Chapter 3
Linear Programming I: Simplex Method

3.1 Minimize $F = -f = 2x_1 + x_2 - 5x_3$

subject to

$$x_1 - 2x_2 + x_3 + x_4 = 8$$
$$-3x_1 + 2x_2 + x_5 = 18$$
$$-2x_1 - x_2 + 2x_3 - x_6 = 4$$

x_1, x_2, x_3 unrestricted ; $x_4 \ge 0,\ x_5 \ge 0,\ x_6 \ge 0$

Define $x_i = x_i^+ - x_i^-$ with $x_i^+ \ge 0,\ x_i^- \ge 0$; $i = 1, 2, 3$

Final problem:

Minimize $f = 2x_1^+ - 2x_1^- + x_2^+ - x_2^- - 5x_3^+ + 5x_3^-$

subject to

$$x_1^+ - x_1^- - 2x_2^+ + 2x_2^- + x_3^+ - x_3^- + x_4 = 8$$
$$-3x_1^+ + 3x_1^- + 2x_2^+ - 2x_2^- + x_5 = 18$$
$$-2x_1^+ + 2x_1^- - x_2^+ + x_2^- + 2x_3^+ - 2x_3^- - x_6 = 4$$
$$x_i^+ \ge 0,\ i = 1, 2, 3 ;\quad x_i^- \ge 0,\ i = 1, 2, 3$$
$$x_i \ge 0,\ i = 4, 5, 6$$

3.2 Minimize $F = -f = -x_1 + 8x_2$

subject to

$$3x_1 + 2x_2 - x_3 = 6$$
$$9x_1 + 7x_2 + x_4 = 108$$
$$-2x_1 + 5x_2 + x_5 = 35$$

Let $x_1 = x_1^+ - x_1^-$, $x_2 = x_2^+ - x_2^-$

$x_i^+ \geq 0$, $x_i^- \geq 0$, $i = 1, 2$

$x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$

Final problem:

Minimize $F = -x_1^+ + x_1^- + 8\,x_2^+ - 8\,x_2^-$

subject to

$$3x_1^+ - 3x_1^- + 2x_2^+ - 2x_2^- - x_3 = 6$$

$$9x_1^+ - 9x_1^- + 7x_2^+ - 7x_2^- + x_4 = 108$$

$$-2x_1^+ + 2x_1^- + 5x_2^+ - 5x_2^- + x_5 = 35$$

$x_i^+ \geq 0$, $i = 1, 2$; $x_i^- \geq 0$, $i = 1, 2$

$x_i \geq 0$, $i = 3, 4, 5$

3.3

$6x_1 - 2x_2 + 3x_3 = 11$ (E_1)

$4x_1 + 7x_2 + x_3 = 21$ (E_2)

$5x_1 + 8x_2 + 9x_3 = 48$ (E_3)

$(E_1) \div 6 \Rightarrow$ $x_1 - \frac{1}{3}x_2 + \frac{1}{2}x_3 = \frac{11}{6}$ (E_4)

$(E_2) - 4(E_4) \Rightarrow$ $\frac{25}{3}x_2 - x_3 = \frac{41}{3}$ (E_5)

$(E_2) \div 4 \Rightarrow$ $x_1 + \frac{7}{2}x_2 + \frac{1}{4}x_3 = \frac{21}{4}$ (E_6)

$(E_3) - 5(E_4) \Rightarrow$ $\frac{29}{3}x_2 + \frac{13}{2}x_3 = \frac{233}{6}$ (E_7)

$\frac{3}{25}(E_5) \Rightarrow$ $x_2 - \frac{3}{25}x_3 = \frac{41}{25}$ (E_8)

$(E_4) + \frac{1}{3}(E_8) \Rightarrow$ $x_1 + \frac{23}{50}x_3 = \frac{119}{50}$ (E_9)

$(E_7) - \frac{29}{3}(E_8) \Rightarrow$ $x_3 = 3$ (E_{10})

$(E_9) - \frac{23}{50}(E_{10}) \Rightarrow$ $x_1 = 1$ (E_{11})

$(E_8) + \frac{3}{25}(E_{10}) \Rightarrow$ $x_2 = 2$ (E_{12})

$\therefore$ $x_1 = 1$, $x_2 = 2$, $x_3 = 3$

3.4 Let reservoir capacity $= x_1$

and water released to irrigation district $= x_2$

Assume : Reservoir is filled by end of wet season and emptied by end of dry season.

Minimize $f = 5.4\,x_2 + 8.4\,x_2 + 25\,x_1 = 25\,x_1 + 13.8\,x_2$

subject to

$$4.5*10^6 + 1.2*10^6 - 0.5*10^6 - x_1 \geq 0.3\,x_2 \quad \text{(wet season)}$$

or $$5.2*10^6 - x_1 - 0.3\,x_2 \geq 0$$

$$1.1*10^6 + 0.3*10^6 - 0.2*10^6 + x_1 \geq 0.7\,x_2 \quad \text{(summer season)}$$

or $$1.2*10^6 + x_1 - 0.7\,x_2 \geq 0$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad 4.5*10^6 \leq x_1$$

3.5

$$4x_1 - 7x_2 + 2x_3 = -8 \qquad (\mathrm{I}_0)$$

$$3x_1 + 4x_2 - 5x_3 = -8 \qquad (\mathrm{II}_0)$$

$$5x_1 + x_2 - 8x_3 = -34 \qquad (\mathrm{III}_0)$$

$$x_1 - \frac{7}{4}x_2 + \frac{1}{2}x_3 = -2 \qquad (\mathrm{I}_1) = \frac{1}{4}(\mathrm{I}_0)$$

$$\frac{37}{4}x_2 - \frac{13}{2}x_3 = -2 \qquad (\mathrm{II}_1) = (\mathrm{II}_0) - 3(\mathrm{I}_1)$$

$$\frac{39}{4}x_2 - \frac{21}{2}x_3 = -24 \qquad (\mathrm{III}_1) = (\mathrm{III}_0) - 5(\mathrm{I}_1)$$

$$x_1 + 0\cdot x_2 - \frac{27}{37}x_3 = -\frac{88}{37} \qquad (\mathrm{I}_2) = (\mathrm{I}_1) + \frac{7}{4}(\mathrm{II}_2)$$

$$+ \quad x_2 - \frac{26}{37}x_3 = -\frac{8}{37} \qquad (\mathrm{II}_2) = \frac{4}{37}(\mathrm{II}_1)$$

$$-\frac{135}{37}x_3 = -\frac{810}{37} \qquad (\mathrm{III}_2) = (\mathrm{III}_1) - \frac{39}{4}(\mathrm{II}_2)$$

$$x_3 = 6 \qquad (\mathrm{III}_3) = -\frac{37}{135}(\mathrm{III}_2)$$

$$x_2 = 4 \qquad (\mathrm{II}_3) = (\mathrm{II}_2) + \frac{26}{37}(\mathrm{III}_3)$$

$$x_1 = 2 \qquad (\mathrm{I}_3) = (\mathrm{I}_2) + \frac{27}{37}(\mathrm{III}_3)$$

3.6

$$2x_1 + x_2 + x_3 = 9 \qquad (\mathrm{I}_0)$$

$$x_1 + x_2 + x_3 = 6 \qquad (\mathrm{II}_0)$$

$$2x_1 + 3x_2 + x_3 = 13 \qquad (\mathrm{III}_0)$$

$$x_1 = 3 \qquad (\mathrm{I}_1) = (\mathrm{I}_0) - (\mathrm{II}_0)$$

$$x_2 = 2 \qquad (\mathrm{II}_1) = \tfrac{1}{2}(\mathrm{III}_0 - \mathrm{I}_0)$$

$$x_1 + 3x_2 + x_3 = 10 \qquad (\mathrm{III}_1) = (\mathrm{III}_0) + (\mathrm{II}_0) - (\mathrm{I}_0)$$

$$x_3 = 1 \qquad (\mathrm{III}_2) = (\mathrm{III}_1) - (\mathrm{I}_1) - 3(\mathrm{II}_1)$$

$$\therefore \quad x_1 = 3, \quad x_2 = 2, \quad x_1 = 1$$

3.7

Maximize $f = 2x_1 + 6x_2$

subject to $g_1 = -x_1 + x_2 - 1 \le 0$

$g_2 = 2x_1 + x_2 - 2 \le 0$

$x_i \ge 0, \quad i = 1, 2$

$g_1 = 0$ at $(-1,0), (0,1), (1,2), (2,3)$

$g_2 = 0$ at $(-1,4), (0,2), (1,0), (2,-2)$

Contours of $f = 2x_1 + 6x_2$:

$f = -2$ at $(-1,0), (1, -\frac{2}{3}), (3, -\frac{4}{3})$

$f = 2$ at $(1,0)$

$f = 0$ at $(0,0), (3,-1)$

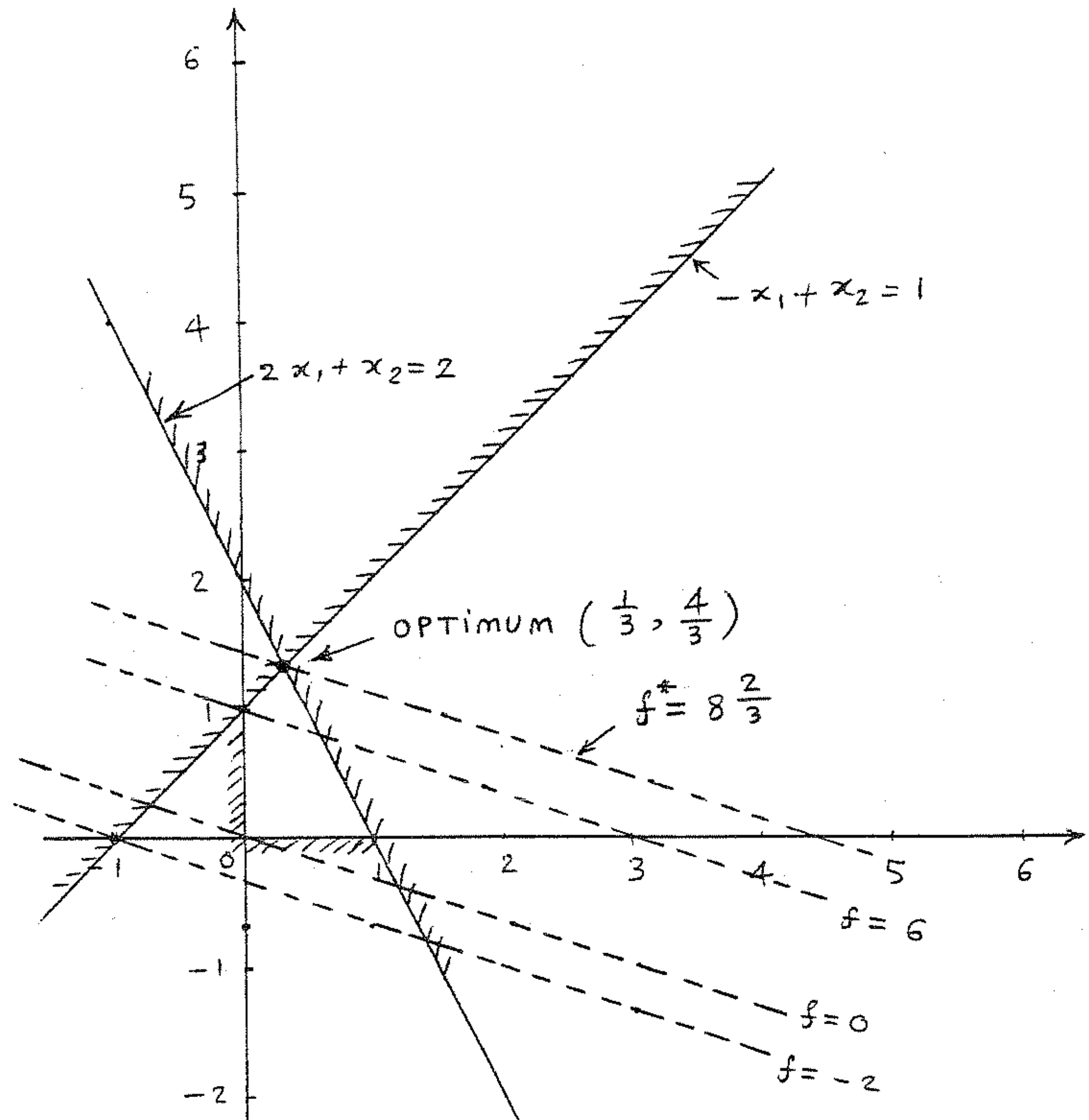
$f = 6$ at $(0,1), (3,0)$

$f = \frac{26}{3}$ at $(\frac{1}{3}, \frac{4}{3}), (2, \frac{7}{9})$

Graphical optimization gives:

$\vec{x}^* = (\frac{1}{3}, \frac{4}{3})$

$f^* = 8\frac{2}{3}$



$-x_1 + x_2 = 1$
$2x_1 + x_2 = 2$
OPTIMUM $\left(\frac{1}{3}, \frac{4}{3}\right)$
$f^* = 8\frac{2}{3}$
$f = 6$
$f = 0$
$f = -2$

3.8 Minimize $f = -3x_1 + 2x_2$

subject to $g_1 = x_1 + x_2 - 5 \le 0$

$0 \le x_1 \le 4$

$1 \le x_2 \le 6$

$g_1 = 0$ at $(0, 5), (5, 0)$

Contours of $f = -3x_1 + 2x_2$:

$f = 0$ at $(2, 3)$, $(0, 0)$

$f = -12$ at $(2, -3)$, $(5, 1\frac{1}{2})$, $(4, 0)$

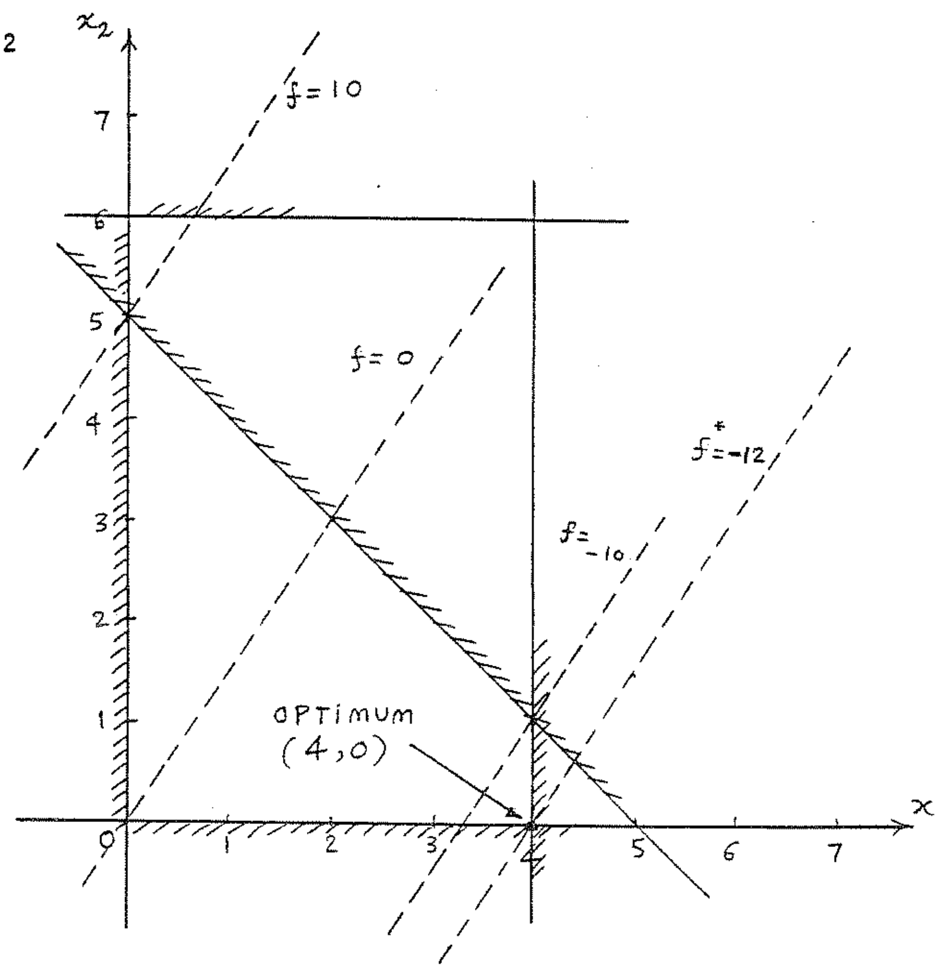
$f = 10$ at $(0, 5)$, $(2, 8)$

$f = -10$ at $(4, 1)$, $(2, -2)$

Graphical optimization gives:

$\vec{x}^* = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

with $f^* = -12$



3.9 Minimize $f = 3x_1 + 2x_2$ subject to $g_1 = 8x_1 + x_2 - 8 \geq 0$

$g_2 = 2x_1 + x_2 - 6 \geq 0$

$g_3 = x_1 + 3x_2 - 6 \geq 0$

$g_4 = x_1 + 6x_2 - 8 \geq 0$

$x_1 \geq 0, \quad x_2 \geq 0$

$g_1 = 0$ at $(0, 8), (1, 0)$

$g_2 = 0$ at $(0, 6), (3, 0)$

$g_3 = 0$ at $(0, 2), (6, 0)$

$g_4 = 0$ at $(0, 4/3), (8, 0)$

contours of $f = 3x_1 + 2x_2$:

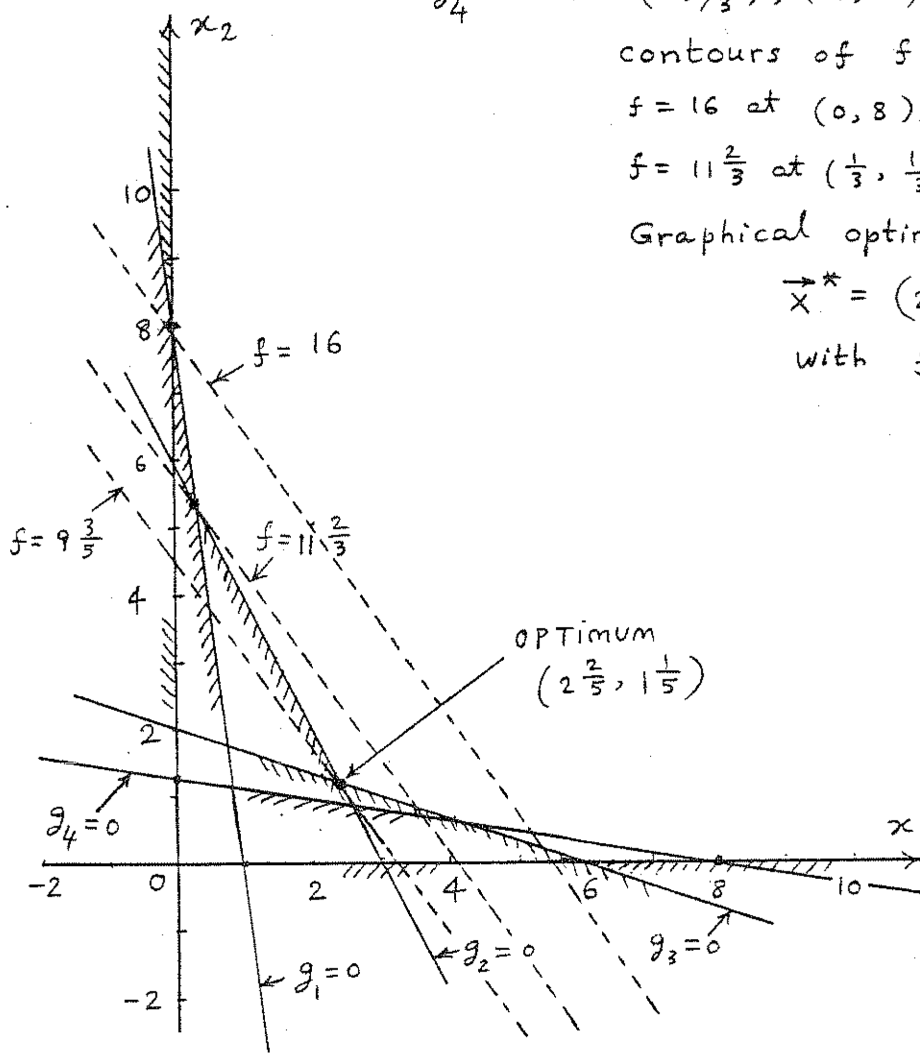
$f = 16$ at $(0, 8), (2, 5)$

$f = 11\frac{2}{3}$ at $(\frac{1}{3}, \frac{16}{3}), (4, -\frac{1}{6})$

Graphical optimization gives:

$\vec{X}^* = (2\frac{2}{5}, 1\frac{1}{5})$

with $f^* = 9\frac{3}{5}$.



3.10 Minimize $f = x_1^2 \, x_2^2$ (1)

subject to $x_1^3 \, x_2^2 \geq e^3$ (2)

$x_1 \, x_2^4 \geq e^4$ (3)

$x_1^2 \, x_2^3 \leq e$ (4)

$x_1 > 0, \; x_2 > 0$ (5)

Taking logarithms, Eqs. (1) to (5) become:

Minimize $F = 2x_3 + 2x_4$ (6)

subject to

$3x_3 + 2x_4 \geq 3$ (7) or $g_1 = 3x_3 + 2x_4 - 3 \geq 0$

$x_3 + 4x_4 \geq 4$ (8) or $g_2 = x_3 + 4x_4 - 4 \geq 0$

$2x_3 + 3x_4 \leq 1$ (9) or $g_3 = 2x_3 + 3x_4 - 1 \leq 0$

$x_3 \geq -\infty$ (10)

$x_4 \geq -\infty$ (11)

where $x_3 = \ln x_1, \quad x_4 = \ln x_2$ (12)

$g_1 = 0$ at $(x_3 = 0, \; x_4 = 1.5)$, $(1, 0)$, $(2, -1.5)$, $(3, -3)$

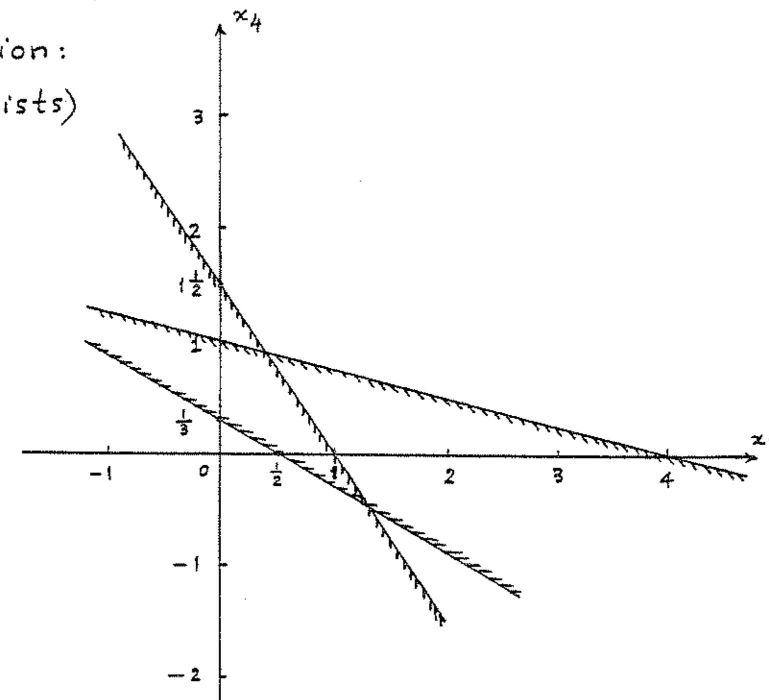
$g_2 = 0$ at $(x_3 = 0, \; x_4 = 1)$, $(4, 0)$

$g_3 = 0$ at $(x_3 = 0, \; x_4 = \frac{1}{3})$, $(0.5, 0)$, $(5, -3)$

Graphical optimization:

(No feasible point exists)

$\therefore$ No solution.



3.11

THEOREM 3.6 *Let S be a closed convex polyhedron. Then the minimum of a linear function over S is attained at an extreme point of S.*

Proof: Suppose $\mathbf{X}^*$ minimizes the objective function $\mathbf{c}^T\mathbf{X}$ over S and an equivalent minimum does not occur at an extreme point.

Then $$\mathbf{c}^T\mathbf{X} < \mathbf{c}^T\mathbf{X}_i^e, \quad i = 1, 2, \ldots, p$$

where $\mathbf{X}_i^e$ are the finite number of extreme points of the polyhedral set.

Thus, for $$0 < \lambda_i < 1,$$

$$\lambda_i \mathbf{c}^T\mathbf{X}^* < \lambda_i \mathbf{c}^T\mathbf{X}_i^e$$

and
$$\sum_{i=1}^{p} \lambda_i \mathbf{c}^T\mathbf{X}^* < \sum_{i=1}^{p} \lambda_i \mathbf{c}^T\mathbf{X}_i^e \tag{E$_1$}$$

Now choose $\lambda_i = \lambda_i^*$ so that

$$\mathbf{X}^* = \sum_{i=1}^{p} \lambda_i^* \mathbf{X}_i^e,\ \lambda_i^* \geqslant 0 \text{ and } \sum_{i=1}^{p} \lambda_i^* = 1$$

This is possible according to the Theorem 3.5. Thus the inequality (E$_1$) becomes

$$\sum_{i=1}^{p} \lambda_i \mathbf{c}^T\mathbf{X}^* = \mathbf{c}^T\mathbf{X}^* < \sum_{i=1}^{p} \lambda_i^* \mathbf{c}^T\mathbf{X}_i^e = \mathbf{c}^T \sum_{i=1}^{p} \lambda_i^* \mathbf{X}_i^e = \mathbf{c}^T\mathbf{X}^*$$

which is a contradiction.

3.12

Maximize $f = 6x + 7y$

subject to
$$g_1 = 7x + 6y - 42 \le 0$$
$$g_2 = 5x + 9y - 45 \le 0$$
$$g_3 = x - y - 4 \le 0$$
$$g_4 = -x \le 0$$
$$g_5 = -y \le 0$$

$g_1 = 0$ at $(0, 7)$, $(6, 0)$

$g_2 = 0$ at $(0, 5)$, $(9, 0)$

$g_3 = 0$ at $(0, -4)$, $(4, 0)$, $(7, 3)$

contours of $f = 6x + 7y$

$f = 24$ at $(4, 0)$ and $(0, 24/7)$

$f = 36$ at $(6, 0)$ and $(0, 36/7)$

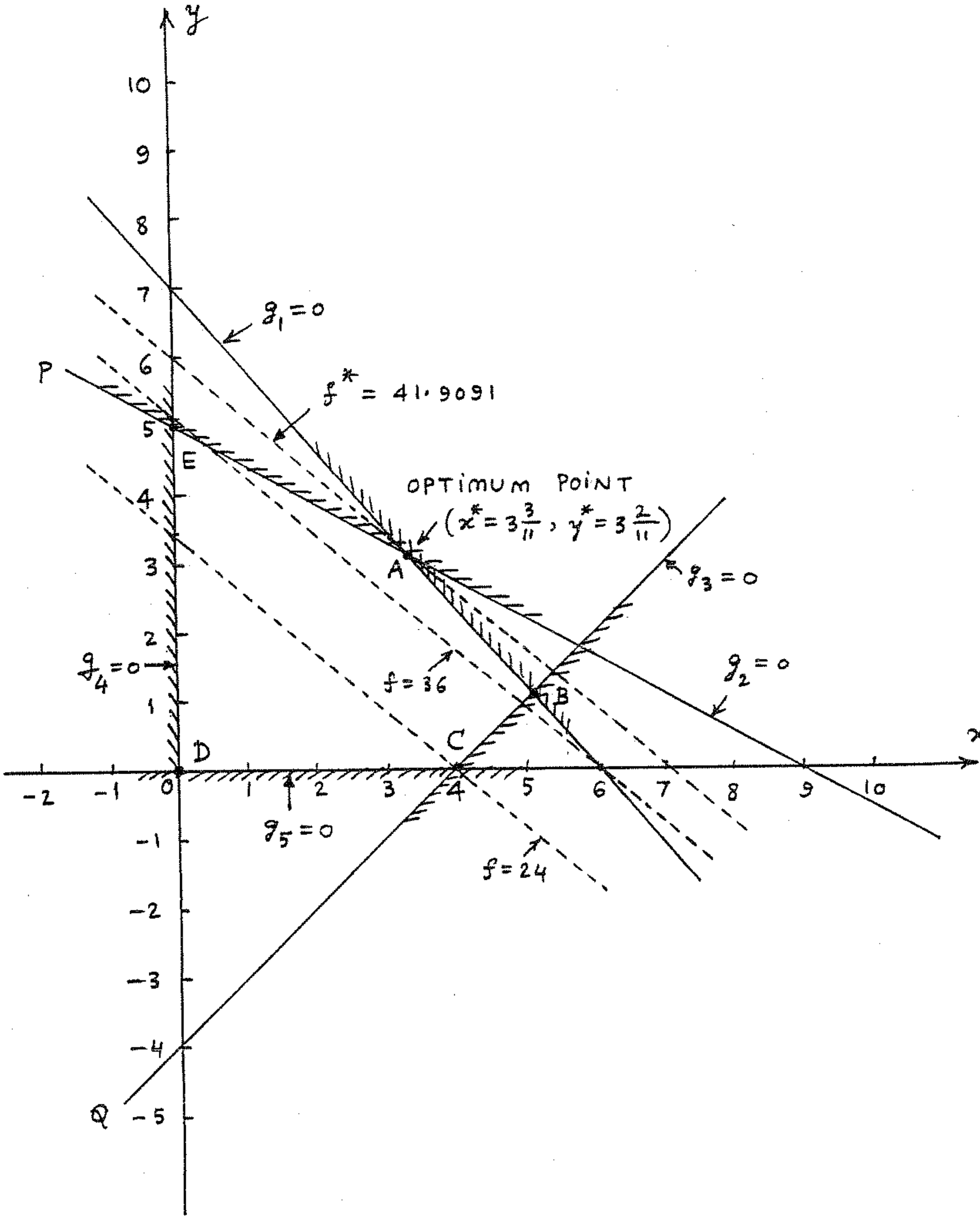
Feasible space = ABCDEA

optimum solution:

$\vec{X}^* =$ Point A $\left(3\frac{3}{11}, 3\frac{2}{11}\right)$

$f^* = 6\left(\frac{36}{11}\right) + 7\left(\frac{35}{11}\right)$

$= 41.9091$



y
x
$g_1 = 0$
$f^* = 41.9091$
OPTIMUM POINT
$(x^* = 3\frac{3}{11},\ y^* = 3\frac{2}{11})$
$g_3 = 0$
$g_2 = 0$
$g_4 = 0$
$g_5 = 0$
$f = 36$
$f = 24$
P
Q
A
B
C
D
E

3.13 This problem is same as Problem 3.12 except that g_4 and g_5 are not present here.

Feasible space = PEABCQ

Optimum solution : $\vec{X}^* =$ Point A $= \begin{Bmatrix} 3\frac{3}{11} \\ 3\frac{2}{11} \end{Bmatrix}$, $f^* = 41.9091$

3.14 Maximize $f = 19x + 7y$

subject to $g_j \le 0$, $j = 1,2,3,4,5$ (as in Problem 3.12).

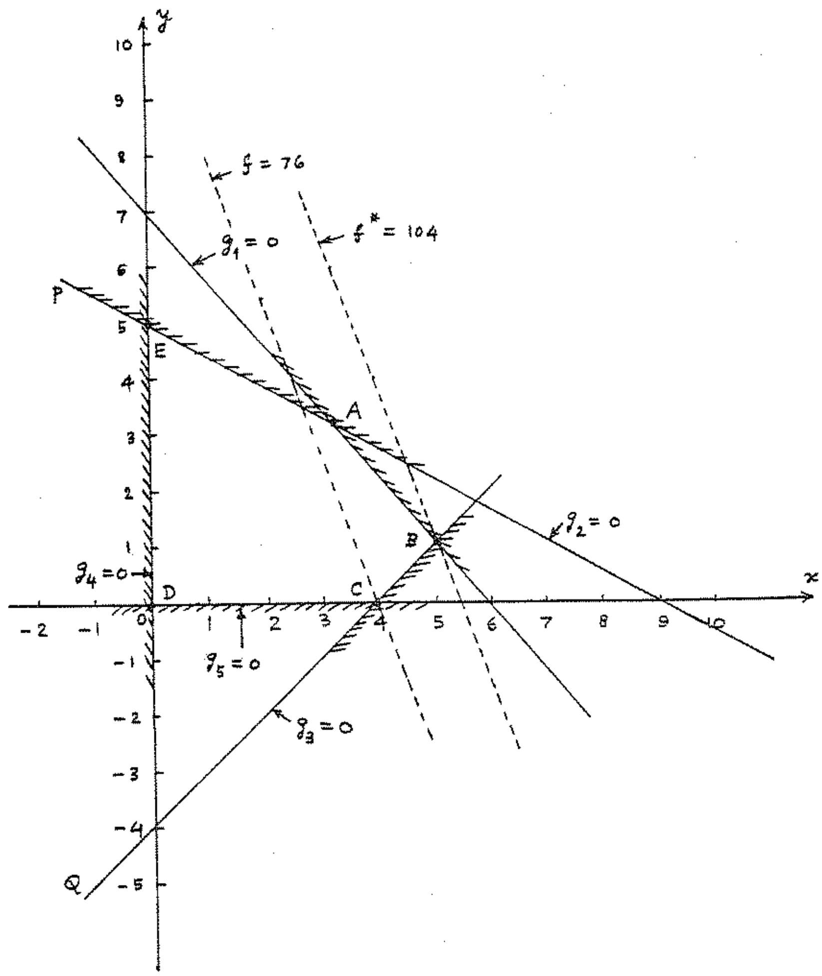
contours of $f = 19x + 7y$:

$f = 76$ at $(4,0)$ and $(2, \frac{38}{7})$

$f = 104$ at $(5\frac{1}{13}, 1\frac{1}{13})$ and $(4,4)$

Feasible space = ABCDEA

Optimum point = B = $(5\frac{1}{13}, 1\frac{1}{13})$, $f^* = 104$.



3.15 This problem is same as Problem 3.14 except that g_4 and g_5 are not present here.

Feasible space = PEABCQ

Optimum point = point B = $\left(5\frac{1}{13}, 1\frac{1}{13}\right)$, $f^* = 104$.

3.16 Maximize $f = x + 2y$

subject to:

$$g_1 = x - y + 8 \geq 0$$
$$g_2 = 5x - y \geq 0$$
$$g_3 = x + y - 8 \geq 0$$
$$g_4 = -x + 6y - 12 \geq 0$$
$$g_5 = -5x - 2y + 68 \geq 0$$
$$g_6 = -x + 10 \geq 0$$
$$g_7 = x \geq 0$$
$$g_8 = y \geq 0$$

$g_1 = 0$ at $(0, 8)$, $(-8, 0)$, $(4, 12)$

$g_2 = 0$ at $(0, 0)$, $(1, 5)$

$g_3 = 0$ at $(0, 8)$, $(8, 0)$

$g_4 = 0$ at $(0, 2)$, $(-12, 0)$, $(6, 3)$

$g_5 = 0$ at $(0, 34)$, $\left(\frac{68}{5}, 0\right)$, $\left(2, \frac{58}{2}\right)$, $(10, 9)$

$g_6 = 0$ at $(10, y)$

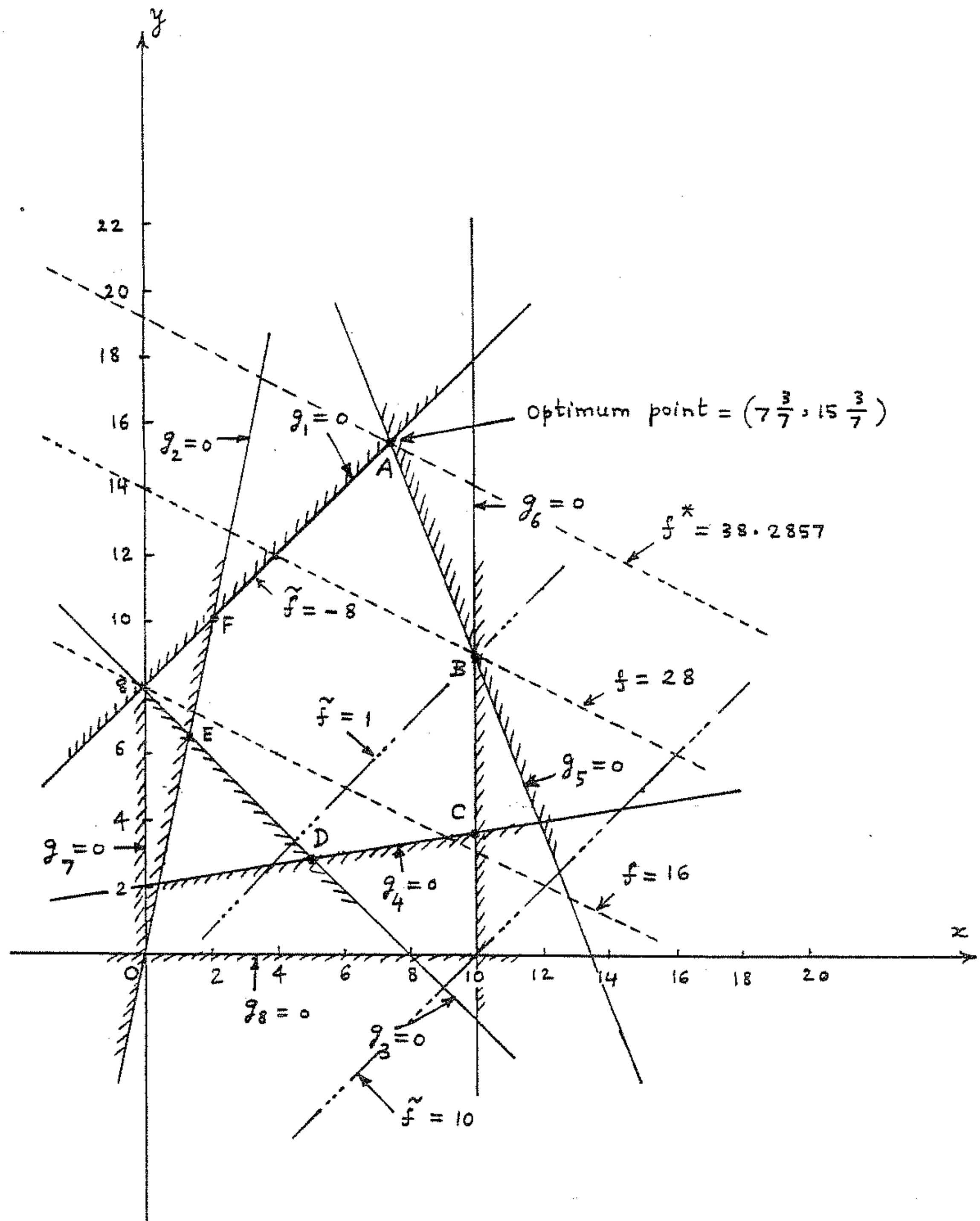
contours of $f = x + 2y$:

$f = 28$ at $(0, 14)$, $(4, 12)$

$f = 16$ at $(0, 8)$, $(4, 6)$

Feasible space: ABCDEFA

Optimum solution: $\vec{X}^* = \left(7\frac{3}{7}, 15\frac{3}{7}\right)$ with $f^* = 38.2857$



y
x
Optimum point = $(7\frac{3}{7}, 15\frac{3}{7})$
$g_1 = 0$
$g_2 = 0$
$g_3 = 0$
$g_4 = 0$
$g_5 = 0$
$g_6 = 0$
$g_7 = 0$
$g_8 = 0$
$f^* = 38.2857$
$f = 28$
$f = 16$
$\tilde{f} = -8$
$\tilde{f} = 1$
$\tilde{f} = 10$
A
B
C
D
E
F
O

3.17 Minimize $\tilde{f} = x - y$

subject to $g_j \geq 0$, $j = 1, 2, \ldots, 8$ (as in Problem 3.16)

Contours of $\tilde{f} = x - y$:

$\tilde{f} = 10$ at $(10, 0)$, $(15, 5)$

$\tilde{f} = 1$ at $(10, 9)$, $(12, 11)$

$\tilde{f} = -8$ at $(4, 12)$, $(8, 16)$

Feasible space = ABCDEFA in figure of Problem 3.16

Optimum solution: All points on "FA"

$\tilde{f}^* = -8$.

3.18 Maximize $f = x + 2y$

subject to $g_1 = x - y + 8 \geq 0$

$g_2 = 5x - y \geq 0$

$g_3 = x + y - 8 \geq 0$

$g_4 = -x + 6y - 12 \geq 0$

$g_5 = 5x + 2y - 68 \geq 0$

$g_6 = -x + 10 \geq 0$

$g_7 = x \geq 0$

$g_8 = y \geq 0$

$g_1 = 0$ at $(0, 8)$, $(4, 12)$

$g_2 = 0$ at $(0, 0)$, $(1, 5)$

$g_3 = 0$ at $(0, 8)$, $(8, 0)$

$g_4 = 0$ at $(0, 2)$, $(6, 3)$

$g_5 = 0$ at $(6, 19)$, $(10, 9)$, $(12, 4)$

$g_6 = 0$ at $(10, y)$

Contours of f:

$f = 46$ at $(10, 18)$, $(12, 17)$

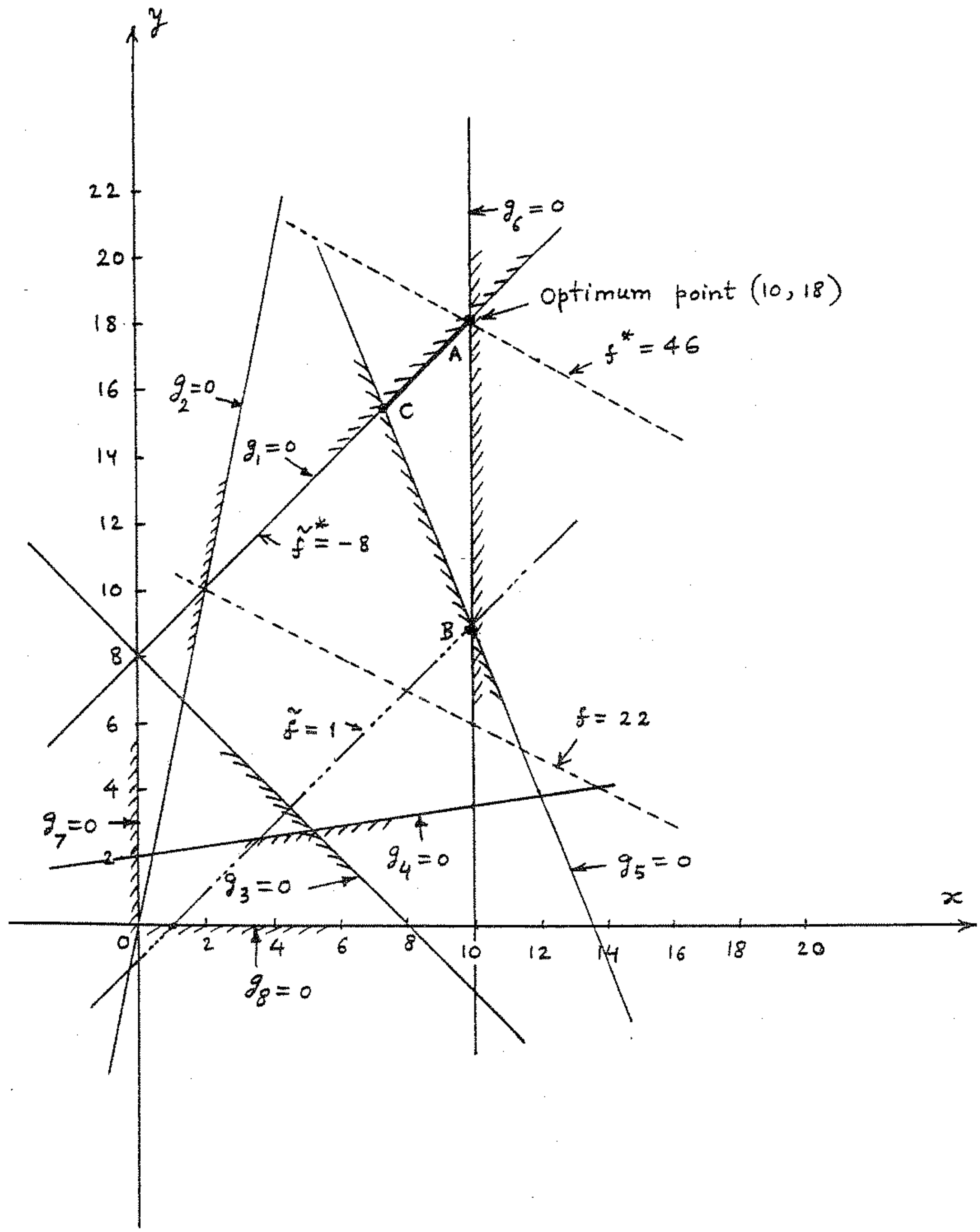
$f = 22$ at $(2, 10)$, $(10, 6)$

Feasible space = ABCA

Optimum point = A

$= (10, 18)$

$f^* = 46$



y
x
Optimum point (10, 18)
f* = 46
A
C
B
g6 = 0
g2 = 0
g1 = 0
f̃* = −8
f̃ = 1
f = 22
g7 = 0
g4 = 0
g5 = 0
g3 = 0
g8 = 0

3.19 Minimize $\tilde{f} = x - y$

subject to $g_j \geq 0$, $j = 1$ to 8 (same as in Problem 3.18)

Contours of $\tilde{f} = x - y$:

$\tilde{f} = -8$ at $(10, 18)$, $(20, 28)$, $(0, 8)$

$\tilde{f} = 1$ at $(10, 9)$, $(1, 0)$

Feasible space = ABCA of figure in Problem 3.18

Optimum point = every point on "CA"

$f^* = -8$

3.20 Maximize $f = x + 3y$

subject to

$$g_1 = -4x + 3y - 12 \leq 0$$
$$g_2 = x + y - 7 \leq 0$$
$$g_3 = x - 4y - 2 \leq 0$$
$$g_4 = -x \leq 0$$
$$g_5 = -y \leq 0$$

$g_1 = 0$ at $(0, 4)$, $(-3, 0)$

$g_2 = 0$ at $(0, 7)$, $(7, 0)$

$g_3 = 0$ at $(0, -\frac{1}{2})$, $(2, 0)$

contours of $f = x + 3y$:

$f = 12$ at $(0, 4)$, $(3, 3)$

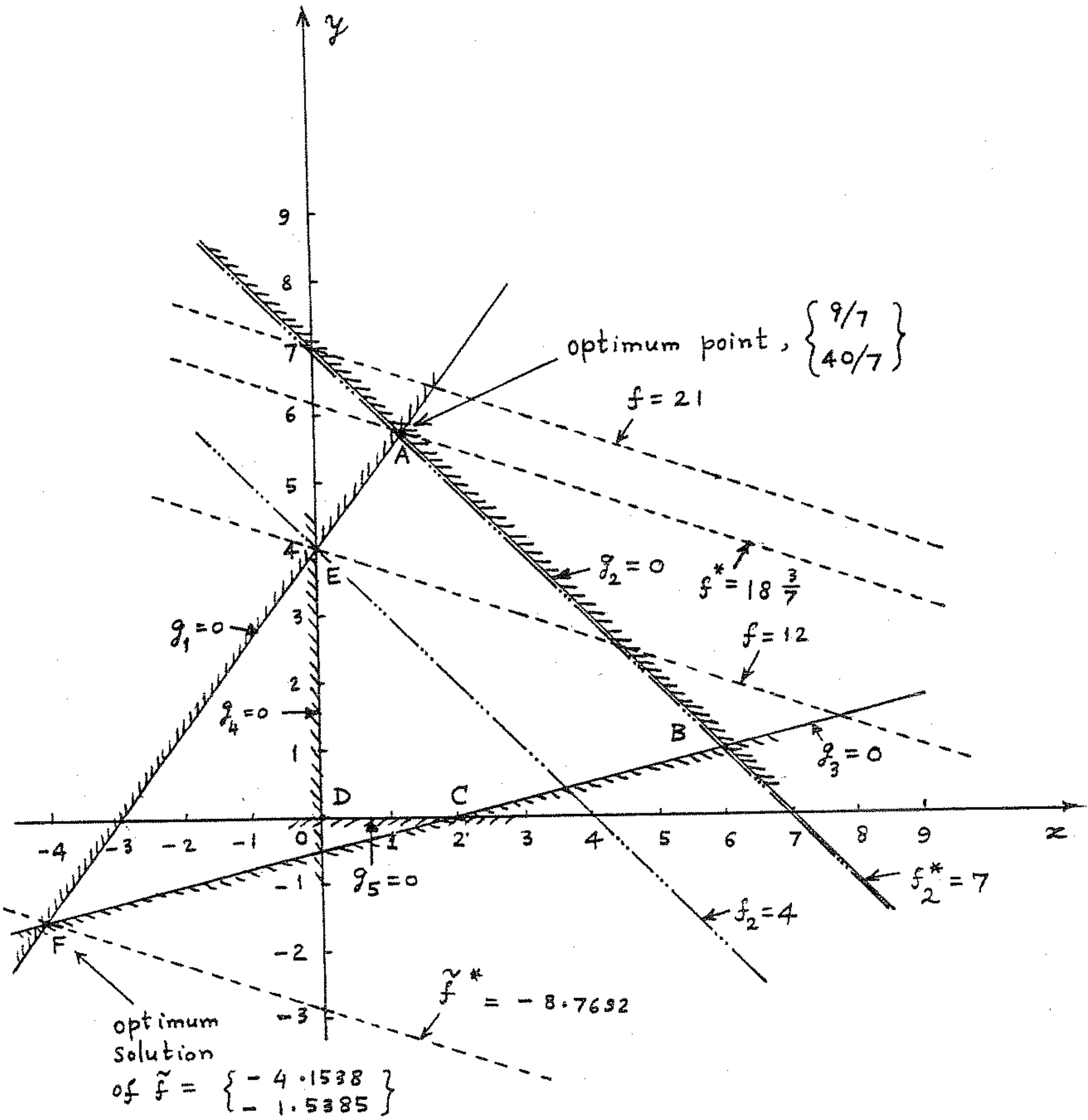
$f = 21$ at $(0, 7)$, $(6, 5)$

Feasible region = ABCDEA

Optimum solution:

Point A $= \vec{x}^* = \begin{Bmatrix} 9/7 \\ 40/7 \end{Bmatrix}$

with $f^* = 18\frac{3}{7}$



y
x
optimum point, $\begin{Bmatrix} 9/7 \\ 40/7 \end{Bmatrix}$
$f = 21$
$f^* = 18\frac{3}{7}$
$f = 12$
$g_2 = 0$
$g_1 = 0$
$g_4 = 0$
$g_3 = 0$
$g_5 = 0$
$f_2^* = 7$
$f_2 = 4$
A
B
C
D
E
F
$\tilde{f}^* = -8.7692$
optimum solution of $\tilde{f} = \begin{Bmatrix} -4.1538 \\ -1.5385 \end{Bmatrix}$

3.21 Minimize $\tilde{f} = x + 3y$

subject to $g_1 = -4x + 3y - 12 \le 0$

$g_2 = x + y - 7 \le 0$

$g_3 = x - 4y - 2 \le 0$

x, y unrestricted in sign.

Feasible space = ABCFEA in the figure of solution of problem 3.20.

Contours of $\tilde{f} = x + 3y$:

$\tilde{f} = 12$ at $(0, 4)$, $(3, 3)$

$\tilde{f}^* = -\frac{54}{13} - \frac{60}{13} = -8.7692$

$\tilde{f} = -8.7692$ at $(x = 0, y = -2.9231)$

Optimum point = F:

$\vec{x}^* = \left(-\frac{54}{13}, -\frac{20}{13}\right)$

Point F:

$$\begin{aligned} -4x + 3y &= 12 \\ 4x - 16y &= 8 \\ \hline -13y &= 20 \end{aligned}$$

$y = -20/13$

$x = 2 + 4y = \frac{26 - 80}{13}$

$= -\frac{54}{13}$

3.22 Maximize $f_2 = x + y$

subject to $g_j \le 0$, $j = 1$ to 5 (same as in Problem 3.20)

Feasible space = ABCDEA in the figure of solution of Problem 3.20

Contours of $f_2 = x + y$:

$f_2 = 4$ at $(0, 4)$, $(4, 0)$

$f_2 = 7$ at $(0, 7)$, $(7, 0)$

Optimum solution: All points between A and B with $f_2^* = 7$, $A = \begin{Bmatrix} 9/7 \\ 40/7 \end{Bmatrix}$, and $B = \begin{Bmatrix} 6 \\ 1 \end{Bmatrix}$.

3.23 Maximize $f = x + 3y$ subject to $g_1 = -4x + 3y - 12 \le 0$

$g_2 = x + y - 7 \le 0$

$g_3 = -x + 4y + 2 \le 0$

$g_4 = -x \le 0$

$g_5 = -y \le 0$

$g_1 = 0$ at $(0, 4), (-3, 0)$

$g_2 = 0$ at $(0, 7), (7, 0)$

$g_3 = 0$ at $(0, -\frac{1}{2}), (2, 0)$

Contours of $f = x + 3y$:

$f = 12$ at $(0, 4), (3, 3)$

$f = 9$ at $(6, 1), (3, 2)$

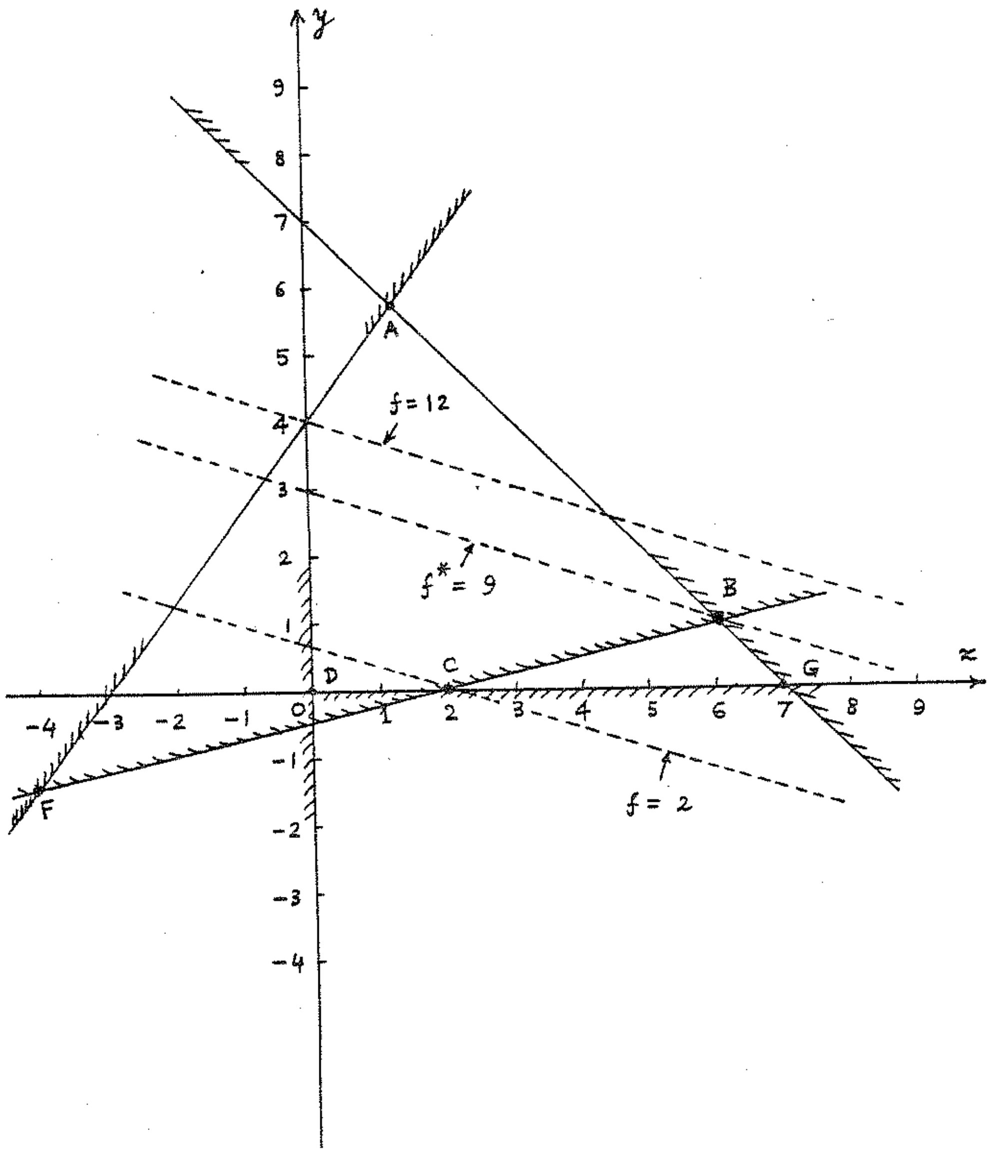
$f = 2$ at $(2, 0), (0, 2/3)$

Feasible space = BGCB.

Optimum point $(\vec{X}^*)$:

$B = \begin{Bmatrix} 6 \\ 1 \end{Bmatrix}$

$f^* = 9$



3.24 Minimize $f = x - 8y$

subject to

$$g_1 = 3x + 2y - 6 \geq 0$$
$$g_2 = -x + y + 6 \geq 0$$
$$g_3 = -9x - 7y + 108 \geq 0$$
$$g_4 = -3x - 7y + 70 \geq 0$$
$$g_5 = 2x - 5y + 35 \geq 0$$
$$g_6 = +x \geq 0$$
$$g_7 = +y \geq 0$$

$g_1 = 0$ at $(0, 3)$, $(2, 0)$

$g_2 = 0$ at $(0, -6)$, $(6, 0)$

$g_3 = 0$ at $(0, \frac{108}{7})$, $(12, 0)$, $(8.1, 5)$

$g_4 = 0$ at $(0, 10)$, $(\frac{70}{3}, 0)$, $(7, 7)$

$g_5 = 0$ at $(0, 7)$, $(-\frac{35}{2}, 0)$, $(5, 9)$

Contours of $f = x - 8y$:

$f = -24$ at $(0, 3)$, $(8, 4)$

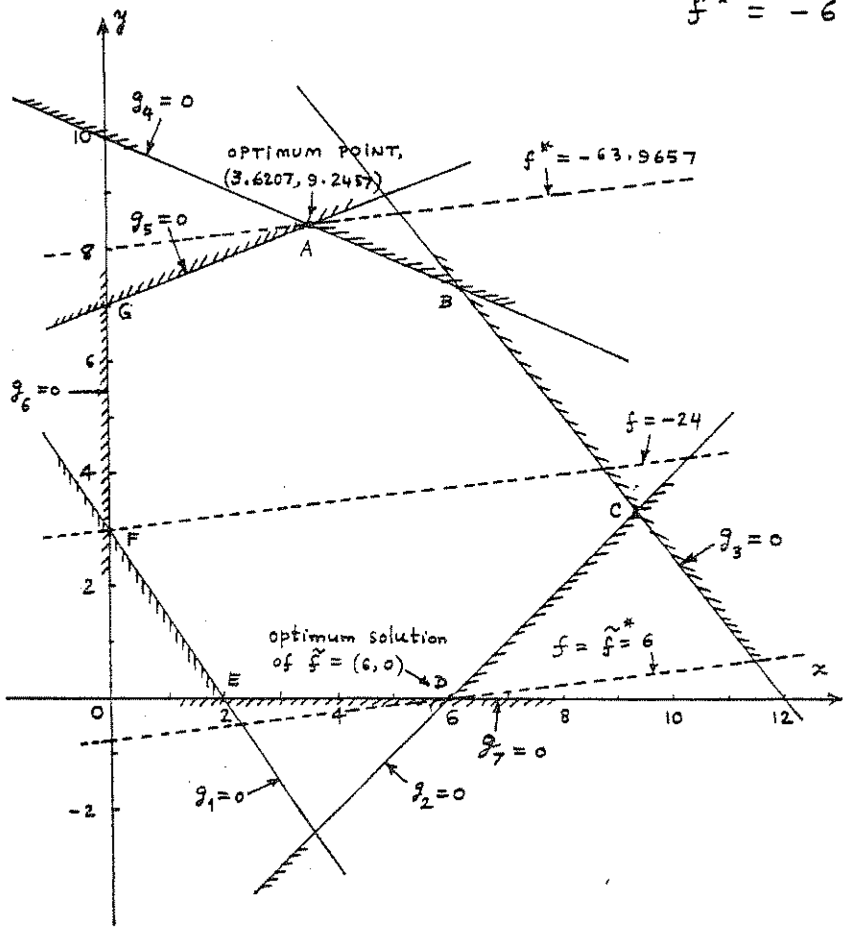
$f = 6$ at $(6, 0)$, $(-2, -1)$

Feasible region: ABCDEFGA

optimum point = A:

$(x^* = 3.6207,\ y^* = 9.2457)$

$f^* = -63.9657$



3.25 Maximize $\tilde{f} = x - 8y$

subject to $g_j \geq 0$, $j = 1$ to 7 (same as in Problem 3.24)

Contours of $\tilde{f} = x - 8y$:

$\tilde{f} = -24$ at $(0,3)$, $(8,4)$

$\tilde{f} = -80$ at $(0,10)$, $(8,11)$

Feasible region: ABCDEFGA in figure of solution of Problem 3.24

optimum point: $D = (6, 0)$ with $\tilde{f}^* = 6$

3.26 Maximize $f = x - 8y$

subject to

$$g_1 = 3x + 2y - 6 \geq 0$$
$$g_2 = -x + y + 6 \geq 0$$
$$g_3 = -9x - 7y + 108 \geq 0$$
$$g_4 = -3x - 7y + 70 \geq 0$$
$$g_5 = 2x - 5y + 35 \geq 0$$
$$g_6 = x \geq 0 \; ; \quad y \text{ is unrestricted in sign}$$

$g_1 = 0$ to $g_6 = 0$ can be plotted as in Problem 3.24.

Feasible space: ABCDEFGHA

Contours of $f = x - 8y$:

$f = -24$ at $(0,3)$, $(8,4)$

$f = 6$ at $(6,0)$, $(-2,-1)$

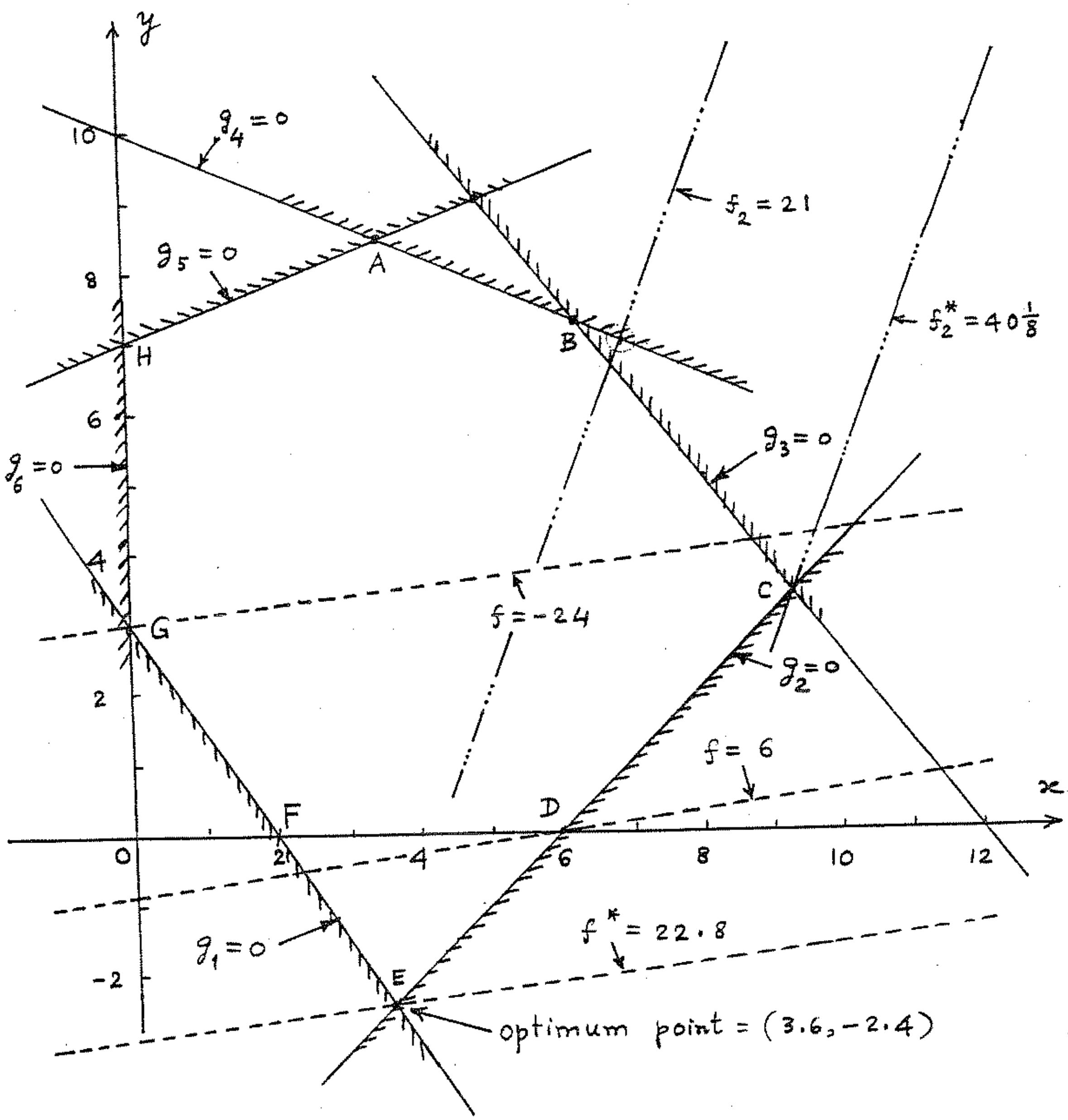
Point E = intersection of $g_1 = 0$ and $g_2 = 0$:

$$\begin{aligned} +3x + 2y &= 6 \\ -2x + 2y &= -12 \\ \hline 5x &= 18 \\ x &= +18/5 \\ y = -6 + x &= -12/5 \end{aligned}$$

optimum point $= E$

$= \vec{X}^* = (3.6, -2.4)$

$f^* = \frac{18}{5} + \frac{96}{5} = \frac{114}{5} = 22\frac{4}{5}$



y
x
$g_4 = 0$
$g_5 = 0$
$g_6 = 0$
$g_3 = 0$
$g_2 = 0$
$g_1 = 0$
$f_2 = 21$
$f_2^* = 40\frac{1}{8}$
$f = -24$
$f = 6$
$f^* = 22.8$
optimum point = (3.6, −2.4)
A
B
C
D
E
F
G
H
10
8
6
4
2
−2
0
2
4
6
8
10
12

3.27 Maximize $f_2 = 5x - 2y$
subject to $g_j \geq 0$; $j = 1, 2, \ldots, 7$ (same as in Problem 3.24)

Feasible region = ABCDFGHA (in the figure of solution of Problem 3.26)

Contours of $f_2 = 5x - 2y$:

$f_2 = 21$ at $(7, 7)$, $(9, 12)$

$f_2 = 5 * \frac{75}{8} - 2 * \frac{27}{8}$

$= (375 - 54)/8 = 321/8$

$= 40\frac{1}{8}$ at $\left(\frac{75}{8}, \frac{27}{8}\right)$

and at $(12, 9\frac{15}{16})$

Point C:

$9x + 7y = 108$

$7x - 7y = 42$

$16x = 150$

$x = 75/8$

$y = x - 6 = \frac{75 - 48}{8}$

$= 27/8$

Optimum solution = Point C = $\left(x^* = \frac{75}{8},\ y^* = \frac{27}{8}\right)$
with $f_2^* = 40\frac{1}{8}$.

3.28 Minimize $f = x - 4y$
subject to
$g_1 = -x + y - 4 \leq 0$
$g_2 = 4x + 5y - 45 \leq 0$
$g_3 = 5x - 2y - 20 \leq 0$
$g_4 = 5x + 2y - 10 \leq 0$
$g_5 = -x \leq 0$
$g_6 = -y \leq 0$

$g_1 = 0$ at $(0, 4)$, $(-4, 0)$, $(2, 6)$

$g_2 = 0$ at $(0, 9)$, $(11\frac{1}{4}, 0)$

$g_3 = 0$ at $(0, -10)$, $(4, 0)$, $(8, 10)$

$g_4 = 0$ at $(0, 5)$, $(2, 0)$

Contours of $f = x - 4y$:

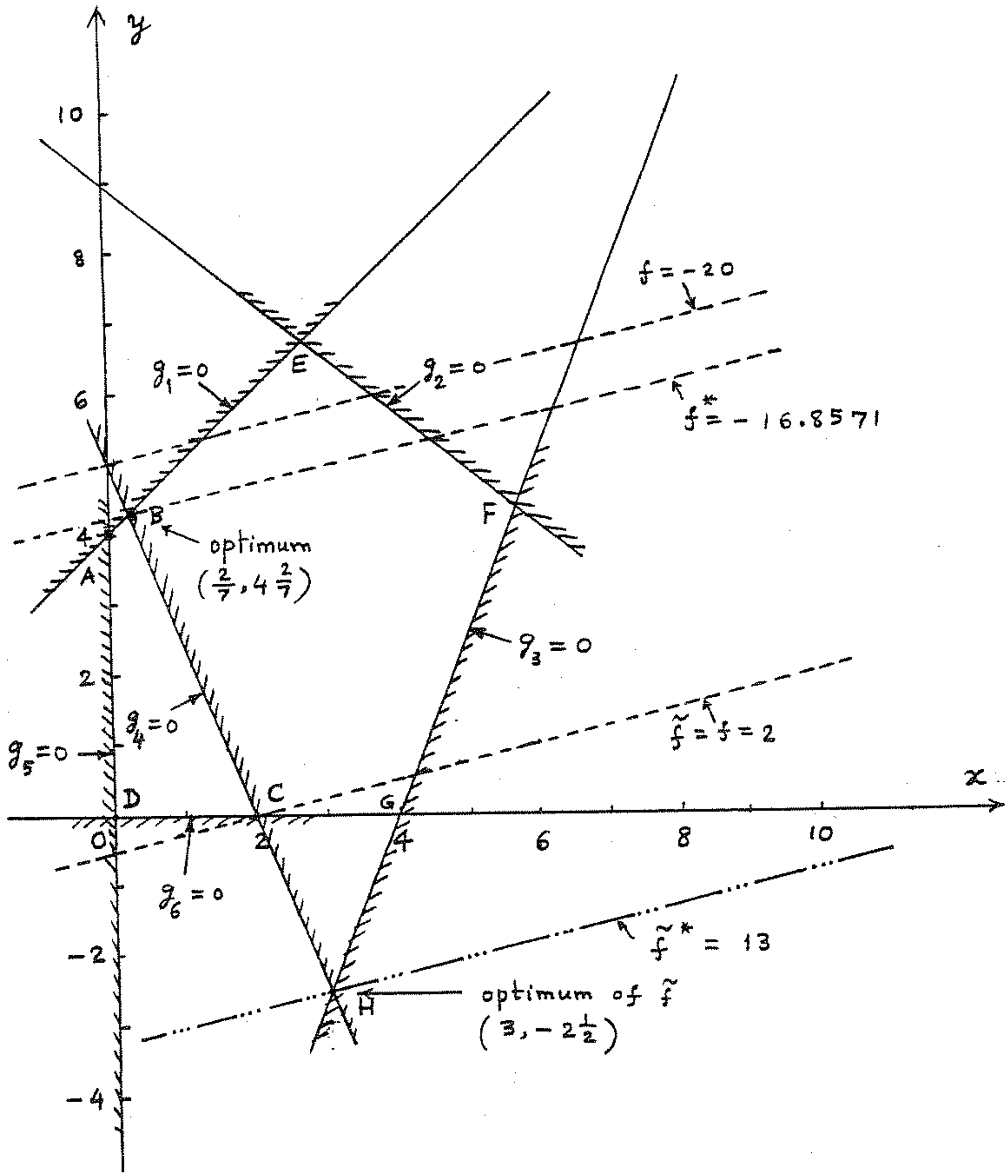
$f = -20$ at $(0, 5)$, $(4, 6)$

$f = 2$ at $(2, 0)$, $(6, 1)$

$f = -16.8571$ at $\left(\frac{2}{7}, \frac{30}{7}\right)$, $(8, 6.2143)$

Feasible space = ABCDA in figure

Optimum point :
$B = \left(\frac{2}{7}, 4\frac{2}{7}\right)$
with $f^* = -16.8571$


y
x
10
8
6
4
2
-2
-4
0
2
4
6
8
10
f = -20
$g_1=0$
$g_2=0$
E
$f^* = -16.8571$
A
B
optimum
$(\frac{2}{7}, 4\frac{2}{7})$
F
$g_3=0$
$g_4=0$
$g_5=0$
$\tilde{f} = f = 2$
D
C
G
$g_6=0$
$\tilde{f}^* = 13$
optimum of $\tilde{f}$
$(3, -2\frac{1}{2})$
H

3.29 Maximize $\tilde{f} = x - 4y$

subject to $g_1 = -x + y - 4 \le 0$

$g_2 = 4x + 5y - 45 \le 0$

$g_3 = 5x - 2y - 20 \le 0$

$g_4 = -5x - 2y + 10 \le 0$

$g_5 = -x \le 0$

y is unrestricted in sign

Feasible space = BEFGHCB in Figure of Problem 3.28.

Point H : intersection of $g_3 = 0$ and $g_4 = 0$:

$$\begin{aligned} 5x - 2y &= 20 \\ -5x - 2y &= -10 \\ \hline 10x = 30, \quad x &= 3 \\ 2y &= 5x - 20 \\ y &= -2\tfrac{1}{2} \end{aligned}$$

$\vec{X}^* =$ Point $H = (3, -2\frac{1}{2})$

$f^* = 3 + 10 = 13$

3.30 Minimize $f = x - 4y$

subject to $g_1 = -x + y - 4 \le 0$

$g_2 = 4x + 5y - 45 \le 0$

$g_3 = 5x - 2y - 20 \le 0$

$g_4 = -5x - 2y + 10 \le 0$

$g_5 = -x \le 0$

$g_6 = -y \le 0$

$g_1 = 0$ at $(0,4), (2,6)$

$g_2 = 0$ at $(0,9), (11\frac{1}{4}, 0)$

$g_3 = 0$ at $(4,0), (8,10)$

$g_4 = 0$ at $(0,5), (2,0)$

Contours of $f = x - 4y$:

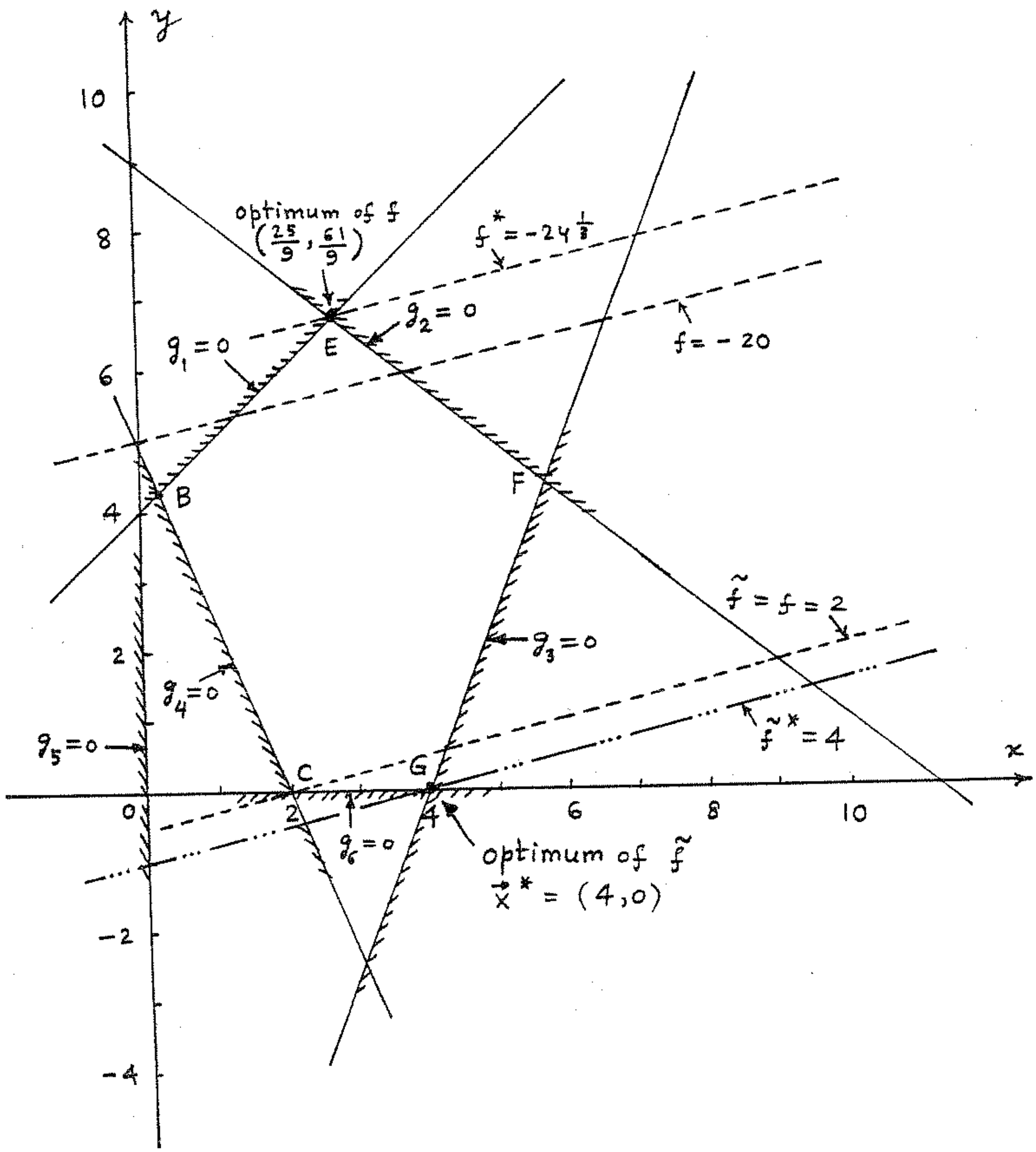
$f = -20$ at $(0,5), (4,6)$

$f = 2$ at $(2,0), (6,1)$

Feasible space = BEFGCB in the figure

Point $E = \left(\frac{25}{9}, \frac{61}{9}\right) = \vec{X}^*$ = optimum point

$f^* = \frac{25}{9} - 4\left(\frac{61}{9}\right) = -\frac{219}{9} = -24\frac{1}{3}$



y
x
10
8
6
4
2
-2
-4
0
2
4
6
8
10
optimum of f
(25/9, 61/9)
E
B
C
G
F
$f^* = -24\frac{1}{3}$
$f = -20$
$g_1 = 0$
$g_2 = 0$
$g_3 = 0$
$g_4 = 0$
$g_5 = 0$
$g_6 = 0$
$\tilde{f} = f = 2$
$\tilde{f}^* = 4$
optimum of $\tilde{f}$
$\vec{x}^* = (4,0)$

3.31 Maximize $\tilde{f} = x - 4y$

subject to $g_j \le 0$, $j = 1, 2, \ldots, 6$ (same as in Problem 3.30)

Feasible space: BEFGCB of figure of problem 3.30

$\tilde{f} = 4$ at $(4, 0)$, $(0, -1)$

optimum point = G = $(4, 0)$ with $f^* = 4$

3.32 Minimize $f = 4x + 5y$

subject to

$$g_1 = 10x + y - 10 \ge 0$$
$$g_2 = 5x + 4y - 20 \ge 0$$
$$g_3 = 3x + 7y - 21 \ge 0$$
$$g_4 = x + 12y - 12 \ge 0$$
$$g_5 = x \ge 0$$
$$g_6 = y \ge 0$$

$g_1 = 0$ at $(0, 10)$, $(1, 0)$

$g_2 = 0$ at $(0, 5)$, $(4, 0)$

$g_3 = 0$ at $(0, 3)$, $(7, 0)$

$g_4 = 0$ at $(0, 1)$, $(12, 0)$

Feasible space: space above ABCDEFG in figure

$f = 50$ at $(0, 10)$, $(12\frac{1}{2}, 0)$

$f = 25$ at $(0, 5)$, $(6\frac{1}{4}, 0)$

Point D = intersection of $g_2 = 0$ and $g_3 = 0$:

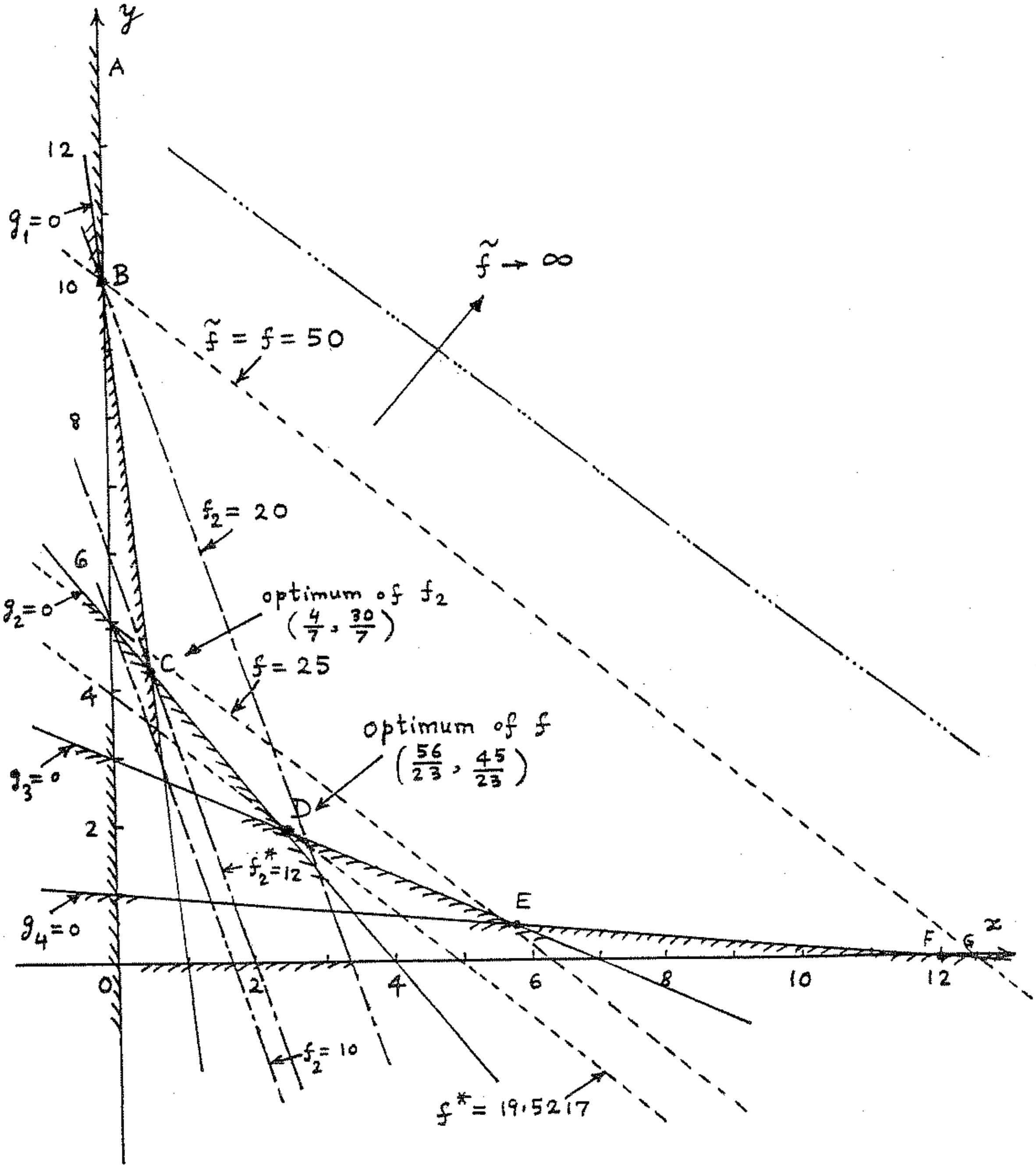
$$= \left(\frac{56}{23}, \frac{45}{23}\right)$$

$$15x + 12y = 60$$
$$15x + 35y = 105$$
$$-23y = -45$$
$$y = \frac{45}{23};$$
$$5x = 20 - \frac{180}{23} = \frac{280}{23}$$
$$x = \frac{56}{23}$$

optimum solution = $\vec{X}^*$ = point D

$$= \left(\frac{56}{23}, \frac{45}{23}\right)$$

$$f^* = \frac{449}{23} = 19.5217$$


y
A
12
$g_1 = 0$
10
B
$\tilde{f} \to \infty$
$\tilde{f} = f = 50$
8
$f_2 = 20$
6
$g_2 = 0$
optimum of f_2
$\left(\frac{4}{7}, \frac{30}{7}\right)$
C
$f = 25$
4
optimum of f
$\left(\frac{56}{23}, \frac{45}{23}\right)$
$g_3 = 0$
2
D
$f_2^* = 12$
E
$g_4 = 0$
F
G
x
0
2
4
6
8
10
12
$f_2 = 10$
$f^* = 19.5217$

3.33 Maximize $\tilde{f} = 4x + 5y$

subject to $g_j \geq 0$, $j = 1, 2, \ldots, 6$ (same as in Problem 3.32)

Optimum solution = unbounded

Feasible space = space above ABCDEFG in figure of Problem 3.32

3.34 Minimize $f_2 = 6x + 2y$

subject to $g_j \geq 0$, $j = 1, 2, \ldots, 6$ (same as in Problem 3.32)

Feasible space = space above ABCDEFG in figure of Problem 3.32

Contours of $f_2 = 6x + 2y$:

$f_2 = 20$ at $(0, 10)$, $(2, 4)$

$f_2 = 10$ at $(0, 5)$, $(2, -1)$

Point C = Optimum of f_2 = intersection of $g_1 = 0$ and $g_2 = 0$:

$= \vec{X}^* = \left(\frac{4}{7}, \frac{30}{7}\right)$

with $f_2^* = 12$

$$10x + y = 10$$
$$10x + 8y = 40$$
$$-7y = -30, \quad y = \frac{30}{7}$$
$$5x = 20 - \frac{120}{7} = \frac{20}{7}$$
$$x = \frac{4}{7}$$

3.35 Minimize $f = 6x + 2y$

subject to $g_1 = 10x + y - 10 \geq 0$

$g_2 = 5x + 4y - 20 \geq 0$

$g_3 = 3x + 7y - 21 \geq 0$

$g_4 = x + 12y - 12 \geq 0$

x, y unrestricted in sign

$g_1 = 0$ at $(0, 10)$, $(1, 0)$

$g_2 = 0$ at $(0, 5)$, $(4, 0)$

$g_3 = 0$ at $(0, 3)$, $(7, 0)$

$g_4 = 0$ at $(0, 1)$, $(12, 0)$

Feasible space: space above ABCDEFG in figure below.

Contours of $f = 6x + 2y$:

$f = 20$ at $(0, 10)$, $(2, 4)$

$f = 10$ at $(0, 5)$, $(2, -1)$

$f = 12$ at $\left(\frac{4}{7}, \frac{30}{7}\right)$, $(1, 3)$

optimum point = point $C = \left(\frac{4}{7}, \frac{30}{7}\right)$ with $f^* = 12$.