Solution Manual for Fault-Tolerant Systems - Israel Koren, Mani Krishna

Solutions to Chapter 2 Exercises

1. The lifetime (measured in years) of a processor is exponentially distributed, with a mean lifetime of 2 years. You are told that a processor failed sometime in the interval $[4, 8]$ years. Given this information, what is the conditional probability that it failed before it was 5 years old?

   Solution:
   Denote the lifetime of the processor by T. Since $E(T) = 2$, $\lambda = 0.5$ and the distribution function of T is $F(t) = 1 - e^{-0.5t}$. Using the conditional probability formula:

$$\begin{aligned}
\text{Prob}\{T < 5 \mid 4 \le T \le 8\} &= \frac{\text{Prob}\{[T < 5] \cap [4 \le T \le 8]\}}{\text{Prob}\{4 \le T \le 8\}} \\
&= \frac{\text{Prob}\{4 \le T < 5\}}{\text{Prob}\{4 \le T \le 8\}} \\
&= \frac{F(5) - F(4)}{F(8) - F(4)} \\
&= \frac{e^{-2} - e^{-2.5}}{e^{-2} - e^{-4}} = 0.455
\end{aligned}$$

2. The lifetime of a processor (measured in years) follows the Weibull distribution, with parameters $\lambda = 0.5$ and $\beta = 0.6$.

   (a) What is the probability that it will fail in its first year of operation?

   (b) Suppose it is still functional after $t = 6$ years of operation. What is the conditional probability that it will fail in the next year?

   (c) Repeat parts (a) and (b) for $\beta = 2$.

(d) Repeat parts (a) and (b) for $\beta = 1$.

Solution:
(a) Denote the lifetime of the processor by T. The distribution function of T is $F(t) = 1 - e^{-0.5t^{0.6}}$. The probability that T is no greater than one year is $F(1) = 0.393$.
(b) We use the conditional probability formula:

$$\begin{aligned} \text{Prob}\{6 < T \leq 7 \mid T > 6\} &= \frac{\text{Prob}\{6 < T \leq 7\}}{\text{Prob}\{T > 6\}} \\ &= \frac{F(7) - F(6)}{1 - F(6)} = 0.045 \end{aligned}$$

3. To get a feel for the failure rates associated with the Weibull distribution, plot them for the following parameter values as a function of the time, t:

   (a) Fix $\lambda = 1$ and plot the failure rate curves for $\beta = 0.5, 1.0, 1.5$.

   (b) Fix $\beta = 1.5$ and plot the failure rate curves for $\lambda = 1, 2, 5$.

Solution:
The failure rate for the Weibull distribution is

$$\lambda(t) = \lambda\beta t^{\beta-1}$$

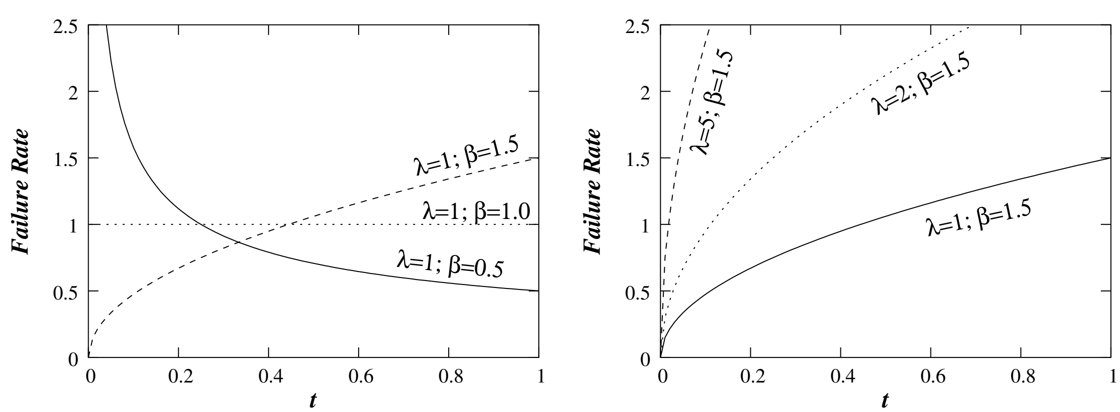

Figure 2.1: The failure rate for the Weibull distribution.

4. Write the expression for the reliability $R_{system}(t)$ of the series/parallel system shown in Figure 2.2, assuming that each of the five modules has a reliability of $R(t)$.

Solution:
The system can be decomposed into a series system consisting of one unit with the leftmost

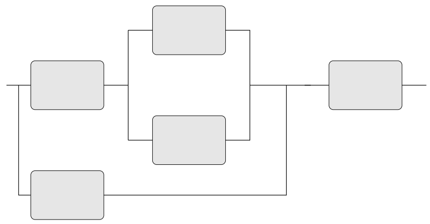

Figure 2.2: A 5-module series-parallel system.

4 blocks and the second unit with the rightmost block. If the reliability of the leftmost 4 blocks is $R_A(t)$, the system reliability is $R_A(t)R(t)$.

Now, we calculate $R_A(t)$. This subsystem consists of a parallel arrangement of one unit consisting of the bottom block and another consisting of the other 3 blocks. If $R_B(t)$ is the reliability of the top 3 blocks, $R_A(t) = 1 - (1 - R_B(t))(1 - R(t))$.

Next, we calculate $R_B(t)$: this subsystem consists of a series arrangement of one block with another consisting of two blocks in parallel. Hence, we have

$$R_B(t) = R(t)(1 - (1 - R(t))^2)$$

Substituting all intermediate results yields

$$R_{system} = R^5(t) - 3R^4(t) + 2R^3(t) + R^2(t)$$

5. The lifetime of each of the seven blocks in Figure 2.3 is exponentially distributed with parameter λ. Derive an expression for the reliability function of the system, $R_{system}(t)$, and plot it over the range $t = [0, 100]$ for $\lambda = 0.02$.

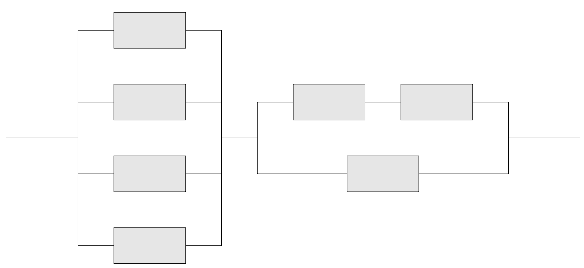

Figure 2.3: A 7-module series-parallel system.

Solution:
As before, we decompose this structure into two substructures connected in series. The left substructure is four blocks in parallel; the right substructure consists of a series arrangement of two blocks in parallel with one block.

The reliability of this system is thus given by:

$$\left[1 - (1 - R(t))^4\right]\left[1 - (1 - R(t))(1 - R^2(t))\right] = R^7(t) - 5R^6(t) + 9R^5(t) - 6R^4(t) - 2R^3(t) + 4R^2(t)$$

where $R(t) = e^{-\lambda t}$.

6. Consider a triplex that produces a one-bit output. Failures that cause the output of a processor to be permanently stuck at 0 or stuck at 1 occur at constant rates λ_0 and λ_1, respectively. The voter never fails. At time t, you carry out a calculation whose correct output is 0. What is the probability that the triplex will produce an incorrect result? (Assume that stuck-at faults are the only ones that a processor can suffer from, and that these are permanent faults; once a processor has its output stuck at some logic value, it remains stuck at that value forever).

Solution:
The result will be incorrect if at least two of the three nodes produce a 1. The probability that a node is stuck by time t at either 0 or 1 is $1 - e^{-(\lambda_0+\lambda_1)t}$, and the probability that it is stuck at 1 (and not at 0) is

$$q(t) = \frac{\lambda_1}{\lambda_0 + \lambda_1}\left(1 - e^{-(\lambda_0+\lambda_1)t}\right)$$

and the answer is

$$q^3(t) + 3q^2(t)(1 - q(t))$$

7. Write the expression for the reliability of a 5MR system and calculate its MTTF. Assume that failures occur as a Poisson process with rate λ per node, that failures are independent and permanent, and that the voter is failure-free.

Solution:
Denote by $r(t) = e^{-\lambda t}$ the reliability of an individual node. The reliability of the 5MR system is

$$\begin{aligned} R_{5MR}(t) &= \sum_{i=3}^{5} \binom{5}{i} r^i(t)(1 - r(t))^{5-i} \\ &= 10r^3(t) - 15r^4(t) + 6r^5(t) \\ &= 10e^{-3\lambda t} - 15e^{-4\lambda t} + 6e^{-5\lambda t} \end{aligned}$$

and the MTTF is

$$\int_{t=0}^{\infty} R_{5MR}(t)dt = \frac{1}{\lambda}\left(\frac{10}{3} - \frac{15}{4} + \frac{6}{5}\right) = \frac{47}{60\lambda}$$

8. Consider an NMR system that produces an eight-bit output. $N = 2m + 1$ for some m. Each processor fails at a constant rate λ and the failures are permanent. A failed processor produces any of the 2^8 possible outputs with equal probability. A majority voter is used to produce the overall output, and the voter is assumed never to fail. What is the probability that, at time t, a majority of the processors produce the same incorrect output after executing some program?

Solution:
The probability that an individual processor is faulty at time t is $q(t) = 1 - e^{-\lambda t}$.

The probability that n_1 out of the N processors are faulty is

$$\binom{N}{n_1} q^{n_1}(t)(1-q(t))^{N-n_1}$$

The probability that exactly n_2 out of the n_1 faulty processors produces the output ω is

$$\binom{n_1}{n_2} 2^{-8n_2}(1-2^{-8})^{n_1-n_2}$$

The probability that a majority of the processors produces ω is

$$\sum_{n_1=m+1}^{N} \sum_{n_2=m+1}^{n_1} \binom{N}{n_1} q^{n_1}(t)(1-q(t))^{N-n_1} \binom{n_1}{n_2} 2^{-8n_2}(1-2^{-8})^{n_1-n_2}$$

The number of incorrect outputs is $2^8 - 1 = 255$. Hence, the answer is

$$255 \sum_{n_1=m+1}^{N} \sum_{n_2=m+1}^{n_1} \binom{N}{n_1} q^{n_1}(t)(1-q(t))^{N-n_1} \binom{n_1}{n_2} 2^{-8n_2}(1-2^{-8})^{n_1-n_2}$$

9. Design a majority voter circuit out of two- and three-input logic gates. Assume that you are voting on one-bit inputs.

   Solution:
   Let the inputs be a, b, c. Then, it is easy to check that the logic expression for the voter output is $\overline{a}bc + a\overline{b}c + ab\overline{c} + abc$, which simplifies to $ab + bc + ac$. The circuit is shown in Figure 2.4.

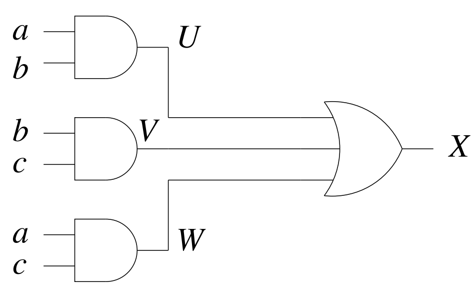

Figure 2.4: Voter circuit diagram.

10. Derive an expression for the reliability of the voter you designed in the previous question. Assume that, for a given time t, the output of each gate is stuck-at-0 or stuck-at-1 with probability P_0 and P_1, respectively (and is fault-free with probability $1 - P_0 - P_1$). What is the probability that the output of your voter circuit is stuck-at-0 (stuck-at-1) given that the 3 inputs to the voter are fault-free and do change between 000 and 111?

Solution:
The voter will be faulty if any of the gates are malfunctioning; since we can implement it using 4 gates, the reliability is $R = (1-p_0-p_1)^4$. Note that $1-R$ is NOT the probability that the voter output will be wrong *every* time.

The probability that the output of the voter will be stuck-at-1 is
$Prob\{X\ is\ stuck-at-1 \mid the\ inputs\ are\ correct\} = Prob\{OR\ gate\ stuck-at-1\} + Prob\{OR\ gate\ is\ fault-free\}\ Prob\{At\ least\ 1\ of\ the\ AND\ gates\ is\ stuck-at-1\} = p_1 + (1-p_0-p_1)[1-(1-p_1)^3]$.
The probability that the output of the voter will be stuck-at-0 is
$Prob\{X\ is\ stuck-at-0 \mid the\ inputs\ are\ correct\} = Prob\{OR\ gate\ stuck-at-0\} + Prob\{OR\ gate\ is\ fault-free\}\ Prob\{All\ AND\ gates\ are\ stuck-at-0\} = p_0 + (1-p_0-p_1)p_0^3$.

11. Show that the MTTF of a parallel system of N modules, each of which suffers permanent failures at a rate λ, is $MTTF_p = \sum_{k=1}^{N} \frac{1}{k\lambda}$.

Solution:
Let the state of the system be the number of modules that are still functional and let T_k be the time spent in state k. Then, $E[T_k] = \frac{1}{k\lambda}$. Since $MTTF = \sum_{k=1}^{N} E[T_k]$, the result follows immediately.
Another way to prove it is as follows:
The reliability of the parallel system is

$$R(t) = 1 - (1-e^{-\lambda t})^N$$

Denoting $x = (1-e^{-\lambda t})$ the above expression can be rewritten as

$$R(t) = (1-x)(1 + \sum_{k=1}^{N-1} x^k)$$

The MTTF is calculated from $\int_0^\infty R(t)dt$. Since $dx = \lambda e^{-\lambda t}dt$, we obtain

$$MTTF = \frac{1}{\lambda}\int_0^1 \sum_{k=0}^{N-1} x^k dx = \frac{1}{\lambda}\left[\sum_{k=0}^{N-1} \frac{x^{k+1}}{k+1}\right]_0^1 = \frac{1}{\lambda}\sum_{k=0}^{N-1}\frac{1}{k+1} = \frac{1}{\lambda}\sum_{k=1}^{N}\frac{1}{k}$$

12. Consider a system consisting of 2 subsystems in series. For improved reliability, you can build subsystem i as a parallel system with k_i units, for $i = 1, 2$. Suppose permanent failures occur at a constant rate λ per unit.

    (a) Derive an expression for the reliability of this system.

    (b) Obtain an expression for the MTTF of this system with $k_1 = 2$ and $k_2 = 3$.

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