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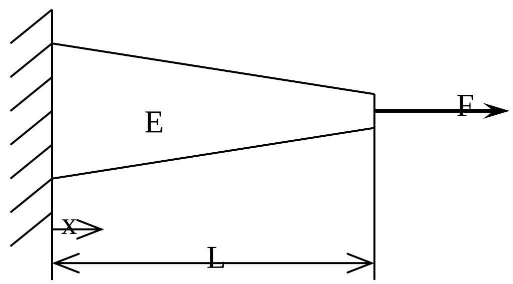
Problem 1.1

Using the principle of virtual work (PVW), find a quadratic displacement function u(x) in 0<x<L of a tapered slender rod of length L, fixed at the origin and loaded axially in tension at the free end. The cross-section area changes lineally and the areas are $A_1 > A_2$ at the fixed and free ends, respectively. The material is homogeneous and isotropic with modulus E.

Solution



The PVW says that:

$$\int_V (\sigma_{ij}\delta\varepsilon_{ij} dV) - \int_S (t_i \delta u_i dS) - \int_V (f_i \delta u_i dV) = 0$$

Assuming *u(x)* as: $u(x) = C_0 + C_1 x + C_2 x^2$

Using the boundary condition $u(x=0) = 0 \xrightarrow{THEN} C_0 = 0$

Because of it's a case of uniaxial load without body forces ($f_i = 0$), the *PVW* can be rewritten as:

$$\int_0^L (\sigma_x \delta\varepsilon_x A_x dx) - F\delta u(x=L) = 0$$

So, we can calculate:

$$\delta u(x) = x\delta C_1 + x^2 \delta C_2$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = C_1 + 2C_2 x$$

$$\delta\varepsilon_x = \delta C_1 + 2x\delta C_2$$

$$\sigma_x = E\varepsilon_x = E(C_1 + 2C_2 x)$$

and

$$A_x = A_1 \frac{(L-x)}{L} + A_2 \frac{x}{L}$$

$$A_x = A_1 - \frac{(A_1 - A_2)}{L} x$$

Now,

$$E\int_0^L (C_1 + 2C_2 x)(\delta C_1 + 2x\delta C_2)\left(A_1 - \frac{(A_1 - A_2)}{L} x\right)dx - F\left(L\delta C_1 + L^2 \delta C_2\right) = 0$$

$$E\int_0^L \left(C_1\delta C_1 + 2C_1 x\delta C_2 + 2C_2 x\delta C_1 + 4C_2 x^2 \delta C_2\right)\left(A_1 - \frac{(A_1 - A_2)}{L} x\right)dx - F\left(L\delta C_1 + L^2 \delta C_2\right) = 0$$

$$E\int_0^L \left(4C_2 x^2 \delta C_2 + 2(C_1\delta C_2 + C_2\delta C_1)x + C_1\delta C_1\right)\left(A_1 - \frac{(A_1 - A_2)}{L} x\right)dx - F\left(L\delta C_1 + L^2 \delta C_2\right) = 0$$

$$EA_1\int_0^L \left(4C_2 x^2 \delta C_2 + 2(C_1\delta C_2 + C_2\delta C_1)x + C_1\delta C_1\right)dx$$
$$- E\frac{(A_1 - A_2)}{L}\int_0^L \left(4C_2 x^3 \delta C_2 + 2(C_1\delta C_2 + C_2\delta C_1)x^2 + C_1\delta C_1 x\right)dx - F\left(L\delta C_1 + L^2 \delta C_2\right) = 0$$

$$EA_1\left(\frac{4}{3}L^3 C_2\delta C_2 + L^2\left(C_1\delta C_2 + C_2\delta C_1\right) + LC_1\delta C_1\right)$$
$$- E\frac{(A_1 - A_2)}{L}\left(L^4 C_2\delta C_2 + \frac{2}{3}L^3(C_1\delta C_2 + C_2\delta C_1) + \frac{1}{2}L^2 C_1\delta C_1\right) - F\left(L\delta C_1 + L^2 \delta C_2\right) = 0$$

Reordering the equation to express it like a linear combination of the variations of *C_i's*

$$\delta C_1\left(EA_1L^2C_2 + EA_1LC_1 - \frac{1}{2}E(A_1 - A_2)LC_1 - \frac{2}{3}E(A_1 - A_2)L^2C_2 - FL\right)$$
$$+ \delta C_2\left(\frac{4}{3}EA_1L^3C_2 + EA_1L^2C_1 - E(A_1 - A_2)L^3C_2 - \frac{2}{3}E(A_1 - A_2)L^2C_1 - FL^2\right) = 0$$

Since *Ci's* are independent variations of *Ci's* values there are two equations involved in the above one. The solution of those equations (2x2 system) will give the values of coefficients *Ci's*.

$$\left(EA_1L-\frac{1}{2}E(A_1-A_2)L\right)C_1+\left(EA_1L^2-\frac{2}{3}E(A_1-A_2)L^2\right)C_2=FL$$

$$\left(EA_1L^2-\frac{2}{3}E(A_1-A_2)L^2\right)C_1+\left(\frac{4}{3}EA_1L^3-E(A_1-A_2)L^3\right)C_2=FL^2$$

Calling

$$\alpha=EA_1-\frac{1}{2}E(A_1-A_2)$$

$$\beta=EA_1-\frac{2}{3}E(A_1-A_2)$$

$$\chi=\frac{4}{3}EA_1-E(A_1-A_2)$$

Then

$$\alpha C_1+\beta LC_2=F$$

$$\beta LC_1+\chi LC_2=F$$

The solution for this system is:

$$C_2=\frac{(\beta-\alpha)F}{(\beta^2-\alpha\chi)L}$$

$$C_1=\frac{(\chi-\beta)F}{(\alpha\chi-\beta^2)}$$

Finally, the displacement solution is:

$$u(x)=\frac{(\chi-\beta)F}{(\alpha\chi-\beta^2)}x+\frac{(\beta-\alpha)F}{(\beta^2-\alpha\chi)L}x^2$$

In terms of the known data

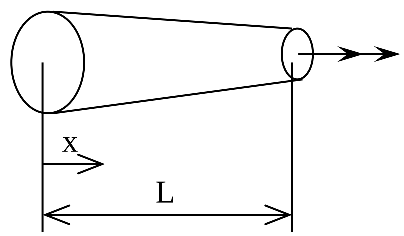
$$u(x)=\frac{6A_2}{E(A_1^2+4A_1A_2+A_2^2)}Fx+\frac{3(A_1-A_2)}{E(A_1^2+4A_1A_2+A_2^2)}\frac{Fx}{L}$$

When $A_1 = A_2 = A$ the well-known result $u(x)=\frac{Fx}{EA}$ is obtained.

Problem 1.2

Using the principle of virtual work (PVW), find a quadratic rotation angle function θ(x) in 0<x<L of a tapered slender shaft of circular cross section and length L, fixed at the origin and loaded by a torque T at the free end. The cross-section area changes lineally and the areas are $A_1 > A_2$ at the fixed and free ends, respectively. The material is homogeneous and isotropic with shear modulus G.

Solution



The PVW says that:

$$\int_V \left(\sigma_{ij}\delta\varepsilon_{ij}dV\right) - \int_S \left(T_i\delta\theta_i dS\right) - \int_V \left(f_i\delta\theta_i dV\right) = 0$$

Assuming θ(x) as: $\theta(x) = C_0 + C_1 x + C_2 x^2$

Using the boundary condition $\theta(x=0) = 0 \xrightarrow{THEN} C_0 = 0$

Because of it's a case of pure torsion without body forces ($f_i = 0$), the *PVW* can be rewritten as:

$$\int_0^L \left(\tau_{xy}\delta\left(\tfrac{1}{2}\gamma_{xy}\right)A_x dx\right) - T\delta\theta(x=L) = 0$$

So, we can calculate:

$$\delta\theta(x) = x\delta C_1 + x^2\delta C_2$$

$$\gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = r_x\left(C_1 + 2C_2 x\right)$$

$$u = 0 \xrightarrow{\frac{\partial}{\partial y}} \frac{\partial u}{\partial y} = 0$$

$$v = r_x\theta(x) \xrightarrow{\frac{\partial}{\partial x}} \frac{\partial v}{\partial x} = r_x\left(C_1 + 2C_2\right)$$

$$\delta\left(\tfrac{1}{2}\gamma_{xy}\right) = \frac{r_x}{2}\left(\delta C_1 + 2x\delta C_2\right)$$

$$\tau_{xy} = G\gamma_{xy} = Gr_x\left(C_1 + 2C_2 x\right)$$

and

$$A_x = A_1 \frac{(L-x)}{L} + A_2 \frac{x}{L}$$
$$A_x = A_1 - \frac{(A_1 - A_2)}{L} x = \pi r_x^2$$
$$r_x^2 = r_1^2 - \frac{(r_1^2 - r_2^2)}{L} x$$

Now,

$$\int_0^L G r_x (C_1 + 2C_2 x) \tfrac{1}{2} r_x (\delta C_1 + 2x\delta C_2) \pi r_x^2 dx - T(L\delta C_1 + L^2 \delta C_2) = 0$$

Solving the integral and organizing the result

$$\delta C_1 (\tfrac{1}{12}\pi G)(2C_1(r_1^4 + r_1^2 r_2^2 + r_2^4) + LC_2(r_1^4 + 2r_1^2 r_2^2 + 3r_2^4))$$
$$+ \delta C_2 (\tfrac{1}{60}\pi G)(5LC_1(r_1^4 + 2r_1^2 r_2^2 + 3r_2^4) + 4L^2 C_2(r_1^4 + 3r_1^2 r_2^2 + 6r_2^4))$$
$$= T\delta C_1 + LT\delta C_2$$

The coefficients of the terms $\delta Ci's$ must be equal on RHS and LHS, therefore a 2x2 system can be written.

$$2\alpha C_1 + \beta L C_2 = \delta$$
$$5\beta C_1 + 4\chi L C_2 = 5\delta$$

with

$$\alpha = r_1^4 + r_1^2 r_2^2 + r_2^4$$
$$\beta = r_1^4 + 2r_1^2 r_2^2 + 3r_2^4$$
$$\chi = r_1^4 + 3r_1^2 r_2^2 + 6r_2^4$$
$$\delta = \frac{12T}{\pi G}$$

Solving the system

$$C_1 = \left(\frac{5\beta - 4\chi}{5\beta^2 - 8\alpha\chi} \right) \delta$$
$$C_2 = \left(\frac{2\alpha - \beta}{5\beta^2 - 8\alpha\chi} \right) \frac{5\delta}{L}$$

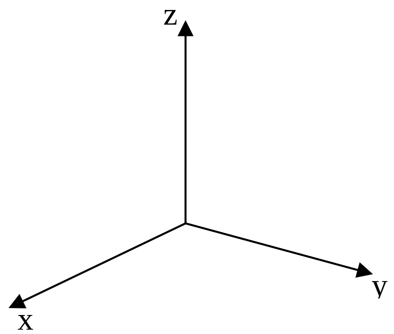
When $r_1 = r_2 = r$ the well-known result $\theta = \dfrac{2Tx}{G\pi r^4}$ is obtained.

Problem 1.3

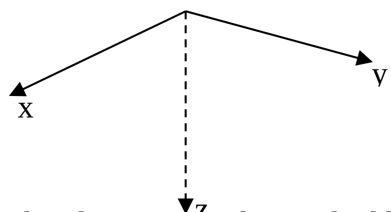
Construct a rotation matrix [a] resulting from three consecutive reflections about (a) the x-y plane, (b) the x-z plane, (c) the y-z plane. The resulting system does not follow the right-hand rule.

Solution

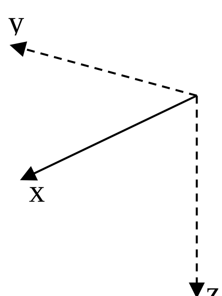
Initial system



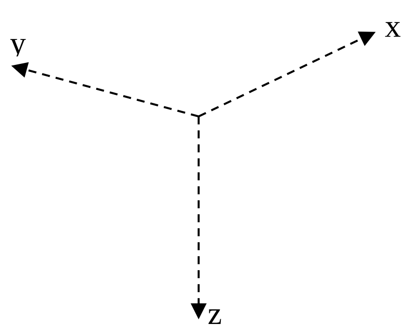
(a) The reflection of z-axis about the plane x-y can be reached by one of the following four rotations: $\theta = (+\pi)$ or $(-\pi)$ about the x-axis or y-axis.



(b) The reflection of y-axis about the plane x-z can be reached by one of the following four rotations: $\theta = (+\pi)$ or $(-\pi)$ about the x-axis or z-axis.



(c) x'-axis represents the reflection of x-axis about the plane y-z. It can be reached by one of the following four rotations: $\theta = (+\pi)$ or $(-\pi)$ about the y-axis or z-axis.



Notice that the new system doesn't accomplish the right hand rule.

Rotation matrix: $a_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Problem 1.4

Construct three rotation matrices [a] for rotations $\theta = \pi$ about (a) the x-axis, (b) the y-axis, (c) the z-axis.

Solution

Rotation about the x-axis

$$a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \xrightarrow{\theta=\pi} a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Rotation about the y-axis

$$a_{ij} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \xrightarrow{\theta=\pi} a_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Rotation about the x-axis

$$a_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\theta=\pi} a_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Problem 1.5

Using

$$\sigma = \begin{bmatrix} 10 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

and [a] of Ex. 1.2, verify that (1.29) yields the same result as (1.26).

Solution

Solved with Mathematica®:

```
T[(_{
     {l1_, m1_, n1_},
     {l2_, m2_, n2_},
     {l3_, m3_, n3_}
    }_)]:=(_{
    {l1², m1², n1², 2 m1 n1, 2 l1  n1, 2 l1  m1},
    {l2², m2², n2², 2 m2  n2, 2 l2  n2, 2 l2  m2},
    {l3², m3², n3², 2 m3  n3, 2 l3  n3, 2 l3  m3},
    {l2 l3, m2 m3, n2 n3, m2 n3+n2 m3, l2 n3+n2 l3, l2 m3 +m2
l3},
    {l1 l3, m1 m3, n1 n3, m1 n3+n1 m3, l1 n3+n1 l3, l1 m3 +m1
l3},
    {l1 l2, m1 m2, n1 n2, m1 n2+n1 m2, l1 n2+n1 l2, l1 m2 +m1 l2}
   }_);
Tbar[T_]:=(_{
     {1, 0, 0, 0, 0, 0},
     {0, 1, 0, 0, 0, 0},
     {0, 0, 1, 0, 0, 0},
     {0, 0, 0, 2, 0, 0},
     {0, 0, 0, 0, 2, 0},
     {0, 0, 0, 0, 0, 2}
    }_).T.(_{
     {1, 0, 0, 0, 0, 0},
     {0, 1, 0, 0, 0, 0},
     {0, 0, 1, 0, 0, 0},
     {0, 0, 0, 1/2, 0, 0},
     {0, 0, 0, 0, 1/2, 0},
     {0, 0, 0, 0, 0, 1/2}
    }_);
Inverse[T[(_{
     {1, 0, 0},
     {0, Cos[π], Sin[π]},
     {0, -Sin[π], Cos[π]}
    }_)]]==Tbar[T[(_{
      {1, 0, 0},
      {0, Cos[π], Sin[π]},
      {0, -Sin[π], Cos[π]}

     }_)]]ᵀ   (* Eq1 .47 [T]^-1Overscript[=, ?][Overscript[T,
_]]^T for (a) *)
```

```
Inverse[T[(_{
     {Cos[π], 0, -Sin[π]},
     {0, 1, 0},
     {Sin[π], 0, Cos[π]}
    }_)]]==Tbar[T[(_{
      {Cos[π], 0, -Sin[π]},
      {0, 1, 0},
      {Sin[π], 0, Cos[π]}
     }_)]]ᵀ (* Eq1 .47 [T]^-1Overscript[=, ?][Overscript[T, _]]^T
for (b) *)
Inverse[T[(_{
     {Cos[π], Sin[π], 0},
     {-Sin[π], Cos[π], 0},
     {0, 0, 1}
    }_)]]==Tbar[T[(_{
      {Cos[π], Sin[π], 0},
      {-Sin[π], Cos[π], 0},
      {0, 0, 1}
     }_)]]ᵀ (* Eq1 .47 [T]^-1Overscript[=, ?][Overscript[T,
_]]^T for (c) *)
```

All three equations yield ``true''.