2-4.1

$$J = \begin{vmatrix} 1-k & k & 0 \\ k & 1-k & 0 \\ kx_3 & 0 & 1+kx_1 \end{vmatrix} = (1-2k)(1+kx_1) > 0$$

For admissible deformation, we take

$1 > 2k \quad \therefore \quad k < \frac{1}{2} \quad$ and $\quad kx_1 > -1$

2-4.2

$$J = \begin{vmatrix} 1 & ax_3 & ax_2 \\ bx_3 & 1 & bx_1 \\ cx_2 & cx_1 & 1 \end{vmatrix} > 0$$

$\therefore \quad 1 > bcx_1^2 + acx_2^2 + abx_3^2 - 2abcx_1x_2x_3$

For finite x_1, x_2, x_3 this is possible for sufficienty small a, b, c.

2-4.3

$$J = \begin{vmatrix} -\frac{(c-y)}{c} \sin\frac{x}{c} & -\cos\frac{x}{c} & 0 \\ \frac{c-y}{c} \cos\frac{x}{c} & -\sin\frac{x}{c} & 0 \\ 0 & 0 & 1 \end{vmatrix} > 0$$

$c \neq 0$

$\therefore \quad 1 - \frac{y}{c} > 0. \quad$ For $c > 0$, $\quad c > y.$

$\therefore \quad c > h. \quad$ For $c < 0$, $\quad c < -h.$

2-4.4

$$J = \begin{vmatrix} 1-k_1x_2 & -k_1x_1 & 0 \\ 2\nu k_2 x_1 & 1+2k_2\nu x_2 & -2\nu k_2 x_3 \\ 0 & k_3\nu x_3 & 1+k_3\nu x_2 \end{vmatrix} > 0$$

Or

$$J = 1 + x_2(\quad) + x_1^2(\quad) + x_1^2(\quad) + x_1^2 x_2(\quad) + x_2^3(\quad) > 0$$

The terms ( ) all contain combinations of k_1, k_2, k_3 and $\nu\,(<\frac{1}{2})$. Hence, if k_1, k_2, k_3 are sufficiently small, $J>0$ for finite x_1, x_2.

2-4.5

$$J = \begin{vmatrix} 1 & -\theta x_3 & -\theta x_2 \\ \theta x_3 & 1 & \theta x_1 \\ \theta x_2\left(\frac{b^2-a^2}{b^2+a^2}\right) & \theta x_1\left(\frac{b^2-a^2}{b^2+a^2}\right) & 1 \end{vmatrix} > 0$$

Or

$$J = 1 - \left(\frac{b^2-a^2}{b^2+a^2}\right)\left[2\theta^3 x_1 x_2 x_3 + \theta^2 x_1^2 - \theta^2 x_2^2\right] + \theta^2 x_3^2 > 0$$

$$\therefore\ (a^2+b^2)(1+\theta^2 x_3^2) > (a^2-b^2)\left[\theta^2(x_2^2-x_1^2) - 2\theta^3 x_1 x_2 x_3\right]$$

since $a^2-b^2>0$, $J>0$ provided

$$\frac{(a^2+b^2)(1+\theta^2 x_3^2)}{a^2-b^2} > \theta^2\left[x_2^2 - x_1^2 - 2\theta x_1 x_2 x_3\right]$$

This condition is satisfied for the usual torsion problem $\theta << 1$ and x_1, x_2, x_3 of typical range.

2-4.6

$$
\begin{aligned}
I_2(\bar{C}) &= \bar{C}_{ij}\bar{C}_{ji} \\
&= a_{i\alpha}a_{j\beta}a_{j\gamma}a_{i\kappa}C_{\alpha\beta}C_{\gamma\kappa} \\
&= a_{\kappa\alpha}a_{\gamma\beta}C_{\alpha\beta}C_{\gamma\kappa} \\
&= C_{\alpha\beta}C_{\alpha\beta} = I_2(C)
\end{aligned}
$$

$$
\begin{aligned}
I_3(\bar{C}) &= \bar{C}_{ij}\bar{C}_{jk}\bar{C}_{ki} \\
&= a_{i\alpha}a_{j\beta}a_{j\gamma}a_{k\kappa}a_{k\chi}a_{i\eta}C_{\alpha\beta}C_{\gamma\kappa}C_{\chi\eta} \\
&= a_{\alpha\eta}a_{\gamma\beta}a_{\chi\kappa}C_{\alpha\beta}C_{\gamma\kappa}C_{\chi\eta} \\
&= C_{\alpha\beta}C_{\beta\chi}C_{\chi\alpha} = I_3(C)
\end{aligned}
$$

2-4.7

By expanding the matrix and finding the determinant, it is straightforward to prove that

$$
I_A = A_{11} + A_{22} + A_{33} = A_{kk}
$$

$$
\begin{aligned}
II_A &= \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \\
&= \frac{1}{2}\left\{(A_{kk})^2 - (A_{ij}A_{ji})\right\}
\end{aligned}
$$

$$
\begin{aligned}
III_A &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\
&= \frac{1}{6}\left\{2(A_{ij}A_{jk}A_{ki}) - 3(A_{kk}A_{ij}A_{ji}) + (A_{kk})^3\right\}
\end{aligned}
$$

2-6.1

By Eq. 2-6.14, given data yields

$$0.004 = 0.002\left(\tfrac{4}{5}\right) + 0.002(0) + (-.002)\left(\tfrac{1}{5}\right) + 2\epsilon_{12}(0) + 2\epsilon_{13}\left(\tfrac{2}{5}\right) + 2\epsilon_{23}(0)$$

$$\therefore \boxed{\epsilon_{13} = 0.0035}$$

$$0.003 = 0.002\left(\tfrac{9}{10}\right) + 0.002\left(\tfrac{1}{10}\right) + 2\epsilon_{12}\left(-\tfrac{3}{10}\right)$$

$$\therefore \boxed{\epsilon_{12} = -0.00166\overline{6}}$$

$$0.001 = .002\left(\tfrac{1}{3}\right) + .002\left(\tfrac{1}{3}\right) + (-.002)\left(\tfrac{1}{3}\right) + 2\epsilon_{12}\left(\tfrac{1}{3}\right) + 2\epsilon_{13}\left(\tfrac{1}{3}\right) + 2\epsilon_{23}\left(\tfrac{1}{3}\right)$$

$$\therefore \boxed{\epsilon_{23} = -0.00133\overline{3}}$$

2-6.2

$$L^2 = x^2 + y^2 \; ; \; L^{*2} = (x + u_1 - u_0)^2 + (y + v_1 - v_0)^2$$

Then

$$\frac{L^* - L}{L} = \frac{\sqrt{(x + u_1 - u_0)^2 + (y + v_1 - v_0)^2} - L}{L}$$

Let $y = 0$, $x = L$. Then

$$\frac{L^* - L}{L} = \frac{\sqrt{(L + u_1 - u_0)^2 + (v_1 - v_0)^2} - L}{L}$$

$$= \sqrt{1 + \frac{2(u_1 - u_0)}{L} + \left(\frac{u_1 - u_0}{L}\right)^2 + \left(\frac{v_1 - v_0}{L}\right)^2} - 1$$

If $u_1 - u_0 << L$, $v_1 - v_0 << L$, binomial expansion yields to first degree terms in u, v

$$\frac{L^* - L}{L} \approx \frac{u_1 - u_0}{L}$$

2-6.3

By Eq. 2-6.7, and given data

$$d\xi_1 = (1+6kx_1)dx_1 + k\,dx_2$$
$$d\xi_2 = (1+4kx_2)dx_2 + k\,dx_3$$
$$d\xi_3 = k\,dx_1 + (1+8kx_3)dx_3$$
$$(ds)^2 = d\xi_1^2 + d\xi_2^2 + d\xi_3^2$$
$$= [(1+6kx_1)dx_1 + k\,dx_2]^2 + [(1+4kx_2)dx_2 + k\,dx_3]^2 + [k\,dx_1 + (1+8kx_3)dx_3]^2$$
$$MF = \tfrac{1}{2}\left[\left(\frac{ds}{dS}\right)^2 - 1\right] = \tfrac{1}{2}\Big\{[(1+6kx_1)n_1 + kn_2]^2 + [(1+4kx_2)n_2 + kn_3]^2 + [(1+8kx_3)n_3 + kn_1]^2 - 1\Big\}$$

Or

$$MF = \tfrac{1}{6}[42k + 155k^2] = 7k + \tfrac{155}{6}k^2$$

2-8.1

a) $$J = \det\left(\delta_{\alpha\beta} + \frac{\partial u_\alpha}{\partial x_\beta}\right) = \begin{vmatrix} 1 & a & -b \\ -a & 1 & c \\ b & -c & 1 \end{vmatrix} = 1 + a^2 + b^2 + c^2 > 0$$

b) By Eq. (2-8.1), $\eta_\alpha\sqrt{1+2MF_i} = (\delta_{\alpha\beta} + u_{\alpha,\beta})n_\beta$ and

Discarding quadratic terms in $\epsilon_{\alpha\beta}$, $MF_i = 0$

$$\therefore \left.\begin{aligned} n_1 &= n_1 + an_2 - bn_3 \\ n_2 &= -an_1 + n_2 + cn_3 \\ n_3 &= bn_1 - cn_2 + n_3 \end{aligned}\right\} \text{ Also, } n_1^2 + n_2^2 + n_3^2 = 1$$

2-8.1 continued pg. 5

2-8.1 cont.

Hence,

$$\left.\begin{aligned} an_2 &= bn_3 \\ an_1 &= cn_3 \\ bn_1 &= cn_2 \end{aligned}\right\} \quad \begin{aligned} n_2 &= \frac{b}{c} n_1 \\ n_3 &= \frac{a}{c} n_1 \end{aligned}$$

$$\therefore \quad n_1^2 + \frac{b^2}{c^2} n_1^2 + \frac{a^2}{c^2} n_1^2 = 1$$

<u>Or</u> $n_1^2 = \dfrac{c^2}{a^2+b^2+c^2}$, $n_2^2 = \dfrac{b^2}{a^2+b^2+c^2}$, $n_3^2 = \dfrac{a^2}{a^2+b^2+c^2}$

When $a=b=c$, $n_1^2 = n_2^2 = n_3^2 = \frac{1}{3}$

2-8.2

a) $$J = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 3(2+1) = 9 > 0$$

b) strain $\equiv MF = \epsilon_{\alpha\beta} N_\alpha N_\beta$

$(N_1, N_2, N_3) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\epsilon_{11} = 4$, $\epsilon_{22} = 1$, $\epsilon_{33} = 2$, $\epsilon_{12} = \frac{3}{2}$, $\epsilon_{23} = -\frac{1}{2}$; $\epsilon_{13} = 0$

$\therefore MF = (4+1+2+3-1)\frac{1}{3} = 3$

$\left(\frac{ds}{ds}\right)^2 = 1 + 2MF = 7$; $ds \approx 2.65\, ds$

Relative elongation $e_i = \dfrac{ds - ds}{ds} \cong 1.65$

c) $\eta_1 = \eta_2 = 0$, $\eta_3 = 1$ Hence, by Eq. (2-8.1)

$3N_1 + N_2 = 0 \Longrightarrow N_1 = -\frac{N_2}{3}$

$N_2 + N_3 = 0 \Longrightarrow N_3 = -N_2$

$-N_2 + 2N_3 = \sqrt{1+2MF}$ MF Not Known

From $N_1^2 + N_2^2 + N_3 = 1$; $N_2^2 = \frac{9}{19}$. Hence,

$N_1^2 = \frac{1}{19}$; $N_3^2 = \frac{9}{19}$. Then, $\sqrt{1+2MF} = -N_2 + 2N_3$

$= \mp\frac{3}{\sqrt{19}} \mp \frac{6}{\sqrt{19}} = \mp\frac{9}{\sqrt{19}}$. $1 + 2MF = \frac{81}{19}$

Or $MF \cong 1.63$

2-8.2 continued pg 6.

2-8.2 cont.

d) Initial angle Θ given by $\cos\Theta = m_\alpha n_\alpha$

$= (1)(\frac{1}{\sqrt{3}}) + (0)(\frac{1}{\sqrt{3}}) + (0)(\frac{1}{\sqrt{3}}) = \frac{1}{\sqrt{3}}$ $\quad \Theta = 54.74°$

By Eq. 2-8.3 ; final angle ϑ given

by $\sqrt{(1+2MF_1)(1+2MF_2)}\ \cos\vartheta = \cos\Theta + 2\epsilon_{\alpha\beta} m_\alpha n_\beta$

By Eq 2-6.13 ; $MF_1 = \epsilon_{11} n_1^2 = 4$ and $MF_2 = 3$

from part b . Therefore ;

$\sqrt{63}\ \cos\vartheta = \frac{1}{\sqrt{3}} + 2[4 + \frac{3}{2}]\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} + \frac{11}{\sqrt{3}} = 4\sqrt{3}$

$\cos\vartheta = \frac{4}{\sqrt{21}} = 0.87$; $\quad \vartheta = 29.21°$

$\therefore \vartheta - \Theta = 29.21 - 54.74° = -25.53°$

2-8.3

a) $J = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 3(8+1) = 27 > 0$

b) $\epsilon_{11} = 4$, $\epsilon_{22} = 8$, $\epsilon_{33} = 2$, $\epsilon_{23} = 1$, $\epsilon_{12} = \epsilon_{13} = 0$

$MF = \epsilon_{\alpha\beta} N_\alpha N_\beta = (4 + 8 + 2 + 2)\frac{1}{3} = \frac{16}{3}$

$\Gamma = \cos\Theta + 2\epsilon_{\alpha\beta} M_\alpha N_\beta$

$\cos\Theta = \frac{1}{\sqrt{3}}(-\frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{3}}(\frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{3}}(0) = 0$

$\Theta = \frac{\pi}{2} \quad \therefore \Gamma = 2\epsilon_{\alpha\beta} M_\alpha N_\beta$

$\Gamma = 2[(4)(\frac{1}{\sqrt{3}})(-\frac{1}{\sqrt{2}}) + (8)(\frac{1}{\sqrt{3}})(\frac{1}{\sqrt{2}}) + 2(\frac{1}{\sqrt{3}})(0) + (1)(\frac{1}{\sqrt{3}})(0) + (1)(\frac{1}{\sqrt{3}})(\frac{1}{\sqrt{2}})]$

$= \frac{10}{\sqrt{6}} = 4.0825$

c) $\eta_1 = \eta_3 = 0 \quad \eta_2 = 1$

$\eta_\gamma = \sqrt{1+2MF} = (\delta_{\alpha\beta} + u_{\alpha,\beta}) N_\beta \qquad 0 = 3N_1 \quad \therefore N_1 = 0$

$\sqrt{1+2MF} = 4N_2 + N_3$

$0 = -N_2 + 2N_3 \Longrightarrow N_2 = 2N_3$

$\therefore N_1^2 + N_2^2 + N_3^2 = 5N_3^2 = 1$; $N_3^2 = \frac{1}{5}$, $N_2^2 = \frac{4}{5}$

$\therefore 1 + 2MF = \frac{81}{5}$, $\quad MF = 7.6$

2-11.1

$\epsilon_{11} = 0.002 \quad , \epsilon_{12} = -0.00166 \quad ; \epsilon_{13} = 0.0035$

$\epsilon_{22} = 0.002 \quad ; \epsilon_{23} = -0.00133 \quad ; \epsilon_{33} = -0.002$

$$\begin{vmatrix} \frac{1}{500} - L & -\frac{1}{600} & \frac{7}{2000} \\ -\frac{1}{600} & \frac{1}{500} - L & -\frac{1}{750} \\ \frac{7}{2000} & -\frac{1}{750} & -\frac{1}{500} - L \end{vmatrix} = 0$$

$$216 \times 10^9 L^3 - 432 \times 10^6 L^2 - 4494 \times 10^3 L + 3228 = 0$$

$L_1 = 0.00536095$

$L_2 = 0.00068842$

$L_3 = -0.00404937$

2-11.2

$\ell = N_1 \; ; \quad m = N_2 \; ; \quad n = N_3$

By the theory of Art. 1-23, let

$H = MF_A - L(N_1^2 + N_2^2 + N_3^2 - 1)$

where L is Lagrange Multiplier and $MF_A = \epsilon_{\alpha\beta} N_\alpha N_\beta$

Hence,

$$\frac{\partial H}{\partial N_\alpha} = 2\epsilon_{\alpha\beta} N_\beta - 2LN_\alpha = 0 \qquad (a)$$

where

$\alpha = 1, 2, 3$ with $N_\alpha N_\alpha = 1$, Eq (a) yields after multiplication by N_α

$L = \epsilon_{\alpha\beta} N_\alpha N_\beta$ = Extreme value of MF_A

where, by Eq (a), necessary and sufficient conditions for nontrivial solution for N_α is

$$\det(\epsilon_{\alpha\beta} - \delta_{\alpha\beta} L) = \begin{vmatrix} \epsilon_{11} - L & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & (\epsilon_{22} - L) & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & (\epsilon_{33} - L) \end{vmatrix} = 0$$

or $L^3 - J_1 L^2 + J_2 L - J_3 = 0$ yields three extreme values of MF_A (see theory of Art. 2-11)

2-11.3

a) $$J = \begin{vmatrix} 2 & -2 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 6(6+6) = 72 > 0$$

Hence, displacement is admissible.

b) $\epsilon_{11} = 6$, $\epsilon_{22} = 6$, $\epsilon_{33} = \frac{35}{2}$, $\epsilon_{12} = \frac{5}{2}$, $\epsilon_{13} = \epsilon_{23} = 0$ (a)

$$F(L) = \begin{vmatrix} (6-L) & \frac{5}{2} & 0 \\ \frac{5}{2} & 6-L & 0 \\ 0 & 0 & (\frac{35}{2}-L) \end{vmatrix} = (\tfrac{35}{2} - L)(36 - 12L + L^2 - \tfrac{25}{4}) = 0$$

$\therefore (\frac{35}{2} - L) = 0$; $4L^2 - 48L + 119 = 0$

$\therefore L_1 = 17.5$, $L_2 = 8.5$; $L_3 = 3.5$ are principal stress.

c) The direction cosine are the solutions $N_\alpha^{(i)}$ of the equations

$$(\epsilon_{11} - L_1) N_1^{(i)} + \epsilon_{12} N_2^{(i)} + \epsilon_{13} N_3^{(i)} = 0$$
$$\epsilon_{12} N_1^{(i)} + (\epsilon_{22} + Li) N_2^{(i)} + \epsilon_{23} N_3^{(i)} = 0 \qquad (b)$$
$$\epsilon_{12} N_1^{(i)} + \epsilon_{23} N_2^{(i)} + (\epsilon_{33} - Li) N_3^{(i)} = 0$$

$$[N_1^{(i)}]^2 + [N_2^{(i)}]^2 + [N_3^{(i)}]^2 = 1 \qquad (c)$$

For $i=1$, $L_1 = 17.5$, Eq (a) and (b) yield

$$-11.5\, N_1^{(1)} + 2.5\, N_2^{(1)} + 0 = 0$$
$$2.5\, N_1^{(1)} + (-11.5)\, N_2^{(1)} + 0 = 0$$
$$0 + 0 + (\tfrac{35}{2} - \tfrac{35}{2})\, N_3^{(1)} = 0$$

The first two of these equations are consistant if and only if $N_1^{(1)} = N_2^{(1)} = 0$

$\therefore N_3^{(1)} = 1$ from Eq (c)

For $i=2$, $L_2 = 8.5$

$$-2.5\, N_1^{(2)} + 2.5\, N_2^{(2)} = 0$$
$$2.5\, N_1^{(2)} - 2.5\, N_2^{(2)} = 0$$
$$9\, N_3^{(2)} = 0 \qquad \therefore N_3^{(2)} = 0$$

$N_1^{(2)} = N_2^{(2)}$ and $[N_1^{(2)}]^2 + [N_2^{(2)}]^2 = 1$

$\therefore N_1^{(2)} = N_2^{(2)} = \pm \frac{1}{\sqrt{2}}$

For $i=3$, $L_3 = 3.5$

$$2.5\, N_1^{(3)} + 2.5\, N_2^{(3)} = 0$$
$$2.5\, N_1^{(3)} + 2.5\, N_2^{(3)} = 0 \qquad \therefore N_1^{(3)} = -N_2^{(3)}$$
$$14\, N_3^{(3)} = 0 \qquad \therefore N_3^{(3)} = 0$$

2-11.3 cont. pg. 9.

2-11.3 cont.

Hence $N_1^{(3)} = \pm\frac{1}{\sqrt{2}}$, $N_2^{(3)} = \mp\frac{1}{\sqrt{2}}$

For right-handed coordinates system the principal axes in the undeformed medium are

$$N_1^{(1)} = 0 \;,\quad N_2^{(1)} = 0 \;,\quad N_3^{(1)} = 1$$
$$N_1^{(2)} = \frac{1}{\sqrt{2}} \;,\quad N_2^{(2)} = \frac{1}{\sqrt{2}} \;,\quad N_3^{(2)} = 0 \qquad (d)$$
$$N_1^{(3)} = -\frac{1}{\sqrt{2}} \;,\quad N_2^{(3)} = \frac{1}{\sqrt{2}} \;,\quad N_3^{(3)} = 0$$

(d) In the undeformed medium, the principal axes are given by $n_\alpha^{(i)}\sqrt{1+2MF_i} = (\delta_{\alpha\beta} + u_{\alpha,\beta})\, N_\beta^{(i)}$ (e)

with the given displacements and with $MF_i = L_i$ we obtain, with Eqs (d) and (e), the pricipal axes in the deformed medium.

$$n_1^{(1)} = n_2^{(1)} = 0 \;,\quad n_3^{(1)} = \pm 1$$
$$n_1^{(2)} = n_3^{(2)} = 0 \;,\quad n_2^{(2)} = \pm 1$$
$$n_2^{(3)} = n_3^{(3)} = 0 \;,\quad n_1^{(3)} = \pm 1$$

2-11.4 The cubic element in the body whose angles are preserved under the strain has sides with directions parallel to the principal directions. Hence, we must compute principal directions. Thus with given strain, $J_1 = 0$, $J_2 = -8$, $J_3 = 8$ and $L^3 - J_1L^2 + J_2L - J_3 = 0$ becomes $L^3 - 8L - 8 = 0$ where common factor 10^{-3} has been removed from $\epsilon_{\alpha\beta}$.

$$\therefore\ L_1 = (1+\sqrt{5})\times 10^{-3} \;,\quad L_2 = (1-\sqrt{5})\times 10^{-3} \;,\quad L_3 = -2\times 10^{-3}$$

Thus Eqs of Art. 2-11 yielded with $L = L_1, L_2, L_3$ respectively.

$$\xi_1 = \frac{1+\sqrt{5}}{\sqrt{2(5+\sqrt{5})}} \;,\quad \xi_2 = 0 \;,\quad \xi_3 = \frac{2}{\sqrt{2(5+\sqrt{5})}}$$

$$\eta_1 = \frac{1-\sqrt{5}}{\sqrt{2(5-\sqrt{5})}} \;,\quad \eta_2 = 0 \;,\quad \eta_3 = \frac{2}{\sqrt{2(5-\sqrt{5})}}$$

$$\zeta_1 = 0 \;,\quad \zeta_2 = 1 \;,\quad \zeta_3 = 0$$

2-11.5

a) $J = \begin{vmatrix} 3 & -1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 > 0$

$\therefore$ Displacement is admissible.

b) By $\epsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\theta,\alpha}\, u_{\theta,\beta})$

$\epsilon_{11} = 6$, $\epsilon_{22} = 2$, $\epsilon_{33} = 1.5$, $\epsilon_{12} = -3.5$, $\epsilon_{13} = 0$; $\epsilon_{23} = 0$

$$F(L) = \begin{vmatrix} (6-L) & -\frac{7}{2} & 0 \\ -\frac{7}{2} & (2-L) & 0 \\ 0 & 0 & (\frac{3}{2}-L) \end{vmatrix} = 0$$

$\therefore (\frac{3}{2} - L)(12 - 8L + L^2 - \frac{49}{4}) = 0$

Hence $L_1 = 4 + \frac{1}{2}\sqrt{65}$, $L_2 = \frac{3}{2}$, $L_3 = 4 - \frac{1}{2}\sqrt{65}$
$= 8.031$ $\quad = 1.5 \quad = -0.0311$

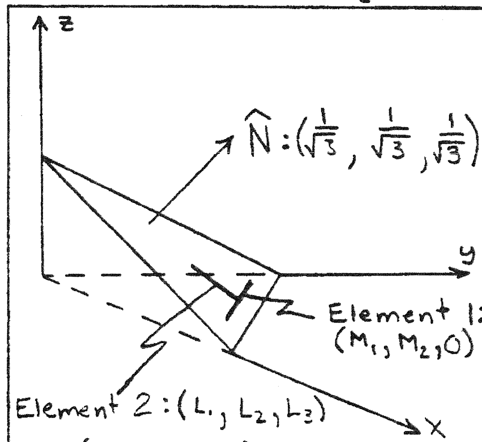
(c) The maximum principal strain is $L_1 = 4 + \frac{1}{2}\sqrt{65} = 8.031$.
The direction cosines (N_1, N_2, N_3) corresponding to L_1 are given by

$$(4 - \sqrt{65})N_1 - 7N_2 = 0$$
$$-7N_1 - (4 + \sqrt{65})N_2 = 0$$
$$N_3 = 0$$
$$N_1^2 + N_2^2 = 1$$

$\therefore \quad N_1 = \dfrac{\pm 7}{\sqrt{130 - 8\sqrt{65}}} = 0.8649$; $\quad N_2 = \dfrac{\pm(4 - \sqrt{65})}{\sqrt{130 - 8\sqrt{65}}} = -0.5019$,

$N_3 = 0$

2-11.6 The elements 1 and 2 are shown in figure 1.



(figure 1.)

The normal to the octahedral plane is $\hat{N}:(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$. Since Element 1 is perpendicular to $\hat{N}$,

$$N_1M_1 + N_2M_2 + N_3M_3 = \frac{1}{\sqrt{3}}(M_1+M_2) = 0$$

$$\therefore \quad M_1 = -M_2$$

Hence, since $M_1^2 + M_2^2 + M_3^2 = M_1^2 + M_2^2 = 1$

$$\therefore M_1 = \pm\frac{1}{\sqrt{2}} \quad , \quad M_2 = \mp\frac{1}{\sqrt{2}}$$

Also, since Element 2 is $\perp$ to $\hat{N}$ and to Element 1

$$\frac{1}{\sqrt{3}}(L_1 + L_2 + L_3) = 0$$

$$M_1L_1 + M_2L_2 = 0$$

$$L_1 - L_2 = 0 \quad \Longrightarrow \quad L_2 = L_1$$

$$L_1 + L_2 + L_3 = 0 \quad \Longrightarrow \quad L_3 = -L_1 - L_2 = -2L_1$$

$$L_1^2 + L_2^2 + L_3^2 = 6L_1^2 = 1$$

$$\therefore L_1 = \pm\frac{1}{\sqrt{6}} \; , \quad L_2 = \pm\frac{1}{\sqrt{6}} \quad , \quad L_3 = \mp\frac{2}{\sqrt{6}}$$

Since Element 1 and 2 are perpendicular and since

$$\epsilon_{11} = \epsilon_1 \; , \; \epsilon_{22} = \epsilon_2 \; , \; \epsilon_{33} = \epsilon_3 \; ; \; \epsilon_{12} = \epsilon_{23} = \epsilon_{13} = 0$$

$$\Gamma_{12} = 2\epsilon_{\alpha\beta}M_\alpha L_\beta = 2(\epsilon_1 M_1 L_1 + \epsilon_2 M_2 L_2) = \frac{1}{\sqrt{3}}(\epsilon_1 - \epsilon_2)$$

$$MF_1 = \epsilon_{\alpha\beta}M_\alpha M_\beta = \epsilon_1 M_1^2 + \epsilon_2 M_2^2 + \epsilon_3 M_3^2 = \frac{1}{2}(\epsilon_1 + \epsilon_2)$$

2-11.7

a) By given relation

$$x_k = (\delta_{kK} + B_{kK}) X_K = \delta_{kK} X_K + B_{kK} X_K = x_k + u_k$$

$$u_k = B_{kK} X_K \quad \text{or} \quad u_k = B_{KL} X_L$$

b) By definition of e_{KL}, ω_{KL}

$$e_{KL} = \frac{1}{2}\left[\frac{\partial u_K}{\partial X_L} + \frac{\partial u_L}{\partial X_K}\right] = \frac{1}{2}\left[B_{KL} + B_{LK}\right]$$

$$\omega_{KL} = \frac{1}{2}\left[\frac{\partial u_L}{\partial X_K} - \frac{\partial u_K}{\partial X_L}\right] = \frac{1}{2}\left[B_{LK} - B_{KL}\right]$$

c) By definition of ϵ_{KL}

$$\epsilon_{KL} = \frac{1}{2}\left[\frac{\partial u_K}{\partial X_L} + \frac{\partial u_L}{\partial X_K} + \frac{\partial u_M}{\partial X_K}\frac{\partial u_M}{\partial X_L}\right]$$

$$\omega_{KL} = \frac{1}{2}\left[B_{KL} + B_{LK} + B_{MK} B_{ML}\right]$$

$\therefore$ For $\epsilon_{KL} \approx e_{KL}$; sum in quadratic terms in B_{KL} must be negligible compared to linear sum $B_{KL} + B_{LK}$.

d) For $B_{KL} = 1$, $\epsilon_{KL} = \frac{1}{2}(1+1+1+1+1) = \frac{5}{2} = e$

and

$$F(L) = \begin{vmatrix} e-L & e & e \\ e & e-L & e \\ e & e & e-L \end{vmatrix}$$

or

$$L^3 - 3eL^2 = 0$$

$\therefore$ The principal strains are $L_1 = 3e$, $L_2 = L_3 = 0$

This result holds for all X_α ; since ϵ_{KL} independent of X_α. The principal directions are given by the relations

2-11.7 cont. on pg 13

2-11.7 cont.

$$(e-L_i)N_1^{(i)} + eN_2^{(i)} + eN_3^{(i)} = 0$$
$$eN_1^{(i)} + (e-L_i)N_2^{(i)} + eN_3^{(i)} = 0$$
$$eN_1^{(i)} + eN_2^{(i)} + (e-L_i)N_3^{(i)} = 0$$
$$\left[N_1^{(i)}\right]^2 + \left[N_2^{(i)}\right]^2 + \left[N_3^{(i)}\right]^2 = 1$$

For $L_1 = 3e = \frac{15}{2}$, $N_1 = N_2 = N_3 = \pm\frac{1}{\sqrt{3}}$

$L_2 = 0$, $L_3 = 0$ is satisfied by any two mutually perpendicular axes that are also perpendicular to direction $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, for example directions $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$, $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

2-11.8

$$F(L) = \begin{vmatrix} e-L & \frac{1}{2}e & \frac{1}{2}e \\ \frac{1}{2}e & e-L & \frac{1}{2}e \\ \frac{1}{2}e & \frac{1}{2}e & e-L \end{vmatrix}$$

or

$$L^3 - 3eL^2 + \frac{9}{4}e^2L - \frac{e^3}{2} = 0$$

$\therefore$ principal strains are :

$$L_1 = 2e \quad , \quad L_2 = L_3 = 0.5e$$

The equations which determine principal directions $N_\alpha^{(i)}$ are.

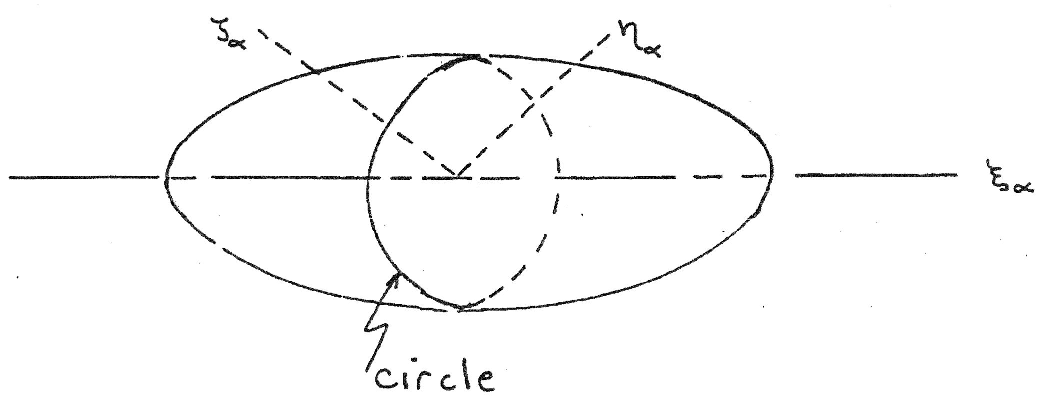
$$(e-L_i)N_1^{(i)} + \tfrac{1}{2}eN_2^{(i)} + \tfrac{1}{2}eN_3^{(i)} = 0$$
$$\tfrac{1}{2}eN_1^{(i)} + (e-L_i)N_2^{(i)} + \tfrac{1}{2}eN_3^{(i)} = 0$$
$$\tfrac{1}{2}eN_1^{(i)} + \tfrac{1}{2}eN_2^{(i)} + (e-L_i)N_3^{(i)} = 0$$
$$\left(N_1^{(i)}\right)^2 + \left(N_2^{(i)}\right)^2 + \left(N_3^{(i)}\right)^2 = 1$$

For $L_1 = 2e$, $\xi_i : \left(\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}\right)$ are principal directions

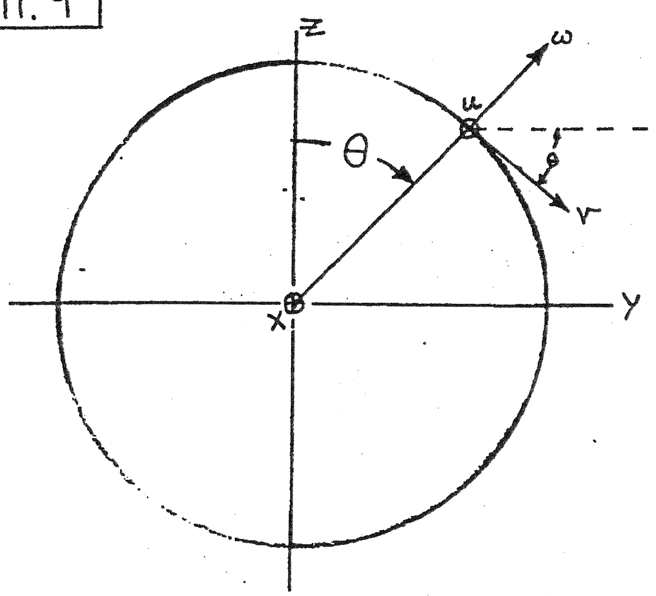
2-11.8 cont.

For $L_2 = L_3 = 0.5e$, we get $\eta_1 + \eta_2 + \eta_3 = 0$, $\zeta_1 + \zeta_2 + \zeta_3 = 0$.
Also, $\eta_1^2 + \eta_2^2 + \eta_3^2 = 1$, $\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 1$.

Any direction cosines that satisfy these conditions define principal directions. Any two mutualy perpendicular directions η_α, ζ_α which are also perpendicular to the direction ξ_α may be taken as principal directions. In other words, the strain ellipsoid (Art 2-10) is an ellipsoid of revolution about the ξ-axis.



2-11.9



$$X^* = x + u$$
$$y^* = y + v\cos\theta + w\sin\theta$$
$$z^* = z - v\sin\theta + w\cos\theta$$

2-13.1 The displacement components are given as

$u = -k_1 xy$, $v = k_2(x^2 + \nu y^2 - \nu z^2)$, $\omega = k_3 \nu yz$.

For small displacement theory volume rotations are given by (where $|e_{\alpha\beta}| << 1$, $|\omega_{\alpha\beta}| << 1$)

$$\bar{\phi}_x \cong \omega_x = \omega_{23} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{\nu z}{2}(k_3 + 2k_2)$$

$$\bar{\phi}_y \cong \omega_y = \omega_{31} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

$$\bar{\phi}_z \cong \omega_z = \omega_{12} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{x}{2}(2k_2 + k_1)$$

or by (where $|\omega_{\alpha\beta}|$ is sufficiently small to allow a Taylor series expansion of $\bar{\phi} = \arctan[\omega_{\alpha\beta} + F(\theta)]$ to first degree terms in $\omega_{\alpha\beta}$)

$$\bar{\phi}_x = \arctan \omega_x = \arctan\left[\frac{\nu z}{2}(k_3 + 2k_2)\right]$$

$$\bar{\phi}_y = \arctan \omega_y = 0$$

$$\bar{\phi}_z = \arctan \omega_z = \arctan\left[\frac{x}{2}(2k_2 + k_1)\right]$$

2-13.2

Given $u = -C_1 zx$, $v = -C_1 zy$, $w = \frac{1}{2}C_1(x^2 + y^2) + C_2 z^2 + C_3$

For $|e_{\alpha\beta}| << 1$, $|\omega_{\alpha\beta}| << 1$

$$\bar{\phi}_x \cong \omega_x = \omega_{23} = \frac{1}{2}(w_y - v_z) = C_1 y$$

$$\bar{\phi}_y \cong \omega_y = \omega_{31} = \frac{1}{2}(u_z - \omega_x) = -C_1 x$$

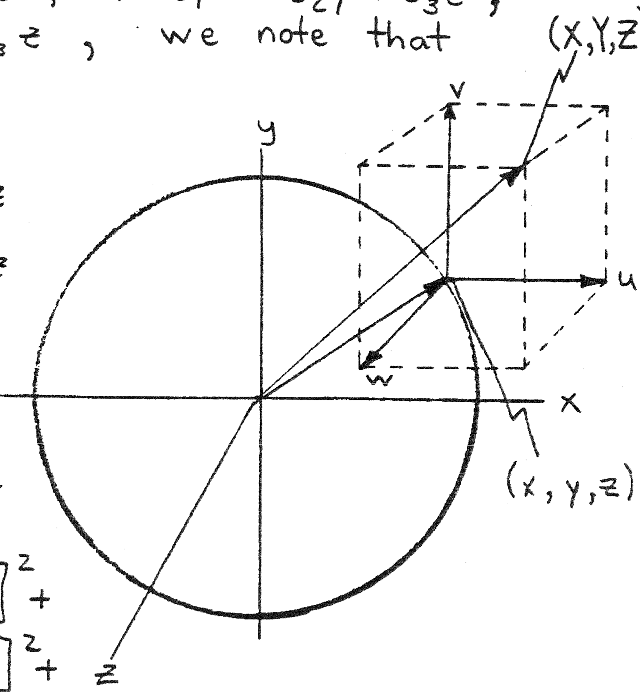
$$\bar{\phi}_z \cong \omega_z = \omega_{12} = \frac{1}{2}(v_x - u_y) = 0$$

2-13.3 Equation of spherical surface of radius (a) is $x^2+y^2+z^2=a^2$ under displacement u, v, w, for a point (p) on sphere $x \Rightarrow X$, $y \Rightarrow Y$, $z \Rightarrow Z$, $a \rightarrow a^*$ where

$$X = x+u, \quad Y = y+v, \quad Z = z+w, \quad a^* = a + \Delta a$$

Since
$$u = a_1 x + a_2 y + a_3 z, \quad v = b_1 x + b_2 y + b_3 z,$$
$$w = c_1 x + c_2 y + c_3 z,$$ we note that

$$X = (1+a_1)x + a_2 y + a_3 z$$
$$Y = b_1 x + (1+b_2)y + b_3 z$$
$$Z = c_1 x + c_2 y + (1+c_3)z$$



Hence, each point p on the spherical surface transforms into a point p^* at distance a^* from origin of coordinate system. Hence,

$$X^2+Y^2+Z^2 = \left[(1+a_1)x + a_2 y + a_3 z\right]^2 + \left[b_1 x + (1+b_2)y + b_3 z\right]^2 + \left[c_1 x + c_2 y + (1+c_3)z\right]^2 = (a^*)^2$$

or $x^2(1+2a_1+a_1^2+b_1^2+c_1^2) + y^2(1+2b_2+a_2^2+b_2^2+c_2^2) + z^2(1+2c_3+a_3^2+b_3^2+c_3^2) + 2xy(a_2+b_1+a_1a_2+b_1b_2+c_1c_2) + 2xz(a_3+c_1+a_1a_3+b_1b_3+c_1c_3) + 2yz(b_3+c_3+a_2a_3+b_2b_3+c_2c_3) = (a^*)$

$$\therefore \; x^2\epsilon_x + y^2\epsilon_y + z^2\epsilon_z + xy\,\gamma_{xy} + xz\,\gamma_{xz} + yz\,\gamma_{xz} = \frac{(a^*)^2 - a^2}{2}$$

where $\epsilon_x, \epsilon_y \ldots$ are constants. This is equation of quadratic surface provided strains remain finite. For $\epsilon_x = \epsilon_y = \epsilon_z = \epsilon$, we obtain

$$x^2+y^2+z^2 + xy\frac{\gamma_{xy}}{\epsilon} + xz\frac{\gamma_{xz}}{\epsilon} + yz\frac{\gamma_{yz}}{\epsilon} = \frac{(a^*)^2 - a^2}{2\epsilon}$$

This is equation of shpere.

$\boxed{2\text{-}13.4}$ We must have

$$J = \begin{vmatrix} 1-\epsilon x_3 & 0 & -\epsilon x_1 \\ 0 & 1-\epsilon x_3 & -\epsilon x_2 \\ \epsilon x_1 & \epsilon x_2 & 1+\epsilon A x_3 \end{vmatrix} > 0$$

$$\therefore\ J = (1-\epsilon x_3)\left[(1-\epsilon x_3)(1+\epsilon A x_3) + (\epsilon x_1)^2 + (\epsilon x_2)^2\right] > 0 \qquad (a)$$

$\therefore$ The condition $\epsilon x_3 = 1$ is excluded.

(a) consider case $1-\epsilon x_3 > 0$, $0 < \epsilon x_3 < 1$

$\therefore\ \epsilon x_3 = 1-\delta$, $0 < \delta < 1$

Hence, Eq (a) becomes

$$\delta[1+A(1-\delta)] > -(\epsilon x_1)^2 - (\epsilon x_2)^2 \qquad (b)$$

Since Eq (b) must hold for all (x_1, x_2)

$$\delta[1+A(1+\delta)] > 0$$

$$\therefore\ A > -\frac{1}{1-\delta} = -\frac{1}{\epsilon x_3}$$

where $0 < \epsilon x_3 < 1$, $\epsilon x_3 \neq 1$. Since $0 < \delta < 1$, $A = -1$ satisfies $J > 0$ for $0 < \epsilon x_3 < 1$.

(b) For case $1-\epsilon x_3 < 0$, we have $\epsilon x_3 > 1$

Let $\epsilon x_3 = 1+\delta$, $\delta > 0$. Then $J > 0$ requires $-\delta[1+A(1+\delta)] + (\epsilon x_1)^2 + (\epsilon x_2)^2 < 0$

$$\therefore\ \delta[1+A(1+\delta)] > (\epsilon x_1)^2 + (\epsilon x_2)^2 \qquad (c)$$

Since Eq (c) must hold for all (x_1, x_2) and for $x_3 > 0$

$$\delta[1+A(1+\delta)] > \infty$$

$\therefore$ For finite δ, $1 + A(1+\delta) > \infty$

$$\therefore\ A > \infty$$

Consequently, we must restrict the problem to case (a).

cont'd

2-13.4 cont'd

(c) with $A=-1$ and $0<\epsilon x_3<1$, for small displacements

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = -\epsilon x_3$$

$$\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0 \quad \text{Hence, } J_1 = -3\epsilon x_3, \; J_2 = 3\epsilon^2 x_3^2$$

$$J_3 = -3\epsilon^3 x_3^3$$

(d) For small displacement theory

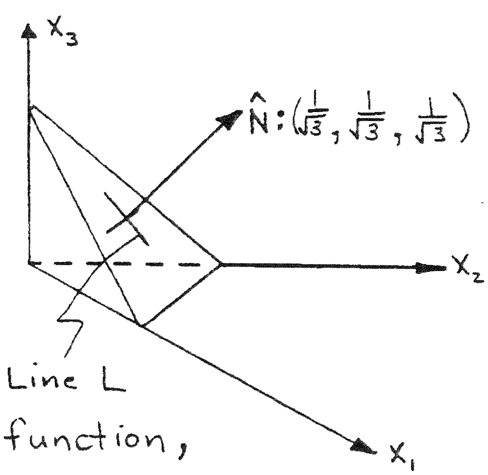
$$\bar{\Phi}_{x_1} \cong \omega_x = \omega_{23} = \tfrac{1}{2}(u_{3,2} - u_{2,3}) = \epsilon x_2$$

$$\bar{\Phi}_{x_2} \cong \omega_y = \omega_{31} = \tfrac{1}{2}(u_{1,3} - u_{3,1}) = -\epsilon x_1$$

$$\bar{\Phi}_{x_3} \cong \omega_z = \omega_{12} = \tfrac{1}{2}(u_{2,1} - u_{1,2}) = 0$$

2-13.5

Consider the octahedral plane normal $\hat{N}$ relative to principal axes x_α. (Fig) Any line in the octahedral plane satisfies the conditions,



$$\hat{N} \cdot \hat{L} = 0$$

or $L_1 + L_2 + L_3 = 0$ (a)

and $L_1^2 + L_2^2 + L_3^2 = 1$ (b)

where $\hat{L}_1:(L_1, L_2, L_3)$ is unit vector in direction of L. Hence, by Lagrange multiplier method, from function,

$$F = T_{LN} + \lambda_1(L_1 + L_2 + L_3) + \lambda_2(L_1^2 + L_2^2 + L_3^2 - 1)$$

where λ_1, λ_2 are Lagrange multipliers.

and $T_{LN} = 2\epsilon_{\alpha\beta} N_\alpha L_\beta = \frac{2}{\sqrt{3}}(L_1\epsilon_1 + L_2\epsilon_2 + L_3\epsilon_3)$ (c)

Since x_α are principal axes. Now

$$\frac{\partial F}{\partial L_1} = \frac{\partial F}{\partial L_2} = \frac{\partial F}{\partial L_3} = 0 \quad \text{yields}$$

cont'd.

2-13.5 cont'd.

$$\frac{2}{\sqrt{3}}\epsilon_1 + \lambda_1 + 2L_1\lambda_2 = 0$$

$$\frac{2}{\sqrt{3}}\epsilon_2 + \lambda_1 + 2L_2\lambda_2 = 0 \qquad \text{(d)}$$

$$\frac{2}{\sqrt{3}}\epsilon_3 + \lambda_1 + 2L_3\lambda_2 = 0$$

Addition of Eq (d) yields, with Eq (a),

$$\frac{2}{\sqrt{3}}(\epsilon_1+\epsilon_2+\epsilon_3) + 3\lambda_1 = 0, \quad \lambda_1 = -\frac{2}{3\sqrt{3}}(\epsilon_1+\epsilon_2+\epsilon_3) \qquad \text{(e)}$$

Multiplication of first, second and third Eq's (d) by L_1, L_2, L_3, respectively and addition yields, with Eq (a), (b), and (c)

$$2\lambda_2 = -\frac{2}{\sqrt{3}}(L_1\epsilon_1 + L_2\epsilon_2 + L_3\epsilon_3) = -T_{LN} \qquad \text{(f)}$$

Hence, Eq's (d), (e) and (f) yield

$$L_1 = \frac{\lambda_1 + \frac{2}{\sqrt{3}}\epsilon_1}{-2\lambda_2} = \frac{2}{3\sqrt{3}\,T_{LN}}\left[2\epsilon_1 - \epsilon_2 - \epsilon_3\right]$$

$$L_2 = \frac{\lambda_1 + \frac{2}{\sqrt{3}}\epsilon_2}{-2\lambda_2} = \frac{2}{3\sqrt{3}\,T_{LN}}\left[2\epsilon_2 - \epsilon_1 - \epsilon_3\right] \qquad \text{(g)}$$

$$L_3 = \frac{\lambda_1 + \frac{2}{\sqrt{3}}\epsilon_3}{-2\lambda_2} = \frac{2}{3\sqrt{3}\,T_{LN}}\left[2\epsilon_3 - \epsilon_1 - \epsilon_2\right]$$

∴ squaring and adding Eq (g), we obtain with Eq (b)

$$T_{LN}^2 = \frac{4}{9}\left[(\epsilon_1-\epsilon_2)^2 + (\epsilon_1-\epsilon_3)^2 + (\epsilon_2-\epsilon_3)^2\right]$$

Hence,

$$T_{LN} = \gamma_{oct} = \frac{2}{3}\left[(\epsilon_1-\epsilon_2)^2 + (\epsilon_1-\epsilon_3)^2 + (\epsilon_2-\epsilon_3)^2\right]^{\frac{1}{2}}$$

2-13.6

(a)
$$J = \begin{vmatrix} \cos kz & -\sin kz & -xk\sin kz - yk\cos kz \\ \sin kz & \cos kz & xk\cos kz - yk\sin kz \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos^2 kz + \sin^2 kz = 1 > 0$$

$\therefore$ Displacement is admissible.

(b) $\epsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta;\alpha} + u_{\theta,\alpha} u_{\theta,\beta})$

$\epsilon_{11} = \epsilon_{22} = \epsilon_{12} = 0$

$\epsilon_{33} = \frac{1}{2}k^2(x^2+y^2)$, $\epsilon_{13} = -\frac{1}{2}ky$

$\epsilon_{23} = \frac{1}{2}kx$

Hence,

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}ky \\ 0 & 0 & \frac{1}{2}kx \\ -\frac{1}{2}ky & \frac{1}{2}kx & \frac{1}{2}k^2(x^2+y^2) \end{pmatrix}$$

(c) The volumetric strain is

$$e = J_1 + 2J_2 + 4J_3$$

where

$$J_1 = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \frac{1}{2}k^2(x^2+y^2)$$

$$J_2 = \epsilon_{11}\epsilon_{22} + \epsilon_{11}\epsilon_{33} + \epsilon_{22}\epsilon_{33} - \epsilon_{12}^2 - \epsilon_{13}^2 - \epsilon_{23}^2$$
$$= -\frac{1}{4}k^2y^2 - \frac{1}{4}k^2x^2 = -\frac{1}{4}k^2(x^2+y^2)$$

$J_3 = 0$. $\therefore$ $e = 0$.

(d) By the second of Eqs. (2-5.3) and the given displacement components, with $u = u_1$, $v = u_2$,

cont'd

2-13.6 cont'd

$w = u_3$ and $x = x_1$, $y = x_2$, $z = x_3$

$$2\omega_{12} = u_{2,1} - u_{1,2} = \sin kz + \sin kz = 2\sin kz$$
$$2\omega_{23} = u_{3,2} - u_{2,3} = -k(x\cos kz) + k(y\sin kz)$$
$$2\omega_{31} = u_{1,3} - u_{3,1} = -k(x\sin kz) - k(y\cos kz)$$

$\therefore$ By Eq (2-13.3).

$$2\bar{\omega} = -\hat{\imath}\, k(x\cos kz - y\sin kz) - \hat{\jmath}\, k(x\sin kz + y\cos kz) + \hat{k}(2\sin kz)$$

2-13.7

By Eq(a) of example 2-13.1 and Eq.(2-5.3)

$$e_{11} = 10C,\ e_{22} = 2C,\ e_{33} = e_{12} = 6C,\ e_{13} = e_{23} = 0$$
$$\omega_{12} = -\omega_{21} = 3C,\ \omega_{23} = -\omega_{32} = \omega_{31} = -\omega_{13} = 0$$

(a) Consider first the line element parallel to the x_3 axis. Then analogous to Eqs. (a) and (b) of section 2-13,

$$\tan\theta_{x_2} = \frac{dx_1}{dx_3} = 0,\quad \tan\theta_{x_2} = \frac{d\xi_1}{d\xi_3},\quad dx_2 = 0$$

and

$$d\xi_1 = (1+e_{11})dx_1 + (e_{31}+\omega_{31})dx_3$$
$$d\xi_3 = (e_{13}+\omega_{13})dx_1 + (1+e_{33})dx_3$$

$$\therefore \tan\theta_{x_2} = \frac{(1+e_{11})dx_1 + (e_{31}+\omega_{31})dx_3}{(e_{13}+\omega_{13})dx_1 + (1+e_{33})dx_3}$$

or

$$\tan\theta_{x_2} = \frac{(1+e_{11})\cancel{\tan\theta}^{\,0} + (e_{31}+\omega_{31})}{(\epsilon_{13}+\omega_{13})\cancel{\tan\theta}^{\,0} + 1 + e_{33}} = \frac{e_{31}+\omega_{31}}{1+e_{33}} = 0$$

$$\therefore \tan\phi_{x_2} = \frac{\tan\theta_{x_2} - \tan\theta_{x_2}}{1 + \tan\theta_{x_2}\tan\theta_{x_2}} = \frac{0-0}{1+0} = 0$$

Similarly, $$\tan\phi_{x_1} = \frac{d\xi_3}{d\xi_2} = \frac{(\epsilon_{23}+\omega_{23})dx_2 + (1+e_{33})dx_3}{(1+e_{22})dx_2 + (e_{32}+\omega_{32})dx_3}$$

cont'd

2-13.7 cont'd

or $\tan\phi_{x_1} = \dfrac{(e_{23}+\omega_{23}) + (1+e_{33})\cancel{\tan\theta_{x_1}}^{\,0}}{(1+e_{22}) + (e_{32}+\omega_{32})\cancel{\tan\theta_{x_1}}^{\,0}} = \dfrac{e_{23}+\omega_{23}}{1+e_{22}} = 0$

2-15.1 This problem is worked out in detail in Art. 2-15. The results are summarized in Eq (2-15.19).

2-15.2 Consider a set of primed axes x'_i such that the transformation matrix between the primed and unprimed axes is a_{ij}. Therefore in terms of the primed coordinates, $u_\alpha = a_{m\alpha} u'_m$

$$\therefore\ \epsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)$$

$$= \frac{1}{2}\left[\frac{\partial(a_{m\alpha}u'_m)}{\partial x'_n}\frac{\partial x'_n}{\partial x_\beta} + \frac{\partial(a_{n\beta}u'_n)}{\partial x'_m}\frac{\partial x'_m}{\partial x_\alpha}\right]$$

But the a_{ij} are constants everywhere; hence

$$\epsilon_{\alpha\beta} = \frac{1}{2} a_{m\alpha} a_{n\beta}\left(\frac{\partial u'_m}{\partial x'_n} + \frac{\partial u'_n}{\partial x'_m}\right) = a_{m\alpha} a_{n\beta} \epsilon'_{mn}$$

$\therefore\ \epsilon_{\alpha\beta}$ transforms as a second-order tensor.

2-15.3

$$\epsilon_{11} = A(L-x_1) \cong \frac{\partial u_1}{\partial x_1} \qquad (a)$$

$$\epsilon_{22} = B(L-x_1) \cong \frac{\partial u_2}{\partial x_2} \qquad (b)$$

$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0 \qquad \text{since } u_3 = 0$$

$$\epsilon_{12} = 0 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \qquad (c)$$

Integration of Eq (a) and (b) yields

$$u_1 = A\left(Lx_1 - \frac{x_1^2}{2}\right) + \Psi(x_2) \qquad (d)$$

$$u_2 = B(Lx_2 - x_1x_2) + \Xi(x_1) \qquad (e)$$

substitution of Eq (d) and (e) into (c) yields

cont'd

2-15.3 cont'd

$$\Psi'(x_2) - Bx_2 + \Phi'(x_1) = 0$$

or $\Psi'(x_2) - Bx_2 = -\Phi'(x_1) = c = \text{const.}$

$\therefore \quad \Psi' - Bx_2 = c \quad , \quad \Phi' = -c$

or $\Psi = \frac{Bx_2^2}{2} + cx_2 + D \quad , \quad \Phi = -cx_1 + E$

where E and D are constants. Hence

$$u_1 = A\left(Lx_1 - \frac{x_1^2}{2}\right) + \frac{Bx_2^2}{2} + cx_2 + D \qquad (f)$$

$$u_2 = B(Lx_2 - x_1x_2) - cx_1 + E$$

$\therefore \quad u_1(0,0) = 0$ yields $D = 0$; $u_2(0,0) = 0$ yields $E = 0$

and $\omega(0,0) = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)_{\substack{x_1=0\\x_2=0}} = 0$ yields

$c = 0$, and Eqs (f) are simplified accordingly.

2-15.4 Given:

$$\epsilon_x = cy(L-x) \cong \frac{\partial u}{\partial x} \qquad (a)$$

$$\epsilon_y = Dy(L-x) \cong \frac{\partial v}{\partial y} \qquad (b)$$

$$\gamma_{xy} = -(c+D)(A^2 - y^2) \cong \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (c)$$

Integration of Eqs (a) and (b) yields

$$u = cy\left(Lx - \frac{x^2}{2}\right) + \Psi(y) \qquad (d)$$

$$v = \frac{1}{2}Dy^2(L-x) + \Phi(x) \qquad (e)$$

substitution of Eqs (e) and (d) into (c) and grouping of terms yields

$$H(x) + G(y) = 0$$

$$H(x) = \Phi'(x) + c\left(Lx - \frac{x^2}{2}\right) = -E \qquad (f)$$

$$G(y) = \Psi'(y) - \frac{1}{2}Dy^2 + (c+D)(A^2 - y^2) = E \qquad (g)$$

where E = constant.

cont'd

2-15.4 cont'd

Hence, integration of Eqs. (f) and (g) yields

$$Y(y) = \frac{1}{6}Dy^3 - (c+D)(A^2y - \frac{1}{3}y^3) + Ey + F$$

$$X(x) = -c\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) - Ex + J$$

where F, J are constants. Hence.

$$u = cy\left(Lx - \frac{x^2}{2}\right) + \left(\frac{1}{3}c + \frac{1}{2}D\right)y^3 + [E - (c+D)A^2]y + F \quad (h)$$

$$v = \frac{1}{2}Dy^2(L-x) - c\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) - Ex + J \quad (i)$$

For $u(0,0) = 0$, $F = 0$; for $v(0,0) = 0$, $J = 0$; and for $\frac{\partial u}{\partial y}(0,0) = 0$, $E = (c+D)A^2$ and Eqs (h) and (i) are modified accordingly.

2-15.5 Given:

$$u_1 = -Kx_1x_2, \quad u_2 = K_2(x_1^2 + \gamma x_2^2 - \gamma x_3^2), \quad u_3 = k_3\gamma x_2 x_3$$

The rotations are

$$\bar{\phi}_x \cong \omega_{23} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) = \frac{1}{2}(k_3\gamma x_3 + 2k_2\gamma x_3)$$
$$= \frac{\gamma x_3}{2}(k_3 + 2k_2)$$

$$\bar{\phi}_y \cong \omega_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(0-0) = 0$$

$$\bar{\phi}_z \cong \omega_{12} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \frac{1}{2}(2k_2x_1 + k_1x_1) = \frac{x_1}{2}(2k_2 + k_1)$$

2-15.6

$$\epsilon_x = \nu c(\ell - z) \cong \frac{\partial u}{\partial x}$$

$$\epsilon_y = \nu c(\ell - z) \cong \frac{\partial v}{\partial y}$$

$$\epsilon_z = \nu c(\ell - z) \cong \frac{\partial w}{\partial z}$$

$$\gamma_{xy} \cong \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \qquad \gamma_{xz} \cong \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0$$

$$\gamma_{yz} \cong \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0$$

Integration yields

$$u = \nu c(\ell - z)x + f(y,z) \quad (1)$$

$$v = \nu c(\ell - z)y + g(x,z) \quad (2)$$

cont'd

2-15.6 cont'd

$$w = -c\left(lz - \frac{z^2}{2}\right) + h(x,y) \qquad (3)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial f(y,z)}{\partial y} + \frac{\partial g(x,z)}{\partial x} = 0 \qquad (4)$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\nu c x + \frac{\partial f(y,z)}{\partial z} + \frac{\partial h(x,y)}{\partial x} = 0 \qquad (5)$$

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\nu c y + \frac{\partial g(x,z)}{\partial z} + \frac{\partial h(x,y)}{\partial y} = 0 \qquad (6)$$

By Eq. (4)

$$\frac{\partial f(y,z)}{\partial y} + \frac{\partial g(x,z)}{\partial x} = 0$$

$$\therefore \quad \frac{\partial f(y,z)}{\partial y} = -\frac{\partial g(x,z)}{\partial x} = Z_1(z)$$

$$f(y,z) = y Z_1(z) + Z_2(z) \qquad (7)$$

$$g(x,z) = -x Z_1(z) + Z_3(z) \qquad (8)$$

$\therefore$ Eq (7) and (5) yield

$$-\nu c x + y Z_1' + Z_2' + \frac{\partial h(x,y)}{\partial x} = 0$$

or $Z_2' = \text{const.} = C_2 \qquad \therefore Z_2 = z C_2 + C_4$

$Z_1' = \text{const.} = C_1 \qquad \therefore Z_1 = z C_1 + C_3$

$$\therefore \quad -\nu c x + C_1 y + C_2 + \frac{\partial h(x,y)}{\partial x} = 0$$

$$\therefore \quad \frac{\partial h(x,y)}{\partial x} - \nu c x = -C_1 y - C_2 \qquad (9)$$

$$h(x,y) = \frac{\nu c x^2}{2} - C_1 x y - C_2 x + Y_1(y)$$

By Eq (6),(8) and (9)

$$-\nu c y - x C_1 + Z_3' - C_1 x + Y_1' = 0$$

$$\therefore \quad Z_3' + Y_1' - \nu c y - 2C_1 x = 0$$

$$\therefore \quad C_1 = 0 \quad , \quad Y' - \nu c y = -Z_3' = \text{const.} = C_5$$

$$\therefore \quad Y = \frac{\nu c y^2}{2} + C_5 y + C_6$$

$$Z_3 = -C_5 z + C_7$$

cont'd

2-15.6 cont'd

$$\therefore \quad u = \nu c(\ell - z)x + c_3 y + c_2 z + c_4$$
$$v = \nu c(\ell - z)y - c_3 x - c_5 z + c_7$$
$$w = -c\left(\ell z - \frac{z^2}{2}\right) + \frac{\nu c x^2}{2} + \frac{\nu c y^2}{2} - c_2 x + c_5 y + c_6$$

The terms

$$c_3 y + c_2 z + c_4$$
$$-c_3 x - c_5 z + c_7$$
$$-c_2 x + c_5 y + c_6$$

represent a rigid body displacement.

For $u(0,0,0) = v(0,0,0) = w(0,0,)$, we require $c_4 = c_7 = c_6 = 0$. For $\omega_x(0,0,0) = 0$, etc., we have

$$\omega_{23} = \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}\left[\nu c y + c_5 + \nu c y + c_5\right]$$

$$\therefore \quad \omega_x(0,0,0) = 0 = c_5$$
$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left[-\nu c x + c_2 - \nu c x + c_2\right]$$
$$\omega_y(0,0,0) = c_2 = 0$$
$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -c_3$$
$$\omega_z(0,0,0) = -c_3 = A$$

For $A = 0$, $c_3 = 0$. Then rigid body displacement vanishes since all terms in rigid body displacement are zero.

2-15.7 proceeding as in problem (2-15.6), we obtain

$$u = A\left(\frac{x^2}{2} - zx\right) - A\frac{y^2}{2} + c_3 y + c_2 z + c_4$$
$$v = A\left(xy - \frac{y^2}{2}\right) - A\frac{z^2}{2} - c_3 x + c_5 z + c_6$$
$$w = A\left(yz + \frac{z^2}{2}\right) + A\frac{x^2}{2} - c_2 x - c_5 y + c_7$$

For $u = v = w = 0$ at $x = y = z = 0$, $c_4 = c_6 = c_7 = 0$

For $\bar{\omega} = 0$ at $x = y = z = 0$, $c_2 = c_3 = c_5 = 0$

$$\therefore \quad u = A\left(\frac{x^2}{2} - \frac{y^2}{2} - xz\right)$$
$$v = A\left(xy - \frac{y^2}{2} - \frac{z^2}{2}\right)$$
$$w = A\left(\frac{x^2}{2} + yz + \frac{z^2}{2}\right)$$

2-15.8

$$u_\alpha = c_{\alpha\beta}\, x_\beta = c_{\alpha\gamma}\, x_\gamma$$
$$\epsilon_{\alpha\beta} = \frac{1}{2}\left[u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\theta,\alpha}\, u_{\theta,\beta}\right]$$
$$\therefore \quad 2\epsilon_{\alpha\beta} = c_{\alpha\beta} + c_{\beta\alpha} + c_{\theta\alpha} c_{\theta\beta}$$

(a) If we take $c_{\alpha\beta} = -c_{\beta\alpha}$, then
$\epsilon_{\alpha\beta} = \frac{1}{2} c_{\theta\alpha} c_{\theta\beta}$ is quadratic in $c_{\alpha\beta}$. Hence the quantities $e_{\alpha\beta}$ vanish.

(b) In case (a) $\epsilon_{\alpha\beta}$ is quadratic in $c_{\alpha\beta}$. Hence, the quantities $e_{\alpha\beta}$ vanish. If we discard quadratic terms in $\epsilon_{\alpha\beta}$, then we discard everything.

2-15.9

Given:
$$u = -(1 - \cos\varphi)\, x - y \sin\varphi$$
$$v = x \sin\varphi - (1 - \cos\varphi)\, y$$
$$w = 0 \quad ; \quad \varphi = \text{const}$$

Then
$$e_{11} = \frac{\partial u}{\partial x} = -(1 - \cos\varphi) \qquad e_{22} = \frac{\partial v}{\partial y} = -(1 - \cos\varphi)$$
$$e_{33} = \frac{\partial w}{\partial z} = 0 \qquad e_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$
$$e_{13} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0, \quad e_{23} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = 0$$

cont'd

2-15.9 cont'd

Although (u, v, w) is a rigid body displacement due to a rotation of angle φ about the z axis, it yields nonzero value of $e_{\alpha\beta}$

(b) $\epsilon_{11} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] = -(1-\cos\varphi) + (1-\cos\varphi) = 0$

similarly $\epsilon_{22} = \epsilon_{33} = \epsilon_{12} = \epsilon_{23}\ \epsilon_{13} = 0$

$\therefore$ Rigid body displacement yields $\epsilon_{\alpha\beta} = 0$

(c) Hence large rigid body rotations ($\cos\varphi$ is not approx. equal to one) produce zero values of $\epsilon_{\alpha\beta}$. However, the linear approximation $e_{\alpha\beta}$ of $\epsilon_{\alpha\beta}$ does not vanish identically.

2-15.10

a) By Fig p 2-15.10 b

$u = \frac{1}{2}y, \quad v = 0, \quad w = 0$

(b) we must use

$$\epsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\theta,\alpha} u_{\theta\beta})$$

$\therefore \epsilon_{11} = 0, \quad \epsilon_{22} = \frac{1}{8}, \quad \epsilon_{12} = \frac{1}{4}$

$\epsilon_{33} = \epsilon_{23} = \epsilon_{13} = 0$ Hence,

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{8} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) In the deformed region the direction of the line is given by $\eta_1 = \eta_3 = 0, \quad \eta_2 = 1$

since $$\eta_\alpha \sqrt{1 + 2MF} = (\delta_{\alpha\beta} + u_{\alpha\beta}) N_\beta$$

cont'd

2-15.10 cont'd

(c) The direction cosines N_β of the line in the undeformed region are given by the equations.

$$N_1 + \tfrac{1}{2}N_2 = 0 \Longrightarrow N_2 = -2N_1$$

$$0 = \eta_2\sqrt{1+2MF} = \sqrt{1+2MF} \Rightarrow MF = -\tfrac{1}{2}$$

$$N_3 = 0$$

$$N_1^2 + N_2^2 = 1$$

$$\therefore \quad N_1 = \mp\frac{1}{\sqrt{5}}, \quad N_2 = \pm\frac{2}{\sqrt{5}}$$

$$\tan\theta = \frac{N_2}{N_1} = -2 \quad ; \quad \theta = \arctan(-2) = 116.56^\circ$$