

1.1 Substitute (1.2-2) into (1.2-1):

$$u_1 = \beta_1 \quad \text{hence } \beta_1 = u_1$$

$$u_2 = \beta_1 + \beta_2 a = u_1 + \beta_2 a \quad \text{hence } \beta_2 = \frac{u_2 - u_1}{a}$$

$$u_3 = \beta_1 + \beta_3 b = u_1 + \beta_3 b \quad \text{hence } \beta_3 = \frac{u_3 - u_1}{b}$$

(1.2-1a) becomes $u = u_1 + \frac{u_2 - u_1}{a} x + \frac{u_3 - u_1}{b} y$
 or $u = \left(1 - \frac{x}{a} - \frac{y}{b}\right) u_1 + \frac{x}{a} u_2 + \frac{y}{b} u_3$

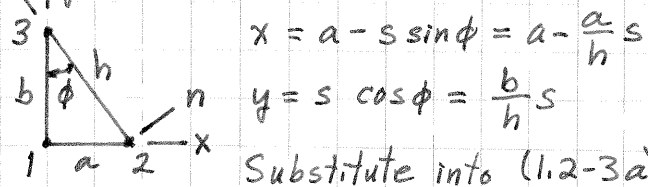
Similarly for v

1.2 On side 1-2, $y=0$; subs. into (1.2-3b):

$$v = \left(1 - \frac{x}{a}\right) v_1 + \frac{x}{a} v_2$$

Depends only on v_1 and v_2 and is a linear function of x , \therefore 1-2 remains straight.

s \setminus y On side 2-3:



$$x = a - s \sin \phi = a - \frac{a}{h} s$$

$$y = s \cos \phi = \frac{b}{h} s$$

Substitute into (1.2-3a):

$$u = \left(1 - 1 + \frac{s}{h} - \frac{s}{h}\right) u_1 + \left(1 - \frac{s}{h}\right) u_2 + \frac{s}{h} u_3$$

$$u = \left(1 - \frac{s}{h}\right) u_2 + \frac{s}{h} u_3, \text{ similarly for } v$$

Now u & v along 2-3 depend only on u_2, v_2, u_3, v_3 & vary linearly with s , \therefore edge remains straight.

1.3 In all parts, substitute coords. into (1.2-1a); v equation is similar.

$$(a) \left. \begin{aligned} u_1 &= \beta_1 + \beta_2 a & (A) \\ u_2 &= \beta_1 + \beta_2 a + \beta_3 b \\ u_3 &= \beta_1 + \beta_3 b & (C) \end{aligned} \right\} \beta_3 = \frac{u_2 - u_1}{b} \quad (B)$$

(B) & (C) yield $\beta_1 = u_1 - u_2 + u_3$ (D)
 (A) & (D) yield $\beta_2 = \frac{u_2 - u_3}{a}$

(1.2-1a) becomes

$$u = (u_1 - u_2 + u_3) + \frac{u_2 - u_3}{a} x + \frac{u_2 - u_1}{b} y$$

$$u = \left(1 - \frac{y}{b}\right) u_1 + \left(-1 + \frac{x}{a} + \frac{y}{b}\right) u_2 + \left(1 - \frac{x}{a}\right) u_3$$

$$(b) \left. \begin{aligned} u_1 &= \beta_1 - \beta_2 a \\ u_2 &= \beta_1 + \beta_2 a \\ u_3 &= \beta_1 + \beta_3 b \end{aligned} \right\} \beta_1 = \frac{u_1 + u_2}{2}$$

$$\beta_2 = \frac{u_2 - u_1}{2a}$$

$$\beta_3 = \frac{u_3 - \beta_1}{b} = \frac{2u_3 - u_1 - u_2}{2b}$$

$$u = \frac{u_1 + u_2}{2} + \frac{u_2 - u_1}{2a} x + \frac{2u_3 - u_1 - u_2}{2b} y$$

$$u = \left(\frac{1}{2} - \frac{x}{2a} - \frac{y}{2b}\right) u_1 + \left(\frac{1}{2} + \frac{x}{2a} - \frac{y}{2b}\right) u_2 + \frac{y}{b} u_3$$

$$(c) \left. \begin{aligned} u_1 &= \beta_1 \\ u_2 &= \beta_1 + \beta_2 a + \beta_3 b \\ u_3 &= \beta_1 - \beta_2 a + \beta_3 b \end{aligned} \right\} \beta_1 = u_1$$

$$\beta_2 = \frac{u_2 - u_3}{2a}$$

Add: $u_2 + u_3 = 2u_1 + 2\beta_3 b, \beta_3 = \frac{u_2 + u_3 - 2u_1}{2b}$

$$u = u_1 + \frac{u_2 - u_3}{2a} x + \frac{u_2 + u_3 - 2u_1}{2b} y$$

$$u = \left(1 - \frac{y}{b}\right) u_1 + \left(\frac{x}{2a} + \frac{y}{2b}\right) u_2 + \left(-\frac{x}{2a} + \frac{y}{2b}\right) u_3$$

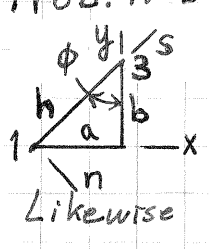
1.4 (a) Same as side 2-3 in Prob. 1.2.

$$(b) \quad x = -a + s \sin \phi = -a + \frac{a}{h} s$$

Use ans. to Prob. 1.3b

$$y = s \cos \phi = \frac{b}{h} s$$

$$u = \left(\frac{1}{2} + \frac{1}{2} - \frac{s}{2h} - \frac{s}{2h}\right) u_1 + \left(\frac{1}{2} - \frac{1}{2} + \frac{s}{2h} - \frac{s}{2h}\right) u_2 + \frac{s}{h} u_3$$



$$u = \left(1 - \frac{s}{h}\right) u_1 + \frac{s}{h} u_3$$

$$\text{Likewise } v = \left(1 - \frac{s}{h}\right) v_1 + \frac{s}{h} v_3$$

Now u & v along 1-3 depend only on u_1, v_1, u_3, v_3 & vary linearly with s , \therefore edge remains straight.

$$(c) \quad x = -s \sin \phi = -\frac{a}{h} s$$

$$y = s \cos \phi = \frac{b}{h} s$$

Use ans. to Prob. 1.3c

$$u = \left(1 - \frac{s}{h}\right) u_1 + \left(-\frac{s}{2h} + \frac{s}{2h}\right) u_2 + \left(\frac{s}{2h} + \frac{s}{2h}\right) u_3$$

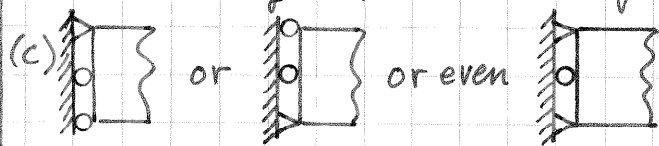
$$u = \left(1 - \frac{s}{h}\right) u_1 + \frac{s}{h} u_3$$

$$\text{Likewise } v = \left(1 - \frac{s}{h}\right) v_1 + \frac{s}{h} v_3$$

Conclusion: as for part (b).

1.5 (a) $\epsilon_x = \frac{\partial u}{\partial x} = \beta_2 + 2\beta_4 x + \beta_5 y$
 $\epsilon_y = \frac{\partial v}{\partial y} = \beta_9 + \beta_{11} x + 2\beta_{12} y$
 $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \beta_3 + \beta_8 + (\beta_5 + 2\beta_{10}) x + (2\beta_6 + \beta_{11}) y$

(b) To permit y-direction displacement associated with the Poisson effect, if all supports at $x=0$ were pins, we would not obtain $\sigma_y = 0$, as beam theory says.



(there is no relative y displacement between top and bottom surfaces)

(d) According to beam theory, $\epsilon_x = c y$, $\epsilon_y = \nu \epsilon_x$, and $\gamma_{xy} = 0$, where $c = \text{const}$.
 Therefore $\beta_2 = \beta_4 = \beta_9 = \beta_{11} = 0$,
 and from the γ_{xy} expression $\beta_3 + \beta_8 = 0$.
 Also, from γ_{xy} , $\beta_6 = 0$ and $\beta_5 + 2\beta_{10} = 0$.

(a) $I = \frac{t}{12} 16^3 = 341t$ $A = 16t$

$\frac{Mc}{I} = \frac{20P(8)}{341t} = 0.469 \frac{P}{t}$

$\frac{P}{A} = \frac{P}{16t} = 0.0625 \frac{P}{t}$ (compressive)

$\sigma_{yA} = 0.469 \frac{P}{t} - 0.0625 \frac{P}{t} = 0.406 \frac{P}{t}$

$\sigma_{yE} = -0.469 \frac{P}{t} - 0.0625 \frac{P}{t} = -0.531 \frac{P}{t}$

(b) Use theory and formulas of Ref. 2.2

$\int \frac{dA}{r} = t \ln \frac{22}{6} = 1.2993t$

$r_n = \frac{16t}{1.2993t} = 12.315$ radius of neutral ax.

$e = R - r_n = 1.685$

At E, $\frac{My}{Aer} = \frac{-20P(12.315-6)}{16t(1.685)6} = -0.780 \frac{P}{t}$

At A, $\frac{My}{Aer} = \frac{-20P(12.315-22)}{16t(1.685)22} = 0.326 \frac{P}{t}$

$\sigma_{yA} = 0.326 \frac{P}{t} - 0.0625 \frac{P}{t} = 0.264 \frac{P}{t}$

$\sigma_{yE} = -0.780 \frac{P}{t} - 0.0625 \frac{P}{t} = -0.843 \frac{P}{t}$

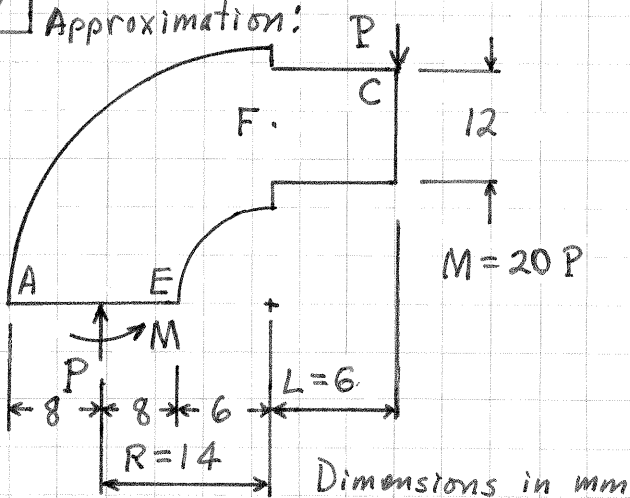
1.6 Should approach, with mesh refinements:

$\sigma_x = 0$ along AB, CD, and at E
 $\sigma_y = 0$ along BC and the straight part of ED
 $\tau_{xy} = 0$ along all of the boundary except the arc of radius r

σ_y and τ_{xy} very large & negative at C

σ_y large and negative at E

1.7 Approximation:

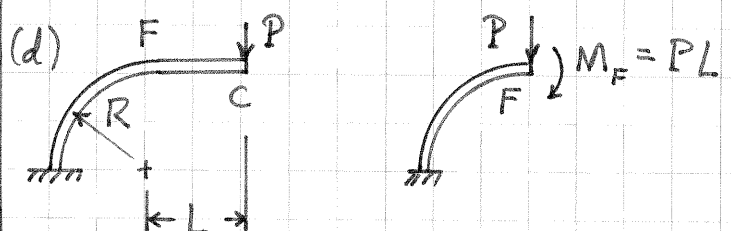


(c) At $r = r_n$, $\int \frac{dA}{r} = t \ln \frac{12.315}{6} = 0.719t$

Largest σ_r is almost at the neutral axis. Let's evaluate at the neutral axis.

$\sigma_r = \frac{M}{Ae r_n} \left(r_n \int_b^{r_n} \frac{dA}{r} - A r \right)$
 $\sigma_r = \frac{-20P}{16t(1.685)t(12.315)} (12.315(0.719t) - 6.315t)$

$\sigma_r = -0.153 \frac{P}{t}$



Using theory & formulas of Ref. 2.1,

$\theta_F = \frac{PR^2}{EI} + \frac{\pi}{2} \frac{M_F R}{EI}$

$V_F = \frac{\pi}{4} \frac{PR^3}{EI} + \frac{M_F R^2}{EI}$

$I = \frac{t}{12} 16^3$

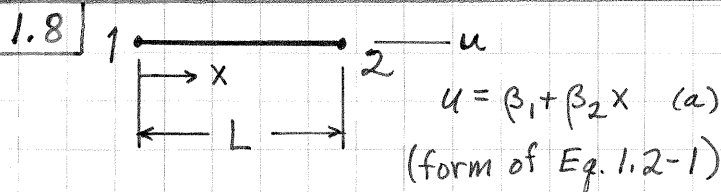
$$V_D = V_F + \theta_F L + \frac{PL^3}{3EI_s} \quad I_s = \frac{t}{12} 12^3$$

$$V_D = \frac{PR^2}{EI} \left(\frac{\pi R}{4} + L + L + \frac{\pi L^2}{2R} \right) + \frac{PL^3}{3EI_s}$$

$$V_D = \frac{P(14)^2}{341Et} \left(\frac{\pi}{4} 14 + 12 + \frac{\pi}{2} \frac{6^2}{14} \right) + \frac{P(6)^3}{3E(144t)} = \frac{16.0 P}{Et} \quad (\text{down})$$

$$\Delta_3 = 0.4554 \frac{PLT}{Eth_0} \quad -1.5\%$$

$$(c) (\sigma_x)_{ex} = -\frac{P}{A} = \frac{-P}{t(h_0 + \frac{3h_0}{L_T}x)} = \frac{-P}{th_0(1 + \frac{3x}{L_T})}$$



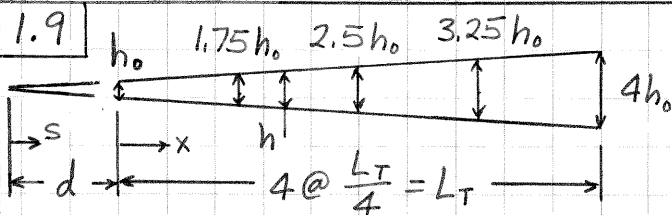
To obtain form of Eq. 1.2-3, evaluate (a) at $x=0$ and at $x=L$.

$$\left. \begin{aligned} x=0: & \quad u_1 = \beta_1 \\ x=L: & \quad u_2 = \beta_1 + \beta_2 L \end{aligned} \right\} \therefore \beta_1 = u_1$$

$$\beta_2 = \frac{u_2 - u_1}{L}$$

Eq. (a) becomes $u = u_1 + \frac{u_2 - u_1}{L} x$

$$\text{or } u = \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2$$



$$(a) \frac{4h_0}{L_T + d} = \frac{h_0}{d}, \quad d = \frac{L_T}{3}$$

$$\frac{h}{s} = \frac{h_0}{d}, \quad h = \frac{h_0}{d} s = \frac{3h_0}{L_T} s$$

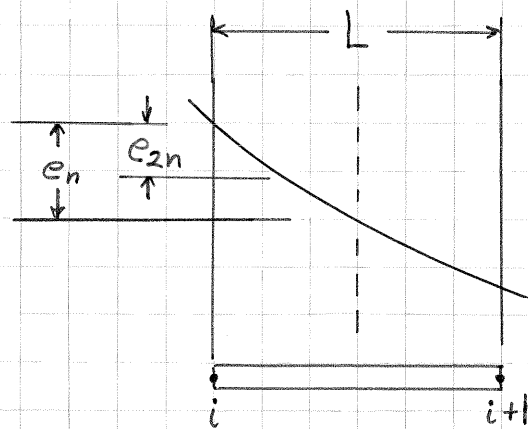
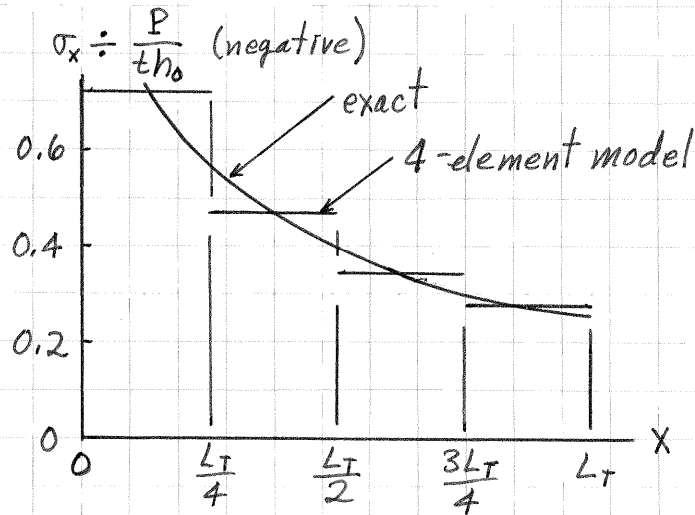
$$\Delta_{ex} = \int_E ds = \int_d^{d+L_T} \frac{P}{Eth} ds = \int_{L_T/3}^{4L_T/3} \frac{P ds}{\frac{3th_0}{L_T} s E}$$

$$\Delta_{ex} = \frac{PL_T}{3Eth_0} \ln 4 = 0.4621 \frac{PL_T}{Eth_0}$$

$$(b) \Delta_1 = \frac{PL_T}{(2.5th_0)E} = 0.4 \frac{PL_T}{th_0 E} \quad -13.4\%$$

$$\Delta_2 = \frac{P(L_T/2)}{(1.75th_0)E} + \frac{P(L_T/2)}{(3.25th_0)E} = 0.4396 \frac{PL_T}{Eth_0} \quad -4.9\%$$

$$\Delta_3 = \frac{P(L_T/4)}{(1.375th_0)E} + \frac{P(L_T/4)}{(2.125th_0)E} + \frac{P(L_T/4)}{(2.875th_0)E} + \frac{P(L_T/4)}{(3.625th_0)E}$$



$e_n = \text{el. nodal } \sigma \text{ error for } n \text{ elements}$
 $e_{2n} = \text{" " " " " } 2n \text{ "}$

We see that when elements are small enough for the curve to be almost a straight line over L , stress error is halved when no. of els. is doubled.