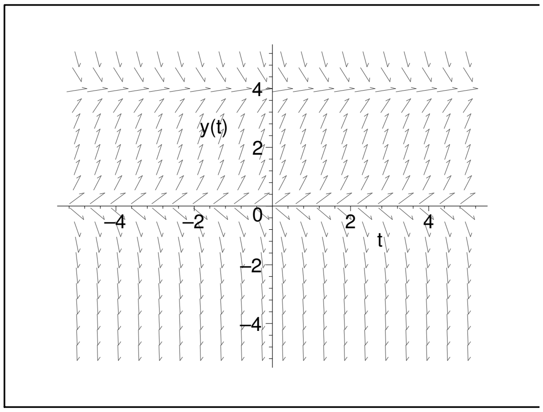
7. For the solutions to satisfy $y \to 3$ as $t \to \infty$, we need $y' < 0$ for $y > 3$ and $y' > 0$ for $y < 3$. The equation $y' = 3 - y$ satisfies these conditions.

8. For the solutions to satisfy $y \to 3/4$ as $t \to \infty$, we need $y' < 0$ for $y > 3/4$ and $y' > 0$ for $y < 3/4$. The equation $y' = 3 - 4y$ satisfies these conditions.

9. For the solutions to satisfy y diverges from 2, we need $y' > 0$ for $y > 2$ and $y' < 0$ for $y < 2$. The equation $y' = y - 2$ satisfies these conditions.

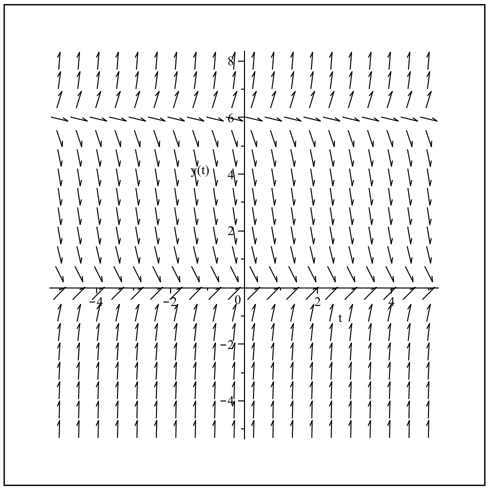
10. For the solutions to satisfy y diverges from $1/3$, we need $y' > 0$ for $y > 1/3$ and $y' < 0$ for $y < 1/3$. The equation $y' = 3y - 1$ satisfies these conditions.

11.



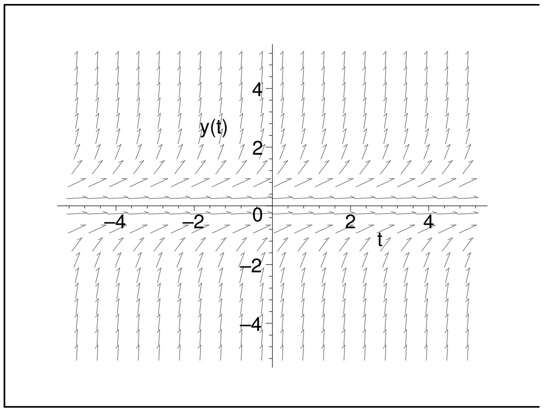
$y = 0$ and $y = 4$ are equilibrium solutions; $y \to 4$ if initial value is positive; y diverges from 0 if initial value is negative.

12.



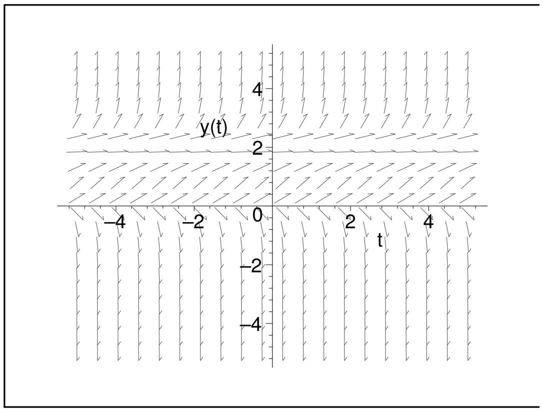
$y = 0$ and $y = 6$ are equilibrium solutions; y diverges from 6 if the initial value is greater than 6; $y \to 0$ if the initial value is less than 6.

13.



$y = 0$ is equilibrium solution; $y \to 0$ if initial value is negative; y diverges from 0 if initial value is positive.

14.



$y = 0$ and $y = 2$ are equilibrium solutions; y diverges from 0 if the initial value is negative; $y \to 2$ if the initial value is between 0 and 2; y diverges from 2 if the initial value is greater than 2.

15. (j)

16. (c)

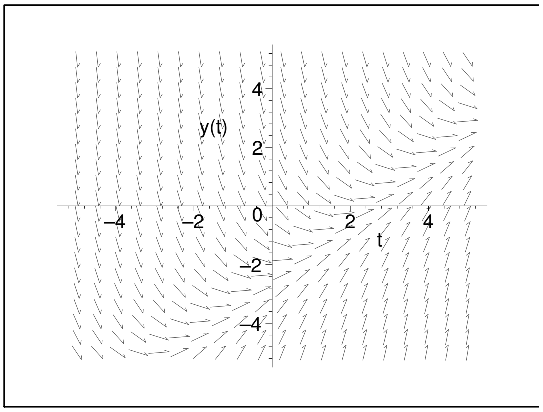
17. (g)

18. (b)

19. (h)

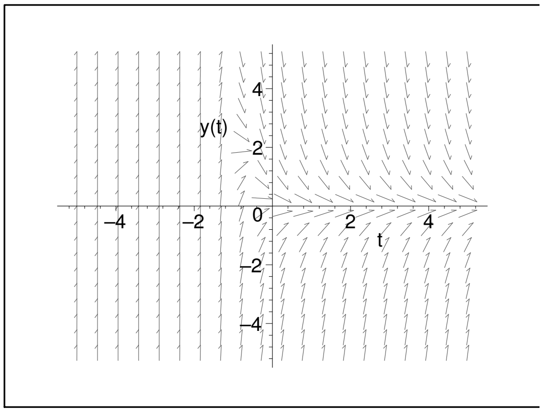
20. (e)

21.



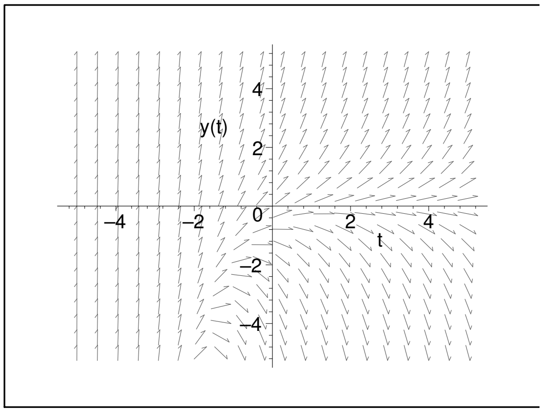
y is asymptotic to $t - 3$ as $t \to \infty$

22.



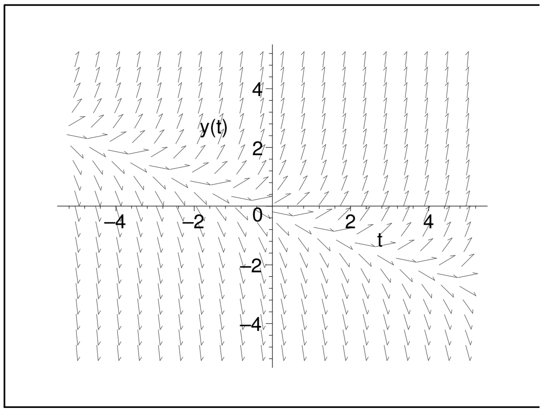
$y \to 0$ as $t \to \infty$.

23.



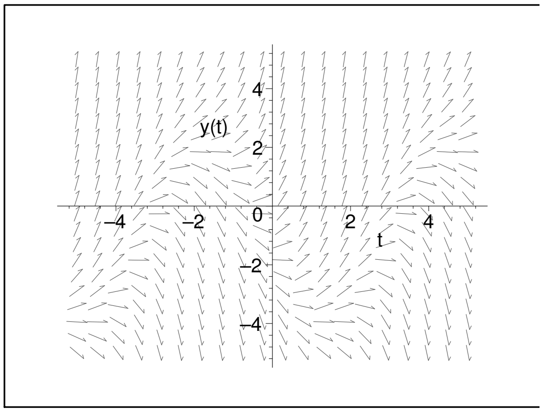
$y \to \infty, 0$, or $-\infty$ depending on the initial value of y

24.



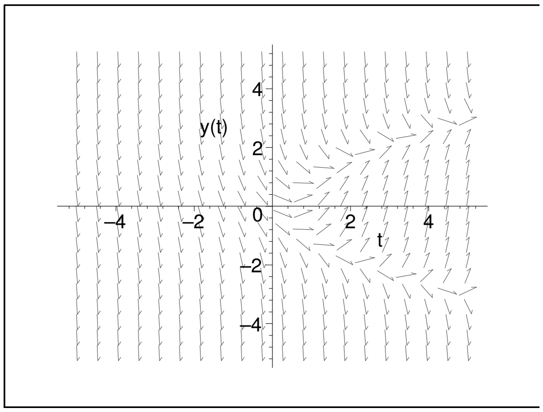
$y \to \infty$ or $-\infty$ depending whether the initial value lies above or below the line $y = -t/2$.

25.



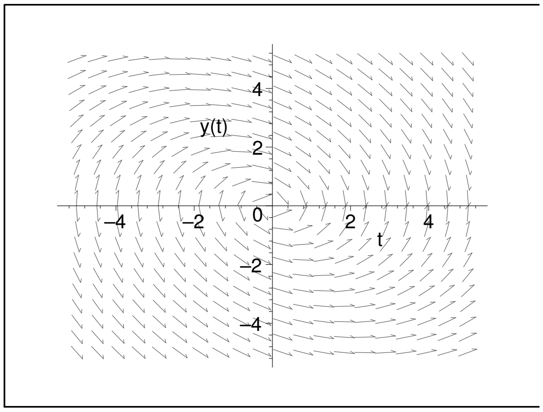
$y \to \infty$ or $-\infty$ or y oscillates depending whether the initial value of y lies above or below the sinusoidal curve.

26.



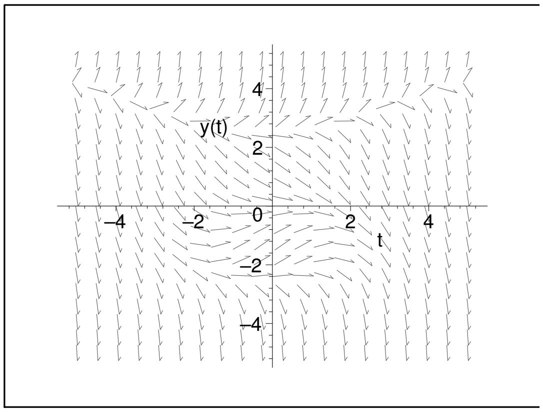
$y \to -\infty$ or is asymptotic to $\sqrt{2t-1}$ depending on the initial value of y.

27.



$y \to 0$ and then fails to exist after some $t_f \geq 0$

28.



$y \to \infty$ or $-\infty$ depending on the initial value of y.

29.

(a) Using the differential equation and the given approximation, we obtain that

$$\frac{u(t_j) - u(t_{j-1})}{\Delta t} = -k(u(t_{j-1}) - T_0).$$

Multiplication by Δt yields $u(t_j) - u(t_{j-1}) = -k\Delta t(u(t_{j-1}) - T_0)$, which gives us $u(t_j) = (1 - k\Delta t)u(t_{j-1}) + k\Delta t T_0$.

(b) We use induction. The statement is true for $n = 1$: $u(t_1) = (1 - k\Delta t)u_0 + kT_0\Delta t$. Suppose the statement is true for n, i.e. that $u(t_n) = (1 - k\Delta t)^n u_0 + kT_0\Delta t \sum_{j=0}^{n-1}(1 - k\Delta t)^j$. This implies that for $n + 1$ we get

$$u(t_{n+1}) = (1 - k\Delta t)u(t_n) + k\Delta t T_0 = (1 - k\Delta t)[(1 - k\Delta t)^n u_0 + kT_0\Delta t \sum_{j=0}^{n-1}(1 - k\Delta t)^j] + k\Delta t T_0 =$$

$$= (1 - k\Delta t)^{n+1}u_0 + kT_0\Delta t \sum_{j=0}^{n}(1 - k\Delta t)^j,$$

which is exactly what we wanted to show. We know that $\sum_{j=0}^{n-1} r^j = 1 + r + \ldots + r^{n-1} = (r^n - 1)/(r - 1) = (1 - r^n)/(1 - r)$; let $r = 1 - k\Delta t$, then $1 - r = k\Delta t$ and we obtain that $u(t_n) = (1 - k\Delta t)^n u_0 + kT_0\Delta t \sum_{j=0}^{n-1}(1 - k\Delta t)^j = (1 - k\Delta t)^n u_0 + T_0(1 - (1 - k\Delta t)^n)$.

(c) $\ln(1 - kt/n)^n = n\ln(1 - kt/n) = \ln(1 - kt/n)/(1/n)$, so using L'Hospital's rule we obtain that the limit of this sequence is the same as the limit of $(1/(1 - kt/n)) \cdot (kt/n^2)/(-1/n^2)$, which is clearly $-kt$ as $n \to \infty$, so the sequence $(1 - kt/n)^n$ converges to e^{-kt} as $n \to \infty$. Let $\Delta t = t/n$ and we obtain immediately that $u(t_n) = (1 - kt/n)^n u_0 + T_0(1 - (1 - kt/n)^n) \to e^{-kt}u_0 + T_0(1 - e^{-kt}) = e^{-kt}(u_0 - T_0) + T_0$ as $n \to \infty$.

30. With

$$\phi(t) = T_0 + \frac{kA}{k^2 + \omega^2}[k\sin(\omega t) + \omega\cos(\omega t)] + ce^{-kt},$$

it is straightforward to see that

$$\phi'(t) + k\phi(t) = kT_0 + kA\sin(\omega t).$$

31. Using the fact that

$$R\sin(\omega t - \delta) = R\cos\delta\sin(\omega t) - R\sin\delta\cos(\omega t)$$

where $R^2\cos^2\delta + R^2\sin^2\delta = R^2 = A^2 + B^2$, the desired result follows.

31. Let $R = \sqrt{A^2 + B^2}$. Using the fact that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$, we obtain that $R\sin(\omega t - \delta) = R\cos\delta\sin\omega t - R\sin\delta\cos\omega t = A\sin\omega t + B\cos\omega t$. The δ value for which $R\cos\delta = A$ and $R\sin\delta = -B$ exists because $R^2 = A^2 + B^2$.

32.

(a) The general solution is $p(t) = 900 + ce^{t/2}$. Plugging in for the initial condition, we have $p(t) = 900 + (p_0 - 900)e^{t/2}$. With $p_0 = 850$, the solution is $p(t) = 900 - 50e^{t/2}$. To find the time when the population becomes extinct, we need to find the time T when $p(T) = 0$. Therefore, $900 = 50e^{T/2}$, which implies $e^{T/2} = 18$, and, therefore, $T = 2\ln 18 \cong 5.78$ months.

(b) Using the general solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, we see that the population will become extinct at the time T when $900 = (900 - p_0)e^{T/2}$. That is, $T = 2\ln[900/(900 - p_0)]$ months.

(c) Using the general solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, we see that the population after 1 year (12 months) will be $p(6) = 900 + (p_0 - 900)e^6$. If we want to know the initial population which will lead to extinction after 1 year, we set $p(6) = 0$ and solve for p_0. Doing so, we have $(900 - p_0)e^6 = 900$ which implies $p_0 = 900(1 - e^{-6}) \cong 897.8$.

33.

(a) The solution of the differential equation $p' = rp$, when $p(0) = p_0$ is $p(t) = p_0e^{rt}$. If the population doubles in 20 days, then $p(20) = p_0e^{20r} = 2p_0$, so $r = \ln 2/20$ (day^{-1}).

(b) The same computation shows that $r = \ln 2/N$ (day^{-1}).

34.

(a) The general solution of the equation is $Q(t) = ce^{-rt}$. Given that $Q(0) = 100$, we have $c = 100$. Assuming that $Q(1) = 82.04$, we have $82.04 = 100e^{-r}$. Solving this equation for r, we have $r = -\ln(82.04/100) = .19796$ per week or $r = 0.02828$ per day.

(b) Using the form of the general solution and r found above, we have $Q(t) = 100e^{-0.02828t}$.

(c) Let T be the time it takes the isotope to decay to half of its original amount. From part (b), we conclude that $.5 = e^{-0.2828T}$ which implies that $T = -\ln(0.5)/0.2828 \cong 24.5$ days.

35.

(a) The direction field is the same as in Problem 1, except the equilibrium solution (where the arrows are horizontal) is at $-mg/\gamma$. We obtain this value by setting $mv' = 0$: $-mg - \gamma v = 0$, so $v = -mg/\gamma$. The direction field shows that the velocity of a falling object does not grow without bound, it approaches this equilibrium velocity. We can also see that the smaller the drag coefficient $\gamma > 0$ is, the higher the terminal velocity the object reaches.

(b) First, $mv' = m(v_0 + mg/\gamma)(-\gamma/m)e^{-\gamma t/m} = -\gamma(v_0 + mg/\gamma)e^{-\gamma t/m}$. Also, $-mg - \gamma v = -mg - \gamma((v_0 + mg\gamma)e^{-\gamma t/m} - mg/\gamma) = -\gamma(v_0 + mg/\gamma)e^{-\gamma t/m}$. So the function satisfies the given differential equation. We can also see that $v(0) = (v_0 + mg/\gamma) - mg/\gamma = v_0$.

(c) The ball reaches its maximum height when $v = 0$. This will happen when $(v_0 + mg/\gamma)e^{-\gamma t/m} = mg/\gamma$. Dividing both sides by $e^{-\gamma t/m}mg/\gamma$, we obtain $v_0\gamma/(mg) + 1 = e^{\gamma t/m}$. Taking the logarithm of both sides and dividing by γ/m we get that $t = t_{\max} = (m/\gamma)\ln(1 + \gamma v_0/(mg))$.

(d) Using the previous parts, $\gamma = -mg/v_{\text{term}} = -0.145{\cdot}9.8/(-33)(\text{kg/sec}) \approx 0.0431(\text{kg/sec})$.

(e) Using the expression for the velocity, we can get the function describing the height of the thrown ball. Because $v = h'$, we get that $h(t) = (-m/\gamma)(v_0 + mg/\gamma)e^{-\gamma t/m} - mgt/\gamma + h_0 + (m/\gamma)(v_0 + mg/\gamma)$, where the constant was chosen to satisfy the initial condition $h(0) = h_0$. Using part (c), the time needed to reach maximum height is $(m/\gamma)\ln(1 + \gamma v_0/(mg))$, by plugging this into the height function we obtain that $h_{\max} \approx 31.16$ (m).

36.

(a) Following the discussion in the text, the equation is given by $mv' = mg - kv^2$.

(b) After a long time, $v' \to 0$. Therefore, $mg - kv^2 \to 0$, or $v \to \sqrt{mg/k}$.

(c) We need to solve the equation $\sqrt{.005 \cdot 9.8/k} = 35$. Solving this equation, we see that $k = 0.0004$ kg/m.

37.

(a) Let $q(t)$ denote the amount of chemical in the pond at time t. The chemical q will be measured in grams and the time t will be measured in hours. The rate at which the chemical is entering the pond is given by 300 gallons/hour $\cdot .01$ grams/gallons $= 300 \cdot 10^{-2}$ grams/hour. The rate at which the chemical leaves the pond is given by 300 gallons/hour $\cdot q/1{,}000{,}000$ grams/gallons $= 300 \cdot q10^{-6}$ grams/hour. Therefore, the differential equation is given by $dq/dt = 300(10^{-2} - q10^{-6})$.

(b) As $t \to \infty$, $10^{-2} - q10^{-6} \to 0$. Therefore, $q \to 10^4$ grams. The limiting amount does not depend on the amount that was present initially.

38. The surface area of a spherical raindrop of radius r is given by $S = 4\pi r^2$. The volume of a spherical raindrop is given by $V = 4\pi r^3/3$. Therefore, we see that the surface area $S = cV^{2/3}$ for some constant c. If the raindrop evaporates at a rate proportional to its surface area, then $dV/dt = -kV^{2/3}$ for some $k > 0$.

39.

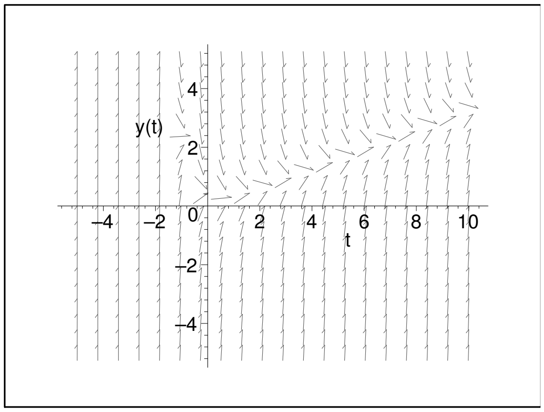
(a) Let $q(t)$ be the total amount of the drug (in milligrams) in the body at a given time t (measured in hours). The drug enters the body at the rate of 5 mg/cm^3 $\cdot$100 cm^3/hr = 500 mg/hr, and the drug leaves the body at the rate of $0.4q$ mg/hr. Therefore, the governing differential equation is given by $dq/dt = 500 - 0.4q$.

(b) If $q > 1250$, then $q' < 0$. If $q < 1250$, then $q' > 0$. Therefore, $q \to 1250$.

1.2 Linear Equations: Method of Integrating Factors

1.

(a)



(b) All solutions seem to converge to an increasing function as $t \to \infty$.

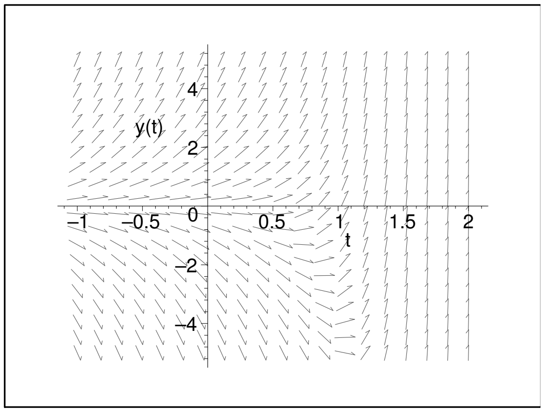
(c) The integrating factor is $\mu(t) = e^{3t}$. Then

$$\begin{aligned} e^{3t}y' + 3e^{3t}y = e^{3t}(t + e^{-2t}) &\implies (e^{3t}y)' = te^{3t} + e^t \\ \implies e^{3t}y = \int (te^{3t} + e^t)\,dt &= \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^t + c \\ \implies y = ce^{-3t} + e^{-2t} + \frac{t}{3} - \frac{1}{9}. \end{aligned}$$

We conclude that y is asymptotic to $t/3 - 1/9$ as $t \to \infty$.

2.

(a)



(b) All slopes eventually become positive, so all solutions will eventually increase without bound.

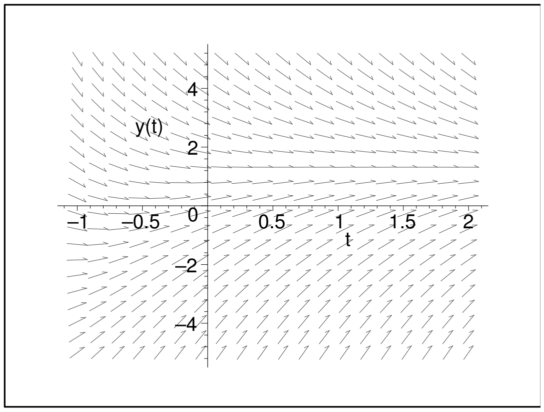
(c) The integrating factor is $\mu(t) = e^{-2t}$. Then

$$\begin{aligned} & e^{-2t}y' - 2e^{-2t}y = e^{-2t}(t^2e^{2t}) \implies (e^{-2t}y)' = t^2 \\ & \implies e^{-2t}y = \int t^2\,dt = \frac{t^3}{3} + c \\ & \implies y = \frac{t^3}{3}e^{2t} + ce^{2t}. \end{aligned}$$

We conclude that y increases exponentially as $t \to \infty$.

3.

(a)



(b) All solutions appear to converge to the function $y(t) = 1$.

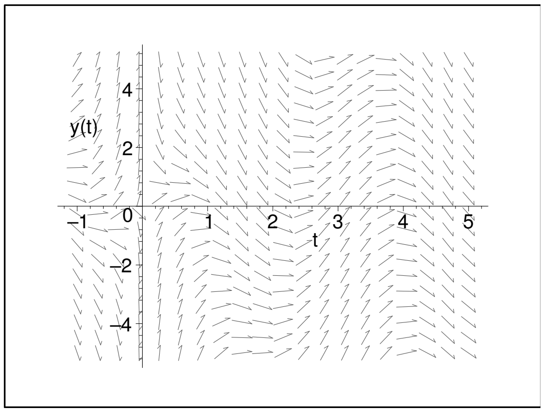
(c) The integrating factor is $\mu(t) = e^t$. Therefore,

$$\begin{aligned} e^t y' + e^t y = t + e^t &\implies (e^t y)' = t + e^t \\ &\implies e^t y = \int (t + e^t)\,dt = \frac{t^2}{2} + e^t + c \\ &\implies y = \frac{t^2}{2} e^{-t} + 1 + ce^{-t}. \end{aligned}$$

Therefore, we conclude that $y \to 1$ as $t \to \infty$.

4.

(a)



(b) The solutions eventually become oscillatory.

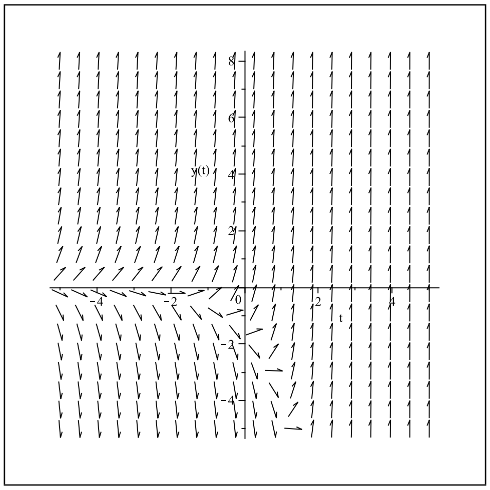
(c) The integrating factor is $\mu(t) = t$. Therefore,

$$\begin{aligned} ty' + y = 3t\cos(2t) &\implies (ty)' = 3t\cos(2t) \\ &\implies ty = \int 3t\cos(2t)\,dt = \frac{3}{4}\cos(2t) + \frac{3}{2}t\sin(2t) + c \\ &\implies y = \frac{3\cos 2t}{4t} + \frac{3\sin 2t}{2} + \frac{c}{t}. \end{aligned}$$

We conclude that y is asymptotic to $(3\sin 2t)/2$ as $t \to \infty$.

5.

(a)



(b) All slopes eventually become positive so all solutions eventually increase without bound.

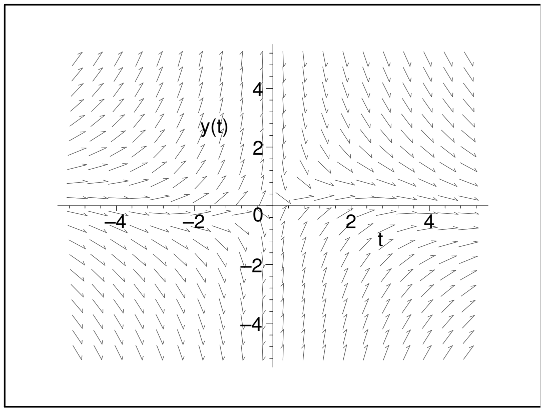
(c) The integrating factor is $\mu(t) = e^{-3t}$. Therefore,

$$e^{-3t}y' - 3e^{-3t}y = 4e^{-2t} \implies (e^{-3t}y)' = 4e^{-2t}$$
$$\implies e^{-3t}y = \int 4e^{-2t}\,dt = -2e^{-2t} + c$$
$$\implies y = -2e^{t} + ce^{3t}.$$

We conclude that y increases or decreases exponentially as $t \to \infty$.

6.

(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $y = 0$.

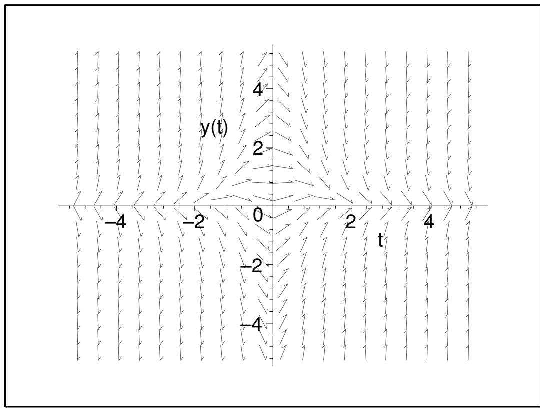
(c) The integrating factor is $\mu(t) = t^2$. Therefore,

$$t^2y' + 2ty = t\sin(t) \implies (t^2y)' = t\sin(t)$$
$$\implies t^2y = \int t\sin(t)\,dt = \sin(t) - t\cos(t) + c$$
$$\implies y = \frac{\sin t - t\cos t + c}{t^2}.$$

We conclude that $y \to 0$ as $t \to \infty$.

7.

(a)

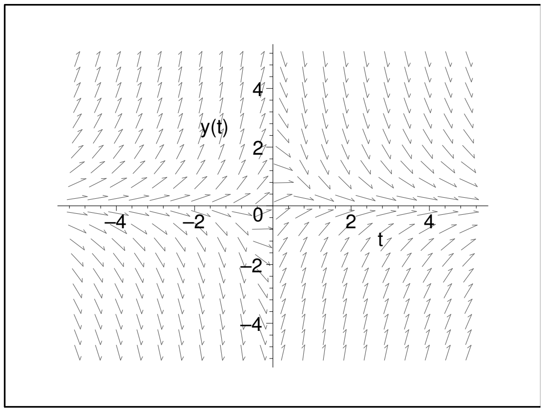


(b) For $t > 0$, all solutions seem to eventually converge to the function $y = 0$.

(c) The integrating factor is $\mu(t) = e^{t^2}$. Therefore, using the techniques shown above, we see that $y(t) = t^2 e^{-t^2} + ce^{-t^2}$. We conclude that $y \to 0$ as $t \to \infty$.

8.

(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $y = 0$.

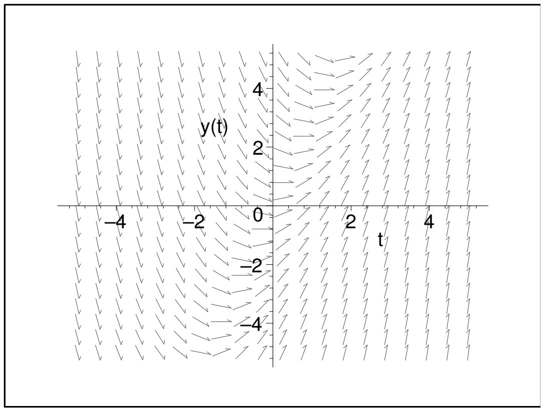
(c) The integrating factor is $\mu(t) = (1 + t^2)^2$. Then

$$\begin{aligned} (1+t^2)^2 y' + 4t(1+t^2)y &= \frac{1}{1+t^2} \\ \implies ((1+t^2)^2 y) &= \int \frac{1}{1+t^2}\,dt \\ \implies y &= (\arctan(t) + c)/(1+t^2)^2. \end{aligned}$$

We conclude that $y \to 0$ as $t \to \infty$.

9.

(a)



(b) All slopes eventually become positive. Therefore, all solutions will increase without bound.

(c) The integrating factor is $\mu(t) = e^{t/2}$. Therefore,

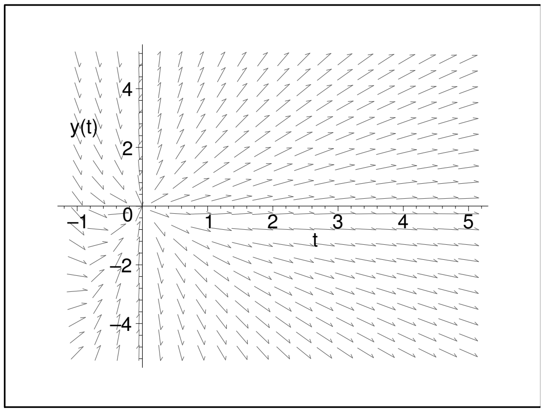
$$2e^{t/2}y' + e^{t/2}y = 3te^{t/2} \quad \Longrightarrow \quad 2e^{t/2}y = \int 3te^{t/2}\,dt = 6te^{t/2} - 12e^{t/2} + c$$

$$\Longrightarrow \quad y = 3t - 6 + ce^{-t/2}.$$

We conclude that $y \to 3t - 6$ as $t \to \infty$.

10.

(a)



(b) For $y > 0$, the slopes are all positive, and, therefore, the corresponding solutions increase without bound. For $y < 0$ almost all solutions have negative slope and therefore decrease without bound.

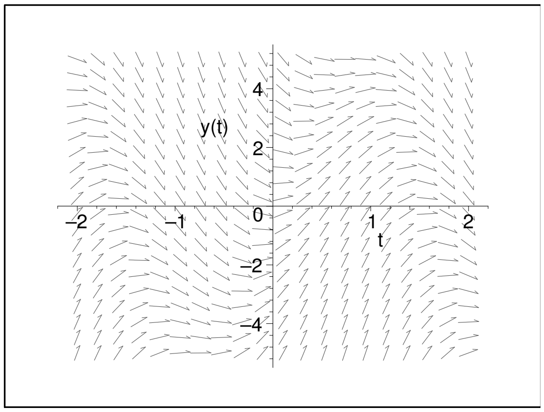
(c) By dividing the equation by t, we see that the integrating factor is $\mu(t) = 1/t$. Therefore,

$$\begin{aligned} y'/t - y/t^2 = te^{-t} &\implies (y/t)' = te^{-t} \\ \implies \frac{y}{t} &= \int te^{-t}\,dt = -te^{-t} - e^{-t} + c \\ \implies y &= -t^2e^{-t} - te^{-t} + ct. \end{aligned}$$

We conclude that $y \to \infty$ if $c > 0$, $y \to -\infty$ if $c < 0$ and $y \to 0$ if $c = 0$.

11.

(a)



(b) The solution appears to be oscillatory.

(c) The integrating factor is $\mu(t) = e^t$. Therefore,

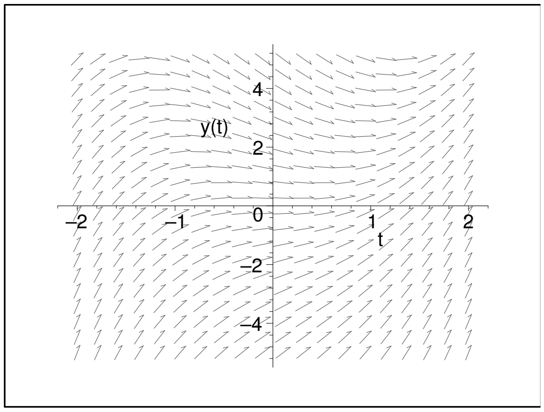
$$e^ty' + e^ty = 5e^t\sin(2t) \implies (e^ty)' = 5e^t\sin(2t)$$

$$\implies e^ty = \int 5e^t\sin(2t)\,dt = -2e^t\cos(2t) + e^t\sin(2t) + c \quad \implies y = -2\cos(2t) + \sin(2t) + ce^{-t}.$$

We conclude that $y \to \sin(2t) - 2\cos(2t)$ as $t \to \infty$.

12.

(a)



(b) All slopes are eventually positive. Therefore, all solutions increase without bound.

(c) The integrating factor is $\mu(t) = e^{t/2}$. Therefore,

$$\begin{aligned} 2e^{t/2}y' + e^{t/2}y = 3t^2e^{t/2} &\implies (2e^{t/2}y)' = 3t^2e^{t/2} \\ &\implies 2e^{t/2}y = \int 3t^2e^{t/2}\,dt = 6t^2e^{t/2} - 24te^{t/2} + 48e^{t/2} + c \\ &\implies y = 3t^2 - 12t + 24 + ce^{-t/2}. \end{aligned}$$

We conclude that y is asymptotic to $3t^2 - 12t + 24$ as $t \to \infty$.

13. The integrating factor is $\mu(t) = e^{-t}$. Therefore,

$$(e^{-t}y)' = 2te^t \implies y = e^t \int 2te^t\,dt = 2te^{2t} - 2e^{2t} + ce^t.$$

The initial condition $y(0) = 1$ implies $-2 + c = 1$. Therefore, $c = 3$ and $y = 3e^t + 2(t-1)e^{2t}$

14. The integrating factor is $\mu(t) = e^{3t}$. Therefore,

$$(e^{3t}y)' = t \implies y = e^{-3t} \int t\,dt = \frac{t^2}{2}e^{-3t} + ce^{-3t}.$$

The initial condition $y(1) = 0$ implies $e^{-3t}/2 + ce^{-3t} = 0$. Therefore, $c = -1/2$, and $y = (t^2 - 1)e^{-3t}/2$.

15. Dividing the equation by t, we see that the integrating factor is $\mu(t) = t^2$. Therefore,

$$(t^2y)' = t^3 - t^2 + t \implies y = t^{-2} \int (t^3 - t^2 + t)\,dt = \left(\frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{c}{t^2}\right).$$

The initial condition $y(1) = 1/2$ implies $c = 1/12$, and $y = (3t^4 - 4t^3 + 6t^2 + 1)/12t^2$.

16. The integrating factor is $\mu(t) = t^2$. Therefore,

$$(t^2y)' = \cos(t) \implies y = t^{-2} \int \cos(t)\,dt = t^{-2}(\sin(t) + c).$$

The initial condition $y(\pi) = 0$ implies $c = 0$ and $y = (\sin t)/t^2$.

17. The integrating factor is $\mu(t) = e^{-4t}$. Therefore,

$$(e^{-4t}y)' = 1 \implies y = e^{4t} \int 1\,dt = e^{4t}(t + c).$$

The initial condition $y(0) = 2$ implies $c = 2$ and $y = (t + 2)e^{4t}$.

18. After dividing by t, we see that the integrating factor is $\mu(t) = t^2$. Therefore,

$$(t^2y)' = 1 \implies y = t^{-2} \int t\sin(t)\,dt = t^{-2}(\sin(t) - t\cos(t) + c).$$

The initial condition $y(\pi/2) = 1$ implies $c = (\pi^2/4) - 1$ and $y = t^{-2}[(\pi^2/4) - 1 - t\cos t + \sin t]$.

19. After dividing by t^3, we see that the integrating factor is $\mu(t) = t^4$. Therefore,

$$(t^4 y)' = te^{-t} \implies y = t^{-4}\int te^{-t}\,dt = t^{-4}(-te^{-t} - e^{-t} + c).$$

The initial condition $y(-1) = 0$ implies $c = 0$ and $y = -(1+t)e^{-t}/t^4, \quad t \neq 0$

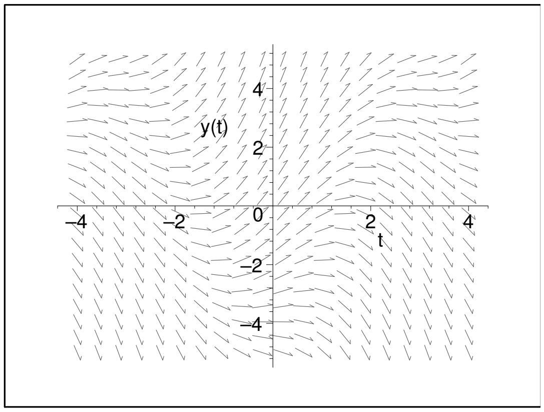
20. After dividing by t, we see that the integrating factor is $\mu(t) = te^t$. Therefore,

$$(te^t y)' = te^t \implies y = t^{-1}e^{-t}\int te^t\,dt = t^{-1}e^{-t}(te^t - e^t + c) = t^{-1}(t - 1 + ce^{-t}).$$

The initial condition $y(\ln 2) = 1$ implies $c = 2$ and $y = (t - 1 + 2e^{-t})/t, \quad t \neq 0.$

21.

(a)



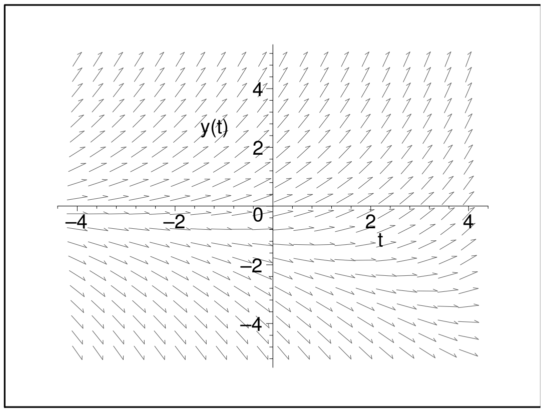
The solutions appear to diverge from an oscillatory solution. It appears that $a_0 \approx -1$. For $a > -1$, the solutions increase without bound. For $a < -1$, the solutions decrease without bound.

(b) The integrating factor is $\mu(t) = e^{-t/2}$. From this, we conclude that the general solution is $y(t) = (8\sin(t) - 4\cos(t))/5 + ce^{t/2}$, where $c = a + 4/5$. The solution will be sinusoidal as long as $c = 0$. The initial condition for the sinusoidal behavior is $y(0) = (8\sin(0) - 4\cos(0))/5 = -4/5$. Therefore, $a_0 = -4/5$.

(c) y oscillates for $a = a_0$

22.

(a)



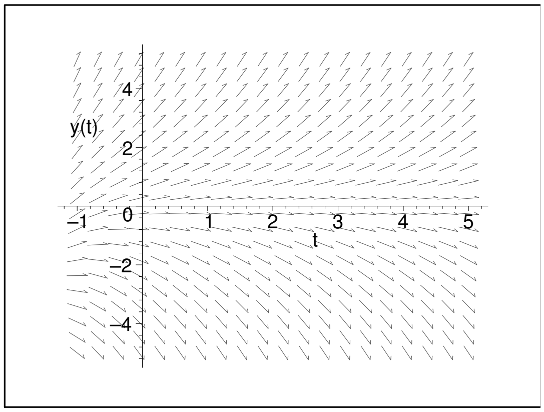
All solutions eventually increase or decrease without bound. The value a_0 appears to be approximately $a_0 = -3$.

(b) The integrating factor is $\mu(t) = e^{-t/2}$, and the general solution is $y(t) = -3e^{t/3} + ce^{t/2}$. The initial condition $y(0) = a$ implies $y = -3e^{t/3} + (a+3)e^{t/2}$. The solution will behave like $(a+3)e^{t/2}$. Therefore, $a_0 = -3$.

(c) $y \to -\infty$ for $a = a_0$.

23.

(a)



Solutions eventually increase or decrease without bound, depending on the initial value a_0. It appears that $a_0 \approx -1/8$.

(b) Dividing the equation by 3, we see that the integrating factor is $\mu(t) = e^{-2t/3}$. Therefore, the solution is $y = [(2 + a(3\pi + 4))e^{2t/3} - 2e^{-\pi t/2}]/(3\pi + 4)$. The solution will eventually behave like $(2 + a(3\pi + 4))e^{2t/3}/(3\pi + 4)$. Therefore, $a_0 = -2/(3\pi + 4)$.

(c) $y \to 0$ for $a = a_0$

24.