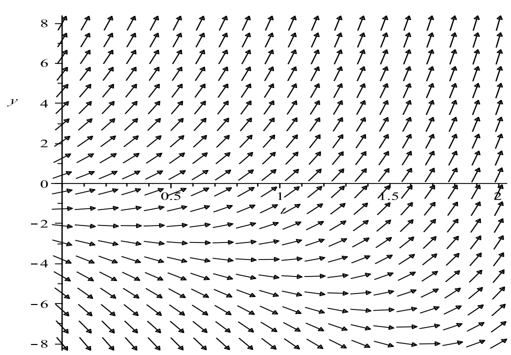
CHAPTER
2

First-Order Differential Equations

2.1

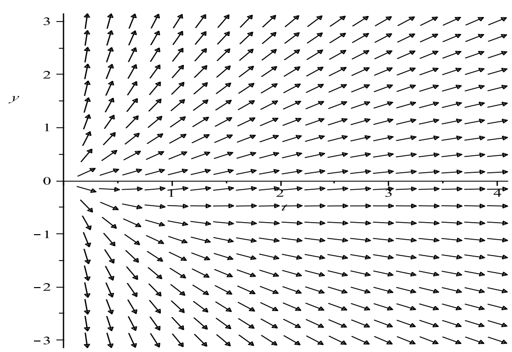
5.(a)



(b) If $y(0) > -3$, solutions eventually have positive slopes, and hence increase without bound. If $y(0) \le -3$, solutions have negative slopes and decrease without bound.

(c) The integrating factor is $\mu(t) = e^{-\int 2dt} = e^{-2t}$. The differential equation can be written as $e^{-2t}y' - 2e^{-2t}y = 3e^{-t}$, that is, $(e^{-2t}y)' = 3e^{-t}$. Integration of both sides of the equation results in the general solution $y(t) = -3e^t + c\,e^{2t}$. It follows that all solutions will increase exponentially if $c > 0$ and will decrease exponentially if $c \leq 0$. Letting $c = 0$ and then $t = 0$, we see that the boundary of these behaviors is at $y(0) = -3$.

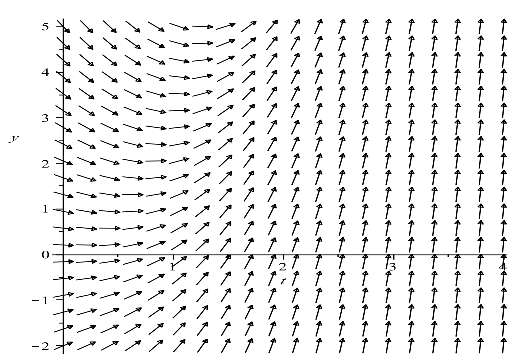
6.(a)



(b) For $y > 0$, the slopes are all positive, and hence the corresponding solutions increase without bound. For $y < 0$, almost all solutions have negative slopes, and hence solutions tend to decrease without bound.

(c) First divide both sides of the equation by t ($t > 0$). From the resulting standard form, the integrating factor is $\mu(t) = e^{-\int (1/t)\,dt} = 1/t$. The differential equation can be written as $y'/t - y/t^2 = t\,e^{-t}$, that is, $(y/t)' = t\,e^{-t}$. Integration leads to the general solution $y(t) = -te^{-t} + c\,t$. For $c \neq 0$, solutions diverge, as implied by the direction field. For the case $c = 0$, the specific solution is $y(t) = -te^{-t}$, which evidently approaches zero as $t \to \infty$.

8.(a)



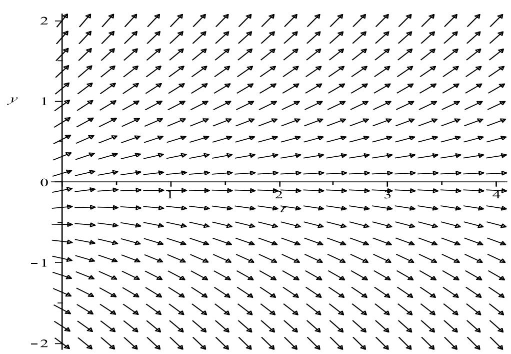
(b) All solutions eventually have positive slopes, and hence increase without bound.

(c) The integrating factor is $\mu(t) = e^{t/2}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t^2/2$, that is, $(e^{t/2}\,y/2)' = 3t^2/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t^2 - 12t + 24 + c\,e^{-t/2}$. It follows that all solutions converge to the specific solution $3t^2 - 12t + 24$.

10. The integrating factor is $\mu(t) = e^{2t}$. After multiplying both sides by $\mu(t)$, the equation can be written as $(e^{2t}\,y)' = t$. Integrating both sides of the equation results in the general solution $y(t) = t^2 e^{-2t}/2 + c\,e^{-2t}$. Invoking the specified condition, we require that $e^{-2}/2 + c\,e^{-2} = 0$. Hence $c = -1/2$, and the solution to the initial value problem is $y(t) = (t^2 - 1)e^{-2t}/2$.

11. The integrating factor is $\mu(t) = e^{\int (2/t)\,dt} = t^2$. Multiplying both sides by $\mu(t)$, the equation can be written as $(t^2\,y)' = \cos t$. Integrating both sides of the equation results in the general solution $y(t) = \sin t/t^2 + c\,t^{-2}$. Substituting $t = \pi$ and setting the value equal to zero gives $c = 0$. Hence the specific solution is $y(t) = \sin t/t^2$.

14.(a)

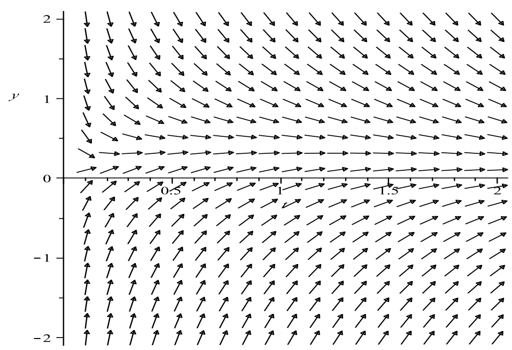


Solutions appear to grow infinitely large in absolute value, with signs depending on the initial value $y(0) = a_0$. The direction field appears horizontal for $a_0 \approx -1/8$.

(b) Dividing both sides of the given equation by 3, the integrating factor is $\mu(t) = e^{-2t/3}$. Multiplying both sides of the original differential equation by $\mu(t)$ and integrating results in $y(t) = (2\,e^{2t/3} - 2\,e^{-\pi t/2} + a(4 + 3\pi)\,e^{2t/3})/(4 + 3\pi)$. The qualitative behavior of the solution is determined by the terms containing $e^{2t/3}$: $2\,e^{2t/3} + a(4 + 3\pi)\,e^{2t/3}$. The nature of the solutions will change when $2 + a(4 + 3\pi) = 0$. Thus the critical initial value is $a_0 = -2/(4 + 3\pi)$.

(c) In addition to the behavior described in part (a), when $y(0) = -2/(4 + 3\pi)$, the solution is $y(t) = (-2\,e^{-\pi t/2})/(4 + 3\pi)$, and that specific solution will converge to $y = 0$.

15.(a)



As $t \to 0$, solutions increase without bound if $y(1) = a > 0.4$, and solutions decrease without bound if $y(1) = a < 0.4$.

(b) The integrating factor is $\mu(t) = e^{\int (t+1)/t\, dt} = t\, e^t$. The general solution of the differential equation is $y(t) = t\, e^{-t} + c\, e^{-t}/t$. Since $y(1) = a$, we have that $1 + c = ae$. That is, $c = ae - 1$. Hence the solution can also be expressed as $y(t) = t\, e^{-t} + (ae - 1)\, e^{-t}/t$. For small values of t, the second term is dominant. Setting $ae - 1 = 0$, the critical value of the parameter is $a_0 = 1/e$.

(c) When $a = 1/e$, the solution is $y(t) = t\, e^{-t}$, which approaches 0 as $t \to 0$.

17. The integrating factor is $\mu(t) = e^{\int (1/2)\, dt} = e^{t/2}$. Therefore the general solution is $y(t) = (4\cos t + 8\sin t)/5 + c\, e^{-t/2}$. Invoking the initial condition, the specific solution is $y(t) = (4\cos t + 8\sin t - 9\, e^{-t/2})/5$. Differentiating, it follows that $y'(t) = (-4\sin t + 8\cos t + 4.5\, e^{-t/2})/5$ and $y''(t) = (-4\cos t - 8\sin t - 2.25\, e^{-t/2})/5$. Setting $y'(t) = 0$, the first solution is $t_1 = 1.3643$, which gives the location of the first stationary point. Since $y''(t_1) < 0$, the first stationary point in a local maximum. The coordinates of the point are $(1.3643\,, 0.82008)$.

18. The integrating factor is $\mu(t) = e^{\int (2/3)\, dt} = e^{2t/3}$, and the differential equation can be written as $(e^{2t/3}\, y)' = e^{2t/3} - t\, e^{2t/3}/2$. The general solution is $y(t) = (21 - 6t)/8 + c\, e^{-2t/3}$. Imposing the initial condition, we have $y(t) = (21 - 6t)/8 + (y_0 - 21/8)e^{-2t/3}$. Since the solution is smooth, the desired intersection will be a point of tangency. Taking the derivative, $y'(t) = -3/4 - (2y_0 - 21/4)e^{-2t/3}/3$. Setting $y'(t) = 0$, the solution is $t_1 = (3/2)\ln\,[(21 - 8y_0)/9]$. Substituting into the solution, the respective value at the stationary point is $y(t_1) = 3/2 + (9/4)\ln 3 - (9/8)\ln(21 - 8y_0)$. Setting this result equal to zero, we obtain the required initial value $y_0 = (21 - 9\, e^{4/3})/8 \approx -1.643$.

19.(a) The integrating factor is $\mu(t) = e^{t/4}$, and the differential equation can be written as $(e^{t/4}\, y)' = 3\, e^{t/4} + 2\, e^{t/4}\cos 2t$. After integration, we get that the general solution is $y(t) = 12 + (8\cos 2t + 64\sin 2t)/65 + ce^{-t/4}$. Invoking the initial condition, $y(0) = 0$, the specific solution is $y(t) = 12 + (8\cos 2t + 64\sin 2t - 788\, e^{-t/4})/65$. As $t \to \infty$, the exponential term will decay, and the solution will oscillate about an average value of 12, with an amplitude of $8/\sqrt{65}$.

(b) Solving $y(t) = 12$, we obtain the desired value $t \approx 10.0658$.

21. The integrating factor is $\mu(t) = e^{-3t/2}$, and the differential equation can be written as $(e^{-3t/2}\, y)' = 3t\, e^{-3t/2} + 2\, e^{-t/2}$. The general solution is $y(t) = -2t - 4/3 - 4\, e^{t} + c\, e^{3t/2}$. Imposing the initial condition, $y(t) = -2t - 4/3 - 4\, e^{t} + (y_0 + 16/3)\, e^{3t/2}$. Now as $t \to \infty$, the term containing $e^{3t/2}$ will dominate the solution. Its sign will determine the divergence properties. Hence the critical value of the initial condition is $y_0 = -16/3$. The corresponding solution, $y(t) = -2t - 4/3 - 4\, e^{t}$, will also decrease without bound.

Note on Problems 24-27:

Let $g(t)$ be given, and consider the function $y(t) = y_1(t) + g(t)$, in which $y_1(t) \to 0$ as $t \to \infty$. Differentiating, $y'(t) = y_1'(t) + g'(t)$. Letting a be a constant, it follows that $y'(t) + ay(t) = y_1'(t) + ay_1(t) + g'(t) + ag(t)$. Note that the hypothesis on the function $y_1(t)$ will be satisfied, if $y_1'(t) + ay_1(t) = 0$. That is, $y_1(t) = c\, e^{-at}$. Hence $y(t) = c\, e^{-at} + g(t)$, which is a solution of the equation $y' + ay = g'(t) + ag(t)$. For convenience, choose $a = 1$.

24. Here $g(t) = 3$, and we consider the linear equation $y' + y = 3$. The integrating factor is $\mu(t) = e^{t}$, and the differential equation can be written as $(e^{t}\, y)' = 3e^{t}$. The general solution is $y(t) = 3 + c\, e^{-t}$.

26. Here $g(t) = 2t - 5$. Consider the linear equation $y' + y = 2 + 2t - 5$. The integrating factor is $\mu(t) = e^{t}$, and the differential equation can be written as $(e^{t}\, y)' = (2t - 3)e^{t}$. The general solution is $y(t) = 2t - 5 + c\, e^{-t}$.

27. $g(t) = 4 - t^2$. Consider the linear equation $y' + y = 4 - 2t - t^2$ .The integrating factor is $\mu(t) = e^{t}$, and the equation can be written as $(e^{t}\, y)' = (4 - 2t - t^2)e^{t}$. The general solution is $y(t) = 4 - t^2 + c\, e^{-t}$.

28.(a) Differentiating y and using the fundamental theorem of calculus we obtain that $y' = Ae^{-\int p(t)dt} \cdot (-p(t))$, and then $y' + p(t)y = 0$.

(b) Differentiating y we obtain that

$$y' = A'(t)e^{-\int p(t)dt} + A(t)e^{-\int p(t)dt} \cdot (-p(t)).$$

If this satisfies the differential equation then

$$y' + p(t)y = A'(t)e^{-\int p(t)dt} = g(t)$$

and the required condition follows.

(c) Let us denote $\mu(t) = e^{\int p(t)dt}$. Then clearly $A(t) = \int \mu(t)g(t)dt$, and after substitution $y = \int \mu(t)g(t)dt \cdot (1/\mu(t))$, which is just Eq. (33).

30. We assume a solution of the form $y = A(t)e^{-\int (1/t)\,dt} = A(t)e^{-\ln t} = A(t)t^{-1}$, where $A(t)$ satisfies $A'(t) = 3t\cos 2t$. This implies that

$$A(t) = \frac{3\cos 2t}{4} + \frac{3t\sin 2t}{2} + c$$

and the solution is

$$y = \frac{3\cos 2t}{4t} + \frac{3\sin 2t}{2} + \frac{c}{t}.$$

2.2

Problems 1 through 16 follow the pattern of the examples worked in this section. The first eight problems, however, do not have an initial condition, so the integration constant c cannot be found.

2. The differential equation may be written as $y^{-2}dy = -\sin x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-1} = \cos x + c$. That is, $(c - \cos x)y = 1$, in which c is an arbitrary constant. Solving for the dependent variable, explicitly, $y(x) = 1/(c - \cos x)$.

3. Write the differential equation as $\cos^{-2} 2y\,dy = \cos^2 x\,dx$, which also can be written as $\sec^2 2y\,dy = \cos^2 x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $\tan 2y = \sin x \cos x + x + c$.

5. The differential equation may be written as $(y + e^y)dy = (x - e^{-x})dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $y^2 + 2\,e^y = x^2 + 2\,e^{-x} + c$.

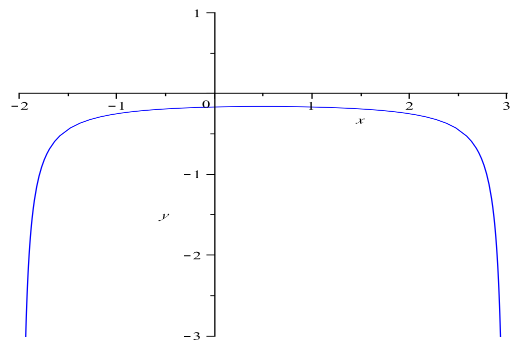
6. Write the differential equation as $(1 + y^2)dy = x^2\,dx$. Integrating both sides of the equation, we obtain the relation $y + y^3/3 = x^3/3 + c$.

7. Write the differential equation as $y^{-1}\,dy = x^{-1}\,dx$. Integrating both sides of the equation, we obtain the relation $\ln|y| = \ln|x| + c$. Solving for y explicitly gives $y(x) = kx$. Note that k may be positive or negative due to the absolute values in the integrated equation.

8. Write the differential equation as $y\,dy = -x\,dx$. Integrating both sides of the equation, we obtain the relation $(1/2)y^2 = -(1/2)x^2 + c$. The explicit form of the solution is $y(x) = \pm\sqrt{x^2 + c}$. The initial condition would then be used to determine whether the positive of negative solution is to be used for a specific initial value problem.

9.(a) The differential equation is separable, with $y^{-2}dy = (1 - 2x)dx$. Integration yields $-y^{-1} = x - x^2 + c$. Substituting $x = 0$ and $y = -1/6$, we find that $c = 6$. Hence the specific solution is $y = 1/(x^2 - x - 6)$.

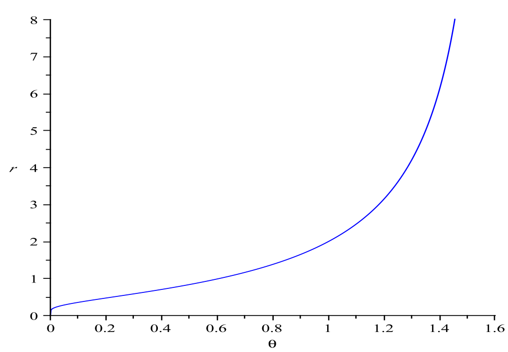
(b)



(c) Note that $x^2 - x - 6 = (x+2)(x-3)$. Hence the solution becomes singular at $x = -2$ and $x = 3$, so the interval of existence is $(-2, 3)$.

12.(a) Write the differential equation as $r^{-2}dr = \theta^{-1}\,d\theta$. Integrating both sides of the equation results in the relation $-r^{-1} = \ln\,\theta + c$. Imposing the condition $r(1) = 2$, we obtain $c = -1/2$. The explicit form of the solution is $r = 2/(1 - 2\,\ln\,\theta)$.

(b)



(c) Clearly, the solution makes sense only if $\theta > 0$. Furthermore, the solution becomes singular when $\ln\,\theta = 1/2$, that is, $\theta = \sqrt{e}$.

18. The differential equation can be written as $(3y^2 - 4)dy = 3x^2dx$. Integrating both sides, we obtain $y^3 - 4y = x^3 + c$. Imposing the initial condition, the specific solution is $y^3 - 4y = x^3 - 1$. Referring back to the differential equation, we find that $y' \to \infty$ as $y \to \pm 2/\sqrt{3}$. The respective values of the abscissas are $x \approx -1.276$, 1.598. Hence the solution is valid for $-1.276 < x < 1.598$.

22.(a) Write the differential equation as $y^{-1}(4-y)^{-1}dy = t(1+t)^{-1}dt$. Integrating both sides of the equation, we obtain $\ln|y| - \ln|y-4| = 4t - 4\ln|1+t| + c$. Taking the exponential of both sides $|y/(y-4)| = c\,e^{4t}/(1+t)^4$. It follows that as $t \to \infty$, $|y/(y-4)| = |1 + 4/(y-4)| \to \infty$. That is, $y(t) \to 4$.

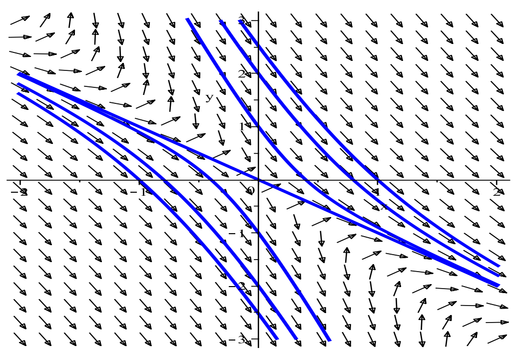
(b) Setting $y(0) = 2$, we obtain that $c = 1$. Based on the initial condition, the solution may be expressed as $y/(y-4) = -e^{4t}/(1+t)^4$. Note that $y/(y-4) < 0$, for all $t \geq 0$. Hence $y < 4$ for all $t \geq 0$. Referring back to the differential equation, it follows that y' is always positive. This means that the solution is monotone increasing. We find that the root of the equation $e^{4t}/(1+t)^4 = 399$ is near $t = 2.844$.

(c) Note the $y(t) = 4$ is an equilibrium solution. Examining the local direction field we see that if $y(0) > 0$, then the corresponding solutions converge to $y = 4$. Referring back to part (a), we have $y/(y-4) = [y_0/(y_0-4)]\, e^{4t}/(1+t)^4$, for $y_0 \neq 4$. Setting $t = 2$, we obtain $y_0/(y_0-4) = (3/e^2)^4 y(2)/(y(2)-4)$. Now since the function $f(y) = y/(y-4)$ is monotone for $y < 4$ and $y > 4$, we need only solve the equations $y_0/(y_0-4) = -399(3/e^2)^4$ and $y_0/(y_0-4) = 401(3/e^2)^4$. The respective solutions are $y_0 = 3.6622$ and $y_0 = 4.4042$.

29.(a) Observe that $-(4x+3y)/(2x+y) = -2 - (y/x)\left[2 + (y/x)\right]^{-1}$. Hence the differential equation is homogeneous.

(b) The substitution $y = x\,v$ results in $v + x\,v' = -2 - v/(2+v)$. The transformed equation is $v' = -(v^2+5v+4)/(2+v)x$. This equation is separable, with general solution $(v+4)^2\,|v+1| = c/x^3$. In terms of the original dependent variable, the solution is $(4x+y)^2\,|x+y| = c$.

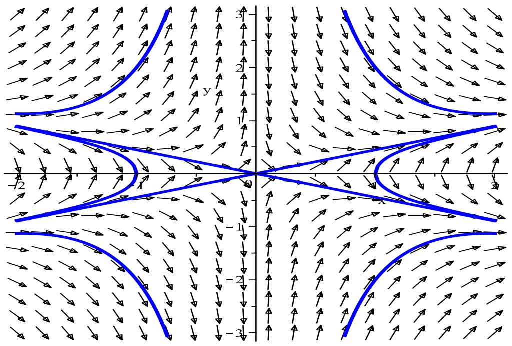
(c) The integral curves are symmetric with respect to the origin.



30.(a) The differential equation can be expressed as $y' = (1/2)(y/x)^{-1} - (3/2)(y/x)$. Hence the equation is homogeneous. The substitution $y = xv$ results in $xv' = (1-5v^2)/2v$. Separating variables, we have $2vdv/(1-5v^2) = dx/x$.

(b) Integrating both sides of the transformed equation yields $-(\ln|1-5v^2|)/5 = \ln|x| + c$, that is, $1 - 5v^2 = c/\,|x|^5$. In terms of the original dependent variable, the general solution is $5y^2 = x^2 - c/\,|x|^3$.

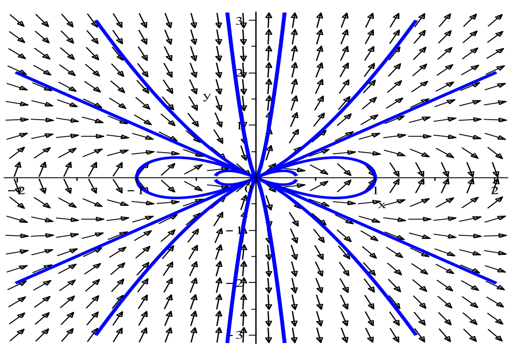
(c) The integral curves are symmetric with respect to the origin.



31.(a) The differential equation can be expressed as $y' = (3/2)(y/x) - (1/2)(y/x)^{-1}$. Hence the equation is homogeneous. The substitution $y = x\,v$ results in $x\,v' = (v^2 - 1)/2v$, that is, $2vdv/(v^2 - 1) = dx/x$.

(b) Integrating both sides of the transformed equation yields $\ln\left|v^2 - 1\right| = \ln|x| + c$, that is, $v^2 - 1 = c\,|x|$. In terms of the original dependent variable, the general solution is $y^2 = c\,x^2\,|x| + x^2$.

(c) The integral curves are symmetric with respect to the origin.



2.3

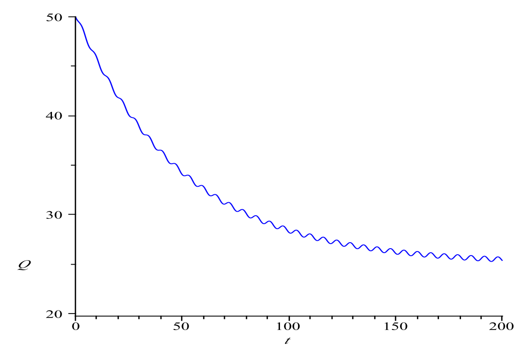
1. Let $Q(t)$ be the amount of dye in the tank at time t. Clearly, $Q(0) = 200$ g. The differential equation governing the amount of dye is $Q'(t) = -2Q(t)/200$. The solution of this separable equation is $Q(t) = Q(0)e^{-t/100} = 200e^{-t/100}$. We need the time T such that $Q(T) = 2$ g. This means we have to solve $2 = 200e^{-T/100}$ and we obtain that $T = -100\ln(1/100) = 100\ln 100 \approx 460.5$ min.

3.(a) Let Q be the amount of salt in the tank. Salt enters the tank of water at a rate of $2\,(1/4)(1 + (1/2)\sin\,t) = 1/2 + (1/4)\sin\,t$ oz/min. It leaves the tank at a rate of $2\,Q/100$ oz/min. Hence the differential equation governing the amount of salt at any time is

$$\frac{dQ}{dt} = \frac{1}{2} + \frac{1}{4}\sin t - \frac{Q}{50}\,.$$

The initial amount of salt is $Q_0 = 50$ oz. The governing differential equation is linear, with integrating factor $\mu(t) = e^{t/50}$. Write the equation as $(e^{t/50}Q)' = e^{t/50}(1/2 + (1/4)\sin t)$. The specific solution is $Q(t) = 25 + (12.5\sin t - 625\cos t + 63150\,e^{-t/50})/2501$ oz.

(b)



(c) The amount of salt approaches a steady state, which is an oscillation of approximate amplitude 1/4 about a level of 25 oz.

4.(a) Using the Principle of Conservation of Energy, the speed v of a particle falling from a height h is given by

$$\frac{1}{2}mv^2 = mgh\,.$$

(b) The outflow rate is (outflow cross-section area)×(outflow velocity): $\alpha\, a\sqrt{2gh}\,$. At any instant, the volume of water in the tank is $V(h) = \int_0^h A(u)du$. The time rate of change of the volume is given by $dV/dt = (dV/dh)(dh/dt) = A(h)dh/dt$. Since the volume is decreasing, $dV/dt = -\alpha\, a\sqrt{2gh}\,$.

(c) With $A(h) = \pi$, $a = 0.01\,\pi$, $\alpha = 0.6$, the differential equation for the water level h is $\pi(dh/dt) = -0.006\,\pi\sqrt{2gh}$, with solution $h(t) = 0.000018gt^2 - 0.006\sqrt{2gh(0)}\,t + h(0)$. Setting $h(0) = 3$ and $g = 9.8$, $h(t) = 0.0001764\,t^2 - 0.046\,t + 3$, resulting in $h(t) = 0$ for $t \approx 130.4$ s.

5.(a) The equation governing the value of the investment is $dS/dt = r\,S$. The value of the investment, at any time, is given by $S(t) = S_0e^{rt}$. Setting $S(T) = 2S_0$, the required time is $T = \ln(2)/r$.

(b) For the case $r = .07$, $T \approx 9.9$ yr.

(c) Referring to part (a), $r = \ln(2)/T$. Setting $T = 8$, the required interest rate is to be approximately $r = 8.66\%$.

8.(a) Using Eq.(15) we have $dS/dt - 0.005S = -(800 + 10t)$, $S(0) = 150,000$. Using an integrating factor and integration by parts we obtain that $S(t) = 560,000 - 410,000e^{0.005t} + 2000t$. Setting $S(t) = 0$ and solving numerically for t yields $t = 146.54$ months.

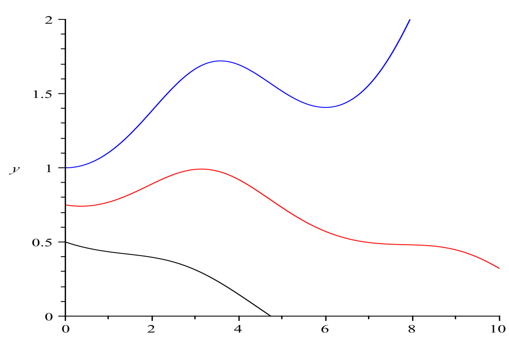
(b) The solution we obtained in part (a) with a general initial condition $S(0) = S_0$ is $S(t) = 560,000 - 560,000e^{0.005t} + S_0e^{0.005t} + 2000t$. Solving the equation $S(240) = 0$ yields $S_0 = 246,758$.

9.(a) Let $Q' = -r\,Q$. The general solution is $Q(t) = Q_0e^{-rt}$. Based on the definition of half-life, consider the equation $Q_0/2 = Q_0e^{-5730\,r}$. It follows that $-5730\,r = \ln(1/2)$, that is, $r = 1.2097 \times 10^{-4}$ per year.

(b) The amount of carbon-14 is given by $Q(t) = Q_0\,e^{-1.2097\times 10^{-4}t}$.

(c) Given that $Q(T) = Q_0/5$, we have the equation $1/5 = e^{-1.2097\times 10^{-4}T}$. Solving for the decay time, the apparent age of the remains is approximately $T = 13,305$ years.

11.(a) The differential equation $dy/dt = r(t)\,y - k$ is linear, with integrating factor $\mu(t) = e^{-\int r(t)dt}$. Write the equation as $(\mu\,y)' = -k\,\mu(t)$. Integration of both sides yields the general solution $y = \left[-k\int \mu(\tau)d\tau + y_0\,\mu(0)\right]/\mu(t)$. In this problem, the integrating factor is $\mu(t) = e^{(\cos\,t - t)/5}$.



(b) The population becomes extinct, if $y(t^*) = 0$, for some $t = t^*$. Referring to part (a), we find that $y(t^*) = 0$ when

$$\int_0^{t^*} e^{(\cos\,\tau - \tau)/5}d\tau = 5\,e^{1/5}y_c.$$

It can be shown that the integral on the left hand side increases monotonically, from zero to a limiting value of approximately 5.0893. Hence extinction can happen only if $5\,e^{1/5}y_0 < 5.0893$. Solving $5e^{1/5}y_c = 5.0893$ yields $y_c = 0.8333$.

(c) Repeating the argument in part (b), it follows that $y(t^*) = 0$ when

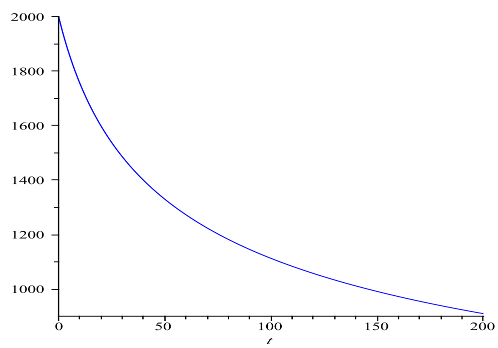
$$\int_0^{t^*} e^{(\cos\,\tau - \tau)/5} d\tau = \frac{1}{k}\, e^{1/5} y_c.$$

Hence extinction can happen only if $e^{1/5} y_0/k < 5.0893$, so $y_c = 4.1667\,k$.

(d) Evidently, y_c is a linear function of the parameter k .

13.(a) The solution of the governing equation satisfies $u^3 = u_0^3/(\,3\,\alpha\,u_0^3\,t + 1\,)$. With the given data, it follows that $u(t) = 2000/\sqrt[3]{6\,t/125 + 1}$.

(b)

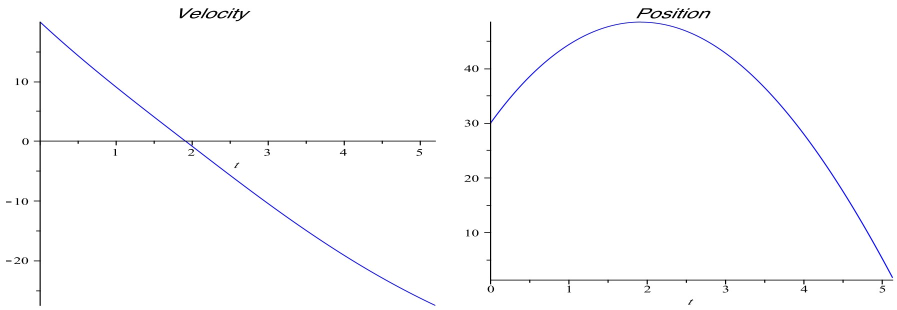


(c) Numerical evaluation results in $u(t) = 600$ for $t \approx 750.77$ s.

18.(a) The differential equation for the upward motion is $mdv/dt = -\mu v^2 - mg$, in which $\mu = 1/1325$. This equation is separable, with $m/(\mu\, v^2 + mg)\, dv = -dt$. Integrating both sides and invoking the initial condition, $v(t) = 44.133\, \tan(0.425 - 0.222\,t)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.916$ s . Integrating $v(t)$, the position is given by $x(t) = 198.75\, \ln[\cos(0.222\,t - 0.425)] + 48.57$. Therefore the maximum height is $x(t_1) = 48.56$ m.

(b) The differential equation for the downward motion is $m\,dv/dt = +\mu v^2 - mg$. This equation is also separable, with $m/(mg - \mu\, v^2)\, dv = -dt$. For convenience, set $t = 0$ at the top of the trajectory. The new initial condition becomes $v(0) = 0$. Integrating both sides and invoking the initial condition, we obtain $\ln((44.13 - v)/(44.13 + v)) = t/2.25$. Solving for the velocity, $v(t) = 44.13(1 - e^{t/2.25})/(1 + e^{t/2.25})$. Integrating $v(t)$, we obtain $x(t) = 99.29\,\ln(e^{t/2.25}/(1 + e^{t/2.25})^2) + 186.2$. To estimate the duration of the downward motion, set $x(t_2) = 0$, resulting in $t_2 = 3.276$ s. Hence the total time that the ball spends in the air is $t_1 + t_2 = 5.192$ s.

(c)



19.(a) Measure the positive direction of motion upward. The equation of motion is given by $mdv/dt = -k\,v - mg$. The initial value problem is $dv/dt = -kv/m - g$, with $v(0) = v_0$. The solution is $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. Setting $v(t_m) = 0$, the maximum height is reached at time $t_m = (m/k)\ln[(mg + k\,v_0)/mg]$. Integrating the velocity, the position of the body is

$$x(t) = -mg\,t/k + \left[(\frac{m}{k})^2 g + \frac{m\,v_0}{k}\right](1 - e^{-kt/m}).$$

Hence the maximum height reached is

$$x_m = x(t_m) = \frac{m\,v_0}{k} - g(\frac{m}{k})^2 \ln\left[\frac{mg + k\,v_0}{mg}\right].$$

(b) Recall that for $\delta \ll 1$, $\ln(1+\delta) = \delta - \delta^2/2 + \delta^3/3 - \delta^4/4 + \ldots$.

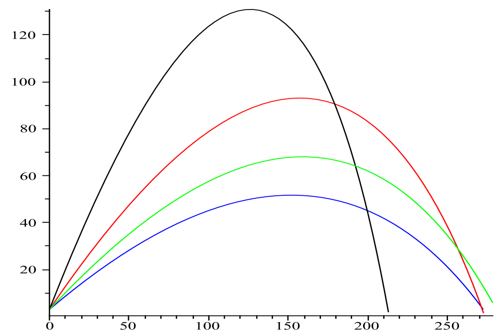
(c) The dimensions of the quantities involved are $[k] = MT^{-1}$, $[v_0] = LT^{-1}$, $[m] = M$ and $[g] = LT^{-2}$. This implies that kv_0/mg is dimensionless.

23.(a) Both equations are linear and separable. Initial conditions: $v(0) = u\cos A$ and $w(0) = u\sin A$. We obtain the solutions $v(t) = (u\cos A)e^{-rt}$ and $w(t) = -g/r + (u\sin A + g/r)e^{-rt}$.

(b) Integrating the solutions in part (a), and invoking the initial conditions, the coordinates are $x(t) = u\cos A(1 - e^{-rt})/r$ and

$$y(t) = -\frac{gt}{r} + \frac{g + ur\,\sin\,A + hr^2}{r^2} - (\frac{u}{r}\sin\,A + \frac{g}{r^2})e^{-rt}.$$

(c)



(d) Let T be the time that it takes the ball to go 350 ft horizontally. Then from above, $e^{-T/5} = (u\cos A - 70)/u\cos A$. At the same time, the height of the ball is given by

$$y(T) = -160T + 803 + 5u\sin A - \frac{(800 + 5u\sin A)(u\cos A - 70)}{u\cos A}.$$

Hence A and u must satisfy the equality

$$800\ln\left[\frac{u\cos A - 70}{u\cos A}\right] + 803 + 5u\sin A - \frac{(800 + 5u\sin A)(u\cos A - 70)}{u\cos A} = 10$$

for the ball to touch the top of the wall. To find the optimal values for u and A, consider u as a function of A and use implicit differentiation in the above equation to find that

$$\frac{du}{dA} = -\frac{u(u^2\cos A - 70u - 11200\sin A)}{11200\cos A}.$$

Solving this equation simultaneously with the above equation yields optimal values for u and A: $u \approx 145.3\,\text{ft/s}$, $A \approx 0.644\,\text{rad}$.

24.(a) Solving equation (i), $y'(x) = \left[(k^2 - y)/y\right]^{1/2}$. The positive answer is chosen, since y is an increasing function of x.

(b) Let $y = k^2\sin^2 t$. Then $dy = 2k^2\sin t\cos t dt$. Substituting into the equation in part (a), we find that

$$\frac{2k^2\sin t\cos t dt}{dx} = \frac{\cos t}{\sin t}.$$

Hence $2k^2\sin^2 t dt = dx$.

(c) Setting $\theta = 2t$, we further obtain $k^2\sin^2(\theta/2)\,d\theta = dx$. Integrating both sides of the equation and noting that $t = \theta = 0$ corresponds to the origin, we obtain the solutions $x(\theta) = k^2(\theta - \sin\theta)/2$ and (from part (b)) $y(\theta) = k^2(1 - \cos\theta)/2$.

(d) Note that $y/x = (1 - \cos\theta)/(\theta - \sin\theta)$. Setting $x = 1$, $y = 2$, the solution of the equation $(1 - \cos\theta)/(\theta - \sin\theta) = 2$ is $\theta \approx 1.401$. Substitution into either of the expressions yields $k \approx 2.193$.

2.4

2. The function $\tan t$ is discontinuous at odd multiples of $\pi/2$. Since $\pi/2 < \pi < 3\pi/2$, the initial value problem has a unique solution on the interval $(\pi/2, 3\pi/2)$.

4. The function $\ln t$ is defined and continuous on the interval $(0\,,\infty)$. At $t = 1$, $\ln t = 0$, so the normal form of the differential equation has a singularity there. Also, $\cot t$ is not defined at integer multiples of π, so the initial value problem will have a solution on the interval $(1, \pi)$.

6. The function $f(t\,,y)$ is discontinuous along the coordinate axes, and on the hyperbola $t^2 - y^2 = 1$. Furthermore,

$$\frac{\partial f}{\partial y} = \frac{\pm 1}{y(1-t^2+y^2)} - 2\frac{y\,\ln|ty|}{(1-t^2+y^2)^2}$$

has the same points of discontinuity.

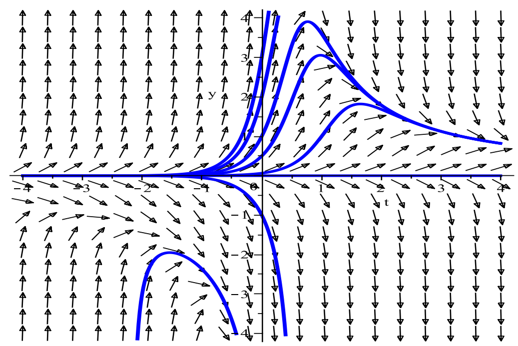
7. $f(t\,,y)$ is continuous everywhere on the plane. The partial derivative $\partial f/\partial y$ is also continuous everywhere.

10. The equation is separable, with $dy/y^2 = 2t\,dt$. Integrating both sides, the solution is given by $y(t) = y_0/(1-y_0t^2)$. For $y_0 > 0$, solutions exist as long as $t^2 < 1/y_0$. For $y_0 \le 0$, solutions are defined for all t.

11. The equation is separable, with $dy/y^3 = -\,dt$. Integrating both sides and invoking the initial condition, $y(t) = y_0/\sqrt{2y_0^2t+1}$. Solutions exist as long as $2y_0^2t+1 > 0$, that is, $2y_0^2t > -1$. If $y_0 \ne 0$, solutions exist for $t > -1/2y_0^2$. If $y_0 = 0$, then the solution $y(t) = 0$ exists for all t.

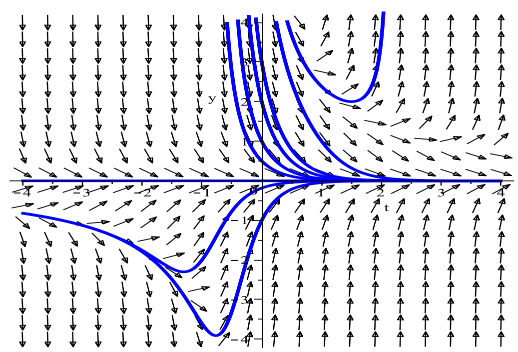
12. The function $f(t\,,y)$ is discontinuous along the straight lines $t = -1$ and $y = 0$. The partial derivative $\partial f/\partial y$ is discontinuous along the same lines. The equation is separable, with $y\,dy = t^2\,dt/(1+t^3)$. Integrating and invoking the initial condition, the solution is $y(t) = \left[(2/3)\ln\left|1+t^3\right| + y_0^2\right]^{1/2}$. Solutions exist as long as $(2/3)\ln\left|1+t^3\right| + y_0^2 \ge 0$, that is, $y_0^2 \ge -(2/3)\ln\left|1+t^3\right|$. For all y_0 (it can be verified that $y_0 = 0$ yields a valid solution, even though Theorem 2.4.2 does not guarantee one), solutions exist as long as $\left|1+t^3\right| \ge e^{-3y_0^2/2}$. From above, we must have $t > -1$. Hence the inequality may be written as $t^3 \ge e^{-3y_0^2/2} - 1$. It follows that the solutions are valid for $(e^{-3y_0^2/2} - 1)^{1/3} < t < \infty$.

14.



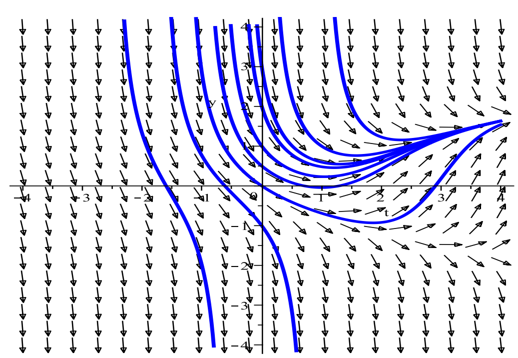
Based on the direction field, and the differential equation, for $y_0 < 0$, the slopes eventually become negative, and hence solutions tend to $-\infty$. For $y_0 > 0$, solutions increase without bound if $t_0 < 0$. Otherwise, the slopes eventually become negative, and solutions tend to zero. Furthermore, $y_0 = 0$ is an equilibrium solution. Note that slopes are zero along the curves $y = 0$ and $ty = 3$.

15.



For initial conditions (t_0, y_0) satisfying $ty < 3$, the respective solutions all tend to zero . For $y_0 \leq 9$, the solutions tend to 0; for $y_0 > 9$, the solutions tend to ∞. Also, $y_0 = 0$ is an equilibrium solution.

16.



Solutions with $t_0 < 0$ all tend to $-\infty$. Solutions with initial conditions (t_0, y_0) to the right of the parabola $t = 1 + y^2$ asymptotically approach the parabola as $t \to \infty$. Integral curves with initial conditions above the parabola (and $y_0 > 0$) also approach the curve. The slopes for solutions with initial conditions below the parabola (and $y_0 < 0$) are all negative. These solutions tend to $-\infty$.

17.(a) No. There is no value of $t_0 \geq 0$ for which $(2/3)(t - t_0)^{2/3}$ satisfies the condition $y(1) = 1$.

(b) Yes. Let $t_0 = 1/2$ in Eq.(19).

(c) For $t_0 > 0$, $|y(2)| \leq (4/3)^{3/2} \approx 1.54$.

20. The assumption is $\phi'(t) + p(t)\phi(t) = 0$. But then $c\phi'(t) + p(t)c\phi(t) = 0$ as well.

22.(a) Recalling Eq.(33) in Section 2.1,

$$y = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)g(s)\,ds + \frac{c}{\mu(t)}.$$

It is evident that $y_1(t) = 1/\mu(t)$ and $y_2(t) = (1/\mu(t)) \int_{t_0}^{t} \mu(s)g(s)\,ds$.

(b) By definition, $1/\mu(t) = e^{-\int p(t)dt}$. Hence $y_1' = -p(t)/\mu(t) = -p(t)y_1$. That is, $y_1' + p(t)y_1 = 0$.

(c) $y_2' = (-p(t)/\mu(t)) \int_0^t \mu(s)g(s)\,ds + \mu(t)g(t)/\mu(t) = -p(t)y_2 + g(t)$. This implies that $y_2' + p(t)y_2 = g(t)$.

25. Since $n = 3$, set $v = y^{-2}$. It follows that $v' = -2y^{-3}y'$ and $y' = -(y^3/2)v'$. Substitution into the differential equation yields $-(y^3/2)v' - \varepsilon y = -\sigma y^3$, which further results in $v' + 2\varepsilon v = 2\sigma$. The latter differential equation is linear, and can be written as $(ve^{2\varepsilon t})' = 2\sigma e^{2\varepsilon t}$. The solution is given by $v(t) = \sigma/\varepsilon + ce^{-2\varepsilon t}$. Converting back to the original dependent variable, $y = \pm v^{-1/2} = \pm(\sigma/\varepsilon + ce^{-2\varepsilon t})^{-1/2}$.

27. The solution of the initial value problem $y_1' + 2y_1 = 0$, $y_1(0) = 1$ is $y_1(t) = e^{-2t}$. Therefore $y(1^-) = y_1(1) = e^{-2}$. On the interval $(1, \infty)$, the differential equation is $y_2' + y_2 = 0$, with $y_2(t) = ce^{-t}$. Therefore $y(1^+) = y_2(1) = ce^{-1}$. Equating the limits $y(1^-) = y(1^+)$, we require that $c = e^{-1}$. Hence the global solution of the initial value problem is

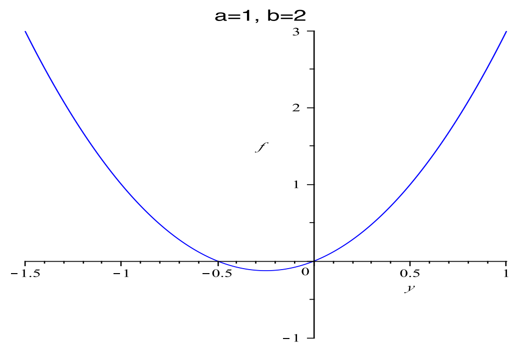
$$y(t) = \begin{cases} e^{-2t}, & 0 \leq t \leq 1 \\ e^{-1-t}, & t > 1 \end{cases}.$$

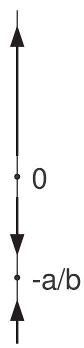
Note the discontinuity of the derivative

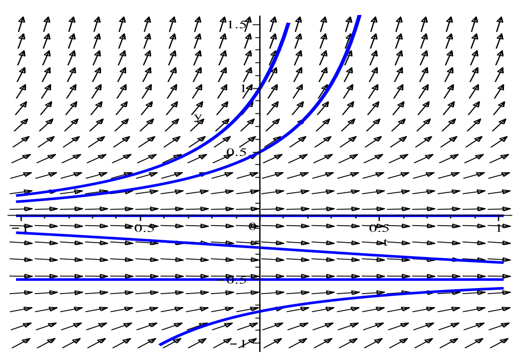
$$y'(t) = \begin{cases} -2e^{-2t}, & 0 < t < 1 \\ -e^{-1-t}, & t > 1 \end{cases}.$$

2.5

1.

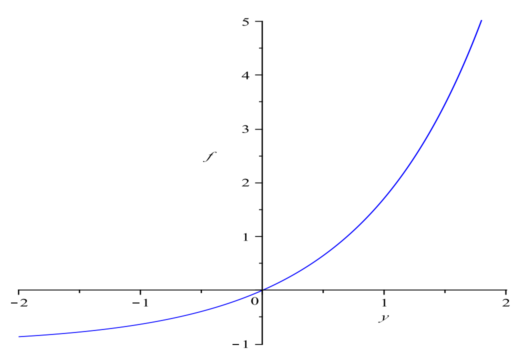


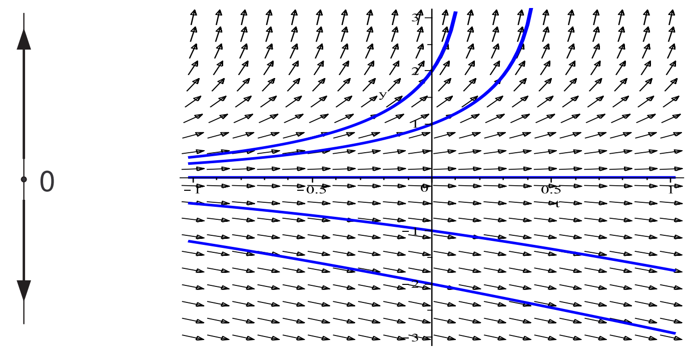




The equilibrium points are $y^* = -a/b$ and $y^* = 0$, and $y' > 0$ when $y > 0$ or $y < -a/b$, and $y' < 0$ when $-a/b < y < 0$, therefore the equilibrium solution $y = -a/b$ is asymptotically stable and the equilibrium solution $y = 0$ is unstable.

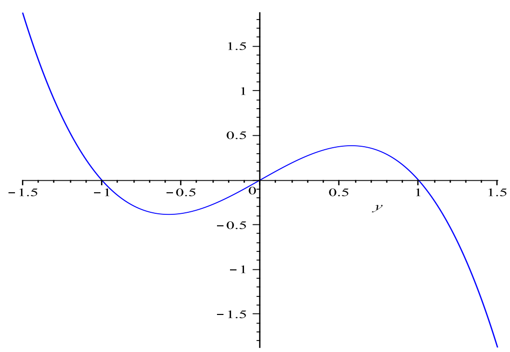
3.

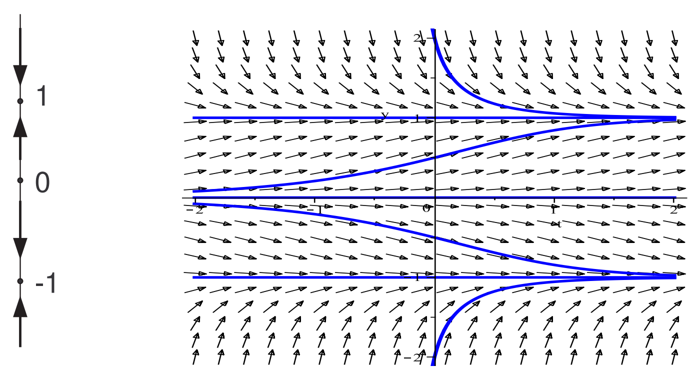




The only equilibrium point is $y^* = 0$, and $y' > 0$ when $y > 0$, $y' < 0$ when $y < 0$, hence the equilibrium solution $y = 0$ is unstable.

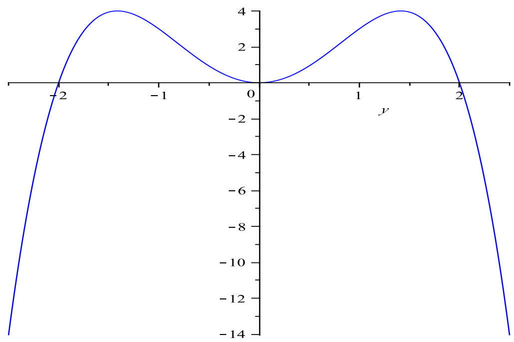
7.

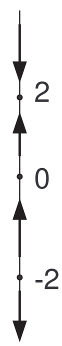


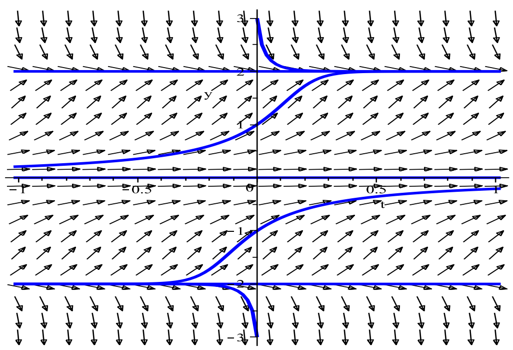


The equilibrium points are $y^* = 0, \pm 1$, and $y' > 0$ for $y < -1$ or $0 < y < 1$ and $y' < 0$ for $-1 < y < 0$ or $y > 1$. The equilibrium solution $y = 0$ is unstable, and the remaining two are asymptotically stable.

8.







The equilibrium points are $y^* = 0, \pm 2$, and $y' < 0$ when $y < -2$ or $y > 2$, and $y' > 0$ for $-2 < y < 0$ or $0 < y < 2$. The equilibrium solutions $y = -2$ and $y = 2$ are unstable and asymptotically stable, respectively. The equilibrium solution $y = 0$ is semistable.

9.