CHAPTER 1

Review of Prerequisite Mathematics

1-1.

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= \|\mathbf{v}\|\ \|\mathbf{w}\|\ \cos\theta = \frac{1}{2}\left[\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - \|\mathbf{v}-\mathbf{w}\|^2\right] \\ &= \frac{1}{2}\left[v_x^2 + v_y^2 + w_x^2 + w_y^2 - (v_x - w_x)^2 - (v_y - w_y)^2\right] \\ &= v_x w_x + v_y w_y\ .\end{aligned}$$

1-2. (a) Begin with $v_1'\mathbf{e}_1' + v_2'\mathbf{e}_2' = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$. Because $\mathbf{e}_1'\cdot\mathbf{e}_2' = 0$, to isolate v_1' on the left hand side, take the dot product of both sides with $\mathbf{e}_1'$. Then

$$\begin{aligned} v_1' &= v_1\mathbf{e}_1\cdot\mathbf{e}_1' + v_2\mathbf{e}_2\cdot\mathbf{e}_1' \\ \Longrightarrow\ a_{11} &= \mathbf{e}_1\cdot\mathbf{e}_1', \quad a_{12} = \mathbf{e}_2\cdot\mathbf{e}_1' \\ \text{and similarly,}\quad a_{21} &= \mathbf{e}_1\cdot\mathbf{e}_2', \quad a_{22} = \mathbf{e}_2\cdot\mathbf{e}_2'\ .\end{aligned}$$

(b) In the one basis,

$$\mathbf{v}\cdot\mathbf{w} = (v_1\mathbf{e}_1 + v_2\mathbf{e}_2)\cdot(w_1\mathbf{e}_1 + w_2\mathbf{e}_2) = v_1 w_1 + v_2 w_2$$

by the orthonormality of the basis vectors. Similarly, using the other basis,

$$\mathbf{v}\cdot\mathbf{w} = v_1' w_1' + v_2' w_2'\ .$$

(c) In the primed basis,

$$\mathbf{v}' + \mathbf{w}' = \left(v_1'\mathbf{e}_1' + v_2'\mathbf{e}_2'\right) + \left(w_1'\mathbf{e}_1' + w_2'\mathbf{e}_2'\right) = \left(v_1' + w_1'\right)\mathbf{e}_1' + \left(v_2' + w_2'\right)\mathbf{e}_2'$$

Substitute for the primed coefficients from part (a),

$$\begin{aligned}\mathbf{v}' + \mathbf{w}' = &(v_1 + w_1)\left(\mathbf{e}_1\cdot\mathbf{e}_1'\right)\ \mathbf{e}_1' + (v_2 + w_2)\left(\mathbf{e}_2\cdot\mathbf{e}_1'\right)\ \mathbf{e}_1' \\ &+ (v_1 + w_1)\left(\mathbf{e}_1\cdot\mathbf{e}_2'\right)\ \mathbf{e}_2' + (v_2 + w_2)\left(\mathbf{e}_2\cdot\mathbf{e}_2'\right)\ \mathbf{e}_2'\end{aligned}$$

and collect terms,

$$= (v_1 + w_1)\underbrace{\left[\left(\mathbf{e}_1\cdot\mathbf{e}_1'\right)\ \mathbf{e}_1' + \left(\mathbf{e}_1\cdot\mathbf{e}_2'\right)\ \mathbf{e}_2'\right]}_{=\mathbf{e}_1} + (v_2 + w_2)\underbrace{\left[\left(\mathbf{e}_2\cdot\mathbf{e}_1'\right)\ \mathbf{e}_1' + \left(\mathbf{e}_2\cdot\mathbf{e}_2'\right)\ \mathbf{e}_2'\right]}_{=\mathbf{e}_2}$$

(d) Apply the scalar to the vector in the primed basis and see what happens to the coefficients after transformation to the original basis. In the primed basis, $c'\mathbf{v} = c'\left(v'_1\mathbf{e}'_1 + v'_2\mathbf{e}'_2\right) = c'v'_1\mathbf{e}'_1 + c'v'_2\mathbf{e}'_2$. Substitute for the primed coefficients from part (a),

$$\begin{aligned} c'v'_1\mathbf{e}'_1 + c'v'_2\mathbf{e}'_2 &= c'\left(v_1\mathbf{e}_1\cdot\mathbf{e}'_1 + v_2\mathbf{e}_2\cdot\mathbf{e}'_1\right)\mathbf{e}'_1 + c'\left(v_1\mathbf{e}_1\cdot\mathbf{e}'_2 + v_2\mathbf{e}_2\cdot\mathbf{e}'_2\right)\mathbf{e}'_2 \\ &= c'v_1\underbrace{\left(\mathbf{e}_1\cdot\mathbf{e}'_1\ \mathbf{e}'_1 + \mathbf{e}_1\cdot\mathbf{e}'_2\ \mathbf{e}'_2\right)}_{=\mathbf{e}_1} + c'v_2\underbrace{\left(\mathbf{e}_2\cdot\mathbf{e}'_1\ \mathbf{e}'_1 + \mathbf{e}_2\cdot\mathbf{e}'_2\ \mathbf{e}'_2\right)}_{=\mathbf{e}_2} = c'\mathbf{v} \end{aligned}$$

The scalar is unchanged by the change of basis.

1-3. (a) Write the function $f(x) = f_e(x) + f_o(x)$. By the definition of even and odd, $f(-x) = f_e(x) - f_o(x)$. Adding these two equations gives f_e, and subtracting gives f_o (1.16).

(b) Write $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$. With f odd, changing variables in the first integral gives $\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-x)\,dx = -\int_{0}^{a} f(x)\,dx$, which cancels the second integral, leading to (1.17).

(c) Write $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$. With f even, changing variables in the first integral gives $\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-x)\,dx = \int_{0}^{a} f(x)\,dx$, which is the same as the second integral and leads to (1.18).

1-4. (a) $f(x)$ is a step function that jumps from 0 to 1 at $x = 1$. $f(-x)$ is a step function that jumps from 1 to 0 at $x = -1$. Adding the two (make a sketch),

$$f_e(x) = \begin{cases} \frac{1}{2}, & |x| > 1 \\ \frac{1}{4}, & |x| = 1 \\ 0, & |x| < 1 \end{cases}$$

and subtracting, $f_o(x) = f_e(x)\ \mathrm{sgn}(x)$.

(b) $f(-x) = e^{x}U(-x)$. Adding the two gives

$$f_e(x) = \frac{1}{2}e^{-|x|},$$

a two-sided exponential. Subtracting gives $f_o(x) = f_e(x)\ \mathrm{sgn}(x)$.

(c) $f(x)$ is a ramp with unit slope between 0 and 1, and zero otherwise (make a sketch). $f(-x)$ is a ramp with slope -1 between -1 and 0, and zero otherwise. Adding the two,

$$f_e(x) = \begin{cases} \frac{|x|}{2}, & |x| < 1 \\ \frac{1}{4}, & |x| = 1 \\ 0, & |x| > 1 \end{cases}$$

Subtracting gives

$$f_o(x) = f_e(x)\ \mathrm{sgn}(x) = \begin{cases} x, & |x| < 1 \\ \pm\frac{1}{4}, & x = \pm 1 \\ 0, & |x| > 1 \end{cases}$$

It may be tempting to think, from these problems, that the odd part is always the even part times a signum, but this is not so.

1-5. The odd part of a function is $f_o(x) = \dfrac{f(x) - f(-x)}{2}$. Its derivative is $f_o'(x) = \dfrac{f'(x) - (-1)f'(-x)}{2} = \dfrac{f'(x) + f'(-x)}{2}$, which is, by definition, the even part of f'. Therefore f_o' is even. The same reasoning, beginning with the definition of $f_e(x)$, shows that $f_e'(x)$ is odd.

1-6. (a) Let $A = B$ in (1.20). Then $\sin 2A = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A$, and similarly, $\cos 2A = \cos^2 A - \sin^2 A$.

(b) Change B to $-B$ in (1.20). Then $\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B) = \sin A \cos B - \cos A \sin B$, and similarly, $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

(c) Write $\cos 2A = \cos^2 A - \sin^2 A$ (part (a)) $= \left(1 - \sin^2 A\right) - \sin^2 A = 1 - 2\sin^2 A$. Solve for $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$. Similarly, $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$.

1-7.

$$\begin{aligned} C \sin(2\pi\nu t + \varphi) &= C \sin(2\pi\nu t)\cos\varphi + C\cos(2\pi\nu t)\sin\varphi \\ &= (C\sin\varphi)\cos(2\pi\nu t) + (C\cos\varphi)\sin(2\pi\nu t)\ . \end{aligned}$$

In this form, $A = C\sin\varphi$ and $B = C\cos\varphi$. We still have $C = \sqrt{A^2 + B^2}$, but $\tan\varphi = A/B$.

1-8. The Taylor series for e^x is $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$. (All derivatives of e^x are e^x, and evaluate to 1 at $x = 0$.). Substitute $x = i\theta$, and

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots}_{\cos\theta} + \underbrace{i\theta - i\frac{\theta^3}{3!} + \ldots}_{i\sin\theta}$$

To verify the Taylor series coefficients for $\sin\theta$,

$$\sin 0 = 0$$
$$\left.\frac{d\sin\theta}{d\theta}\right|_0 = \cos 0 = 1$$
$$\left.\frac{d^2\sin\theta}{d\theta^2}\right|_0 = -\sin 0 = 0$$
$$\left.\frac{d^3\sin\theta}{d\theta^3}\right|_0 = -\cos 0 = -1$$
$$\vdots$$

and similarly for $\cos\theta$.

1-9. The easier way is

$$|\exp(i\theta)|^2 = \exp(i\theta)\exp(-i\theta) = \exp(0) = 1,$$

but using the Euler equations,

$$|\exp(i\theta)|^2 = |\cos\theta + i\sin\theta|^2 = \cos^2\theta + \sin^2\theta = 1$$

.

1-10. We want to show $S_N = \sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}$. It is easy to see that $S_1 = 1$ and $S_2 = \frac{1-x^2}{1-x} = 1+x$. Now, $S_K = \sum_{n=0}^{K-1} x^n = \sum_{n=0}^{K-2} x^n + x^{K-1} = S_{K-1} + x^{K-1}$, and assuming $S_{K-1} = \frac{1-x^{K-1}}{1-x}$,

$$S_K = \frac{1-x^{K-1}}{1-x} + x^{K-1} = \frac{1-x^{K-1}+(1-x)x^{K-1}}{1-x} = \frac{1-x^K}{1-x}\ .$$

1-11. (a) $\frac{1-e^{2x}}{x}$ is of the form $0/0$ as $x \to 0$, so apply L'Hospital's rule,

$$\lim_{x\to 0}\frac{1-e^{2x}}{x} = \lim_{x\to 0}\frac{-2e^{2x}}{1} = -2$$

(b) $\frac{1-\cos\pi x}{\pi x}$ is of the form $0/0$ as $x \to 0$, so apply L'Hospital's rule,

$$\lim_{x\to 0}\frac{1-\cos\pi x}{\pi x} = \lim_{x\to 0}\frac{\pi\sin\pi x}{\pi} = 0$$

1-12. (a) Let $u = x$, $dv = e^{ax}dx$, then

$$\int xe^{ax}\,dx = \frac{1}{a}xe^{ax} - \int \frac{1}{a}e^{ax}\,dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax} = \frac{ax-1}{a^2}e^{ax}\ .$$

(b) Let $u = x^2$, $dv = \cos bx$, then $du = 2x dx$ and $v = \frac{1}{b}\sin bx$, and

$$\int x^2 \cos bx\,dx = \frac{x^2}{b}\sin bx - \int \frac{2}{b}x \sin bx\,dx.$$

Now let $u = \frac{2x}{b}$, $dv = \sin bx\,dx$, then $du = \frac{2}{b}\,dx$ and $v = -\frac{1}{b}\cos bx$, and

$$\begin{aligned}\int x^2 \cos bx\,dx &= \frac{x^2}{b}\sin bx + \frac{2x}{b^2}\cos bx - \int \frac{2}{b^2}\cos bx\,dx \\ &= \frac{x^2}{b}\sin bx + \frac{2x}{b^2}\cos bx - \frac{2}{b^3}\sin bx = \frac{b^2x^2-2}{b^3}\sin bx + \frac{2x}{b^2}\cos bx\ .\end{aligned}$$

1-13. (a) $x^{-1/2}$ grows more slowly than $1/x$ as $x \to 0$, so it is integrable. The integral is

$$\int_0^1 \frac{dx}{x^{1/2}} = 2x^{1/2}\Big|_0^1 = 2\ .$$

(b) $\dfrac{1}{1+x^2}$ decays more rapidly than $1/x$ as $|x| \to \infty$, so it is integrable. Using an integral table or symbolic integrator,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \arctan x\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

1-14. (a) The integrand is positive and even, and grows faster than $1/x$ as $x \to 0$, so it is not integrable.

(b) The integrand grows faster than $1/x$ as $x \to 0$, but because it is the integral of an odd function over a symmetric interval, we expect the integral to be zero. Calculate the Cauchy principal value,

$$\begin{aligned}\mathcal{P}\int_{-1}^{1} \frac{dx}{x^3} &= \lim_{\epsilon\to 0}\left[\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^{1} \frac{dx}{x^3}\right] \\ &= \lim_{\epsilon\to 0}\left[-\frac{1}{2}\left(\frac{1}{\epsilon^2} - \frac{1}{(-1)^2}\right) - \frac{1}{2}\left(1 - \frac{1}{\epsilon^2}\right)\right] = 0\end{aligned}$$

(c) Using the result of part (b), we may write

$$\mathcal{P}\int_{-1}^{2} \frac{dx}{x^3} = \underbrace{\mathcal{P}\int_{-1}^{1} \frac{dx}{x^3}}_{0} + \int_{1}^{2} \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_1^2 = \frac{3}{8}\ .$$

CHAPTER 2

Vector Spaces

2-1. (a) It is not a vector space, because there cannot be an additive inverse if all the signals are positive. Other than that, it works.

(b) This is a vector space. The key point is closure under addition, and the sum of two signals $A\cos(\omega t + \alpha) + B\cos(\omega t + \beta)$ has frequency ω (use Equation 1.21).

(c) This is not a vector space. The set is not closed under addition (can get pixel values greater than 255). Also, there is no additive inverse.

(d) *Invalid problem, don't assign.* This appears to be better that (c), because the set is now closed under addition and under scalar multiplication. However, there is an algebraic problem that was glossed over in the text. The definition of vector space used here effectively assumes that the scalars are either real or complex numbers. Any other set of scalars must comprise what is called a *field*, with certain addition and multiplication properties that these numbers lack.

(e) One-sided signals do comprise a vector space. The key point is closure under addition and scalar multiplication. Adding two one-sided signals results in a one-sided signal; multiplication by a scalar results in a one-sided signal.

(f) One-sided signals that decay exponentially do comprise a vector space. Again, closure is the key point. The sum of two exponentially decaying functions decays exponentially (dominated by the slower decay).

(g) Piecewise-constant functions comprise a vector space. The sum of two piecewise-constant functions is also piecewise constant, and the product of a scalar with a piecewise-constant function is piecewise constant. This is the key point.

(h) Waves on a 1m string with fixed ends are functions $f(x)$ defined on an interval $[0, 1]$ with $f(0) = f(1) = 0$. This set of functions is closed under addition and scalar multiplication, because the boundary conditions are still satisfied. This is the key point in establishing the set as a vector space.

(i) If two signals f and g have zero average value, their sum also has zero average value, and the product with a scalar, cf, also has zero average value. This is the key point in establishing the set as a vector space.

(j) The set of functions with unit area is not a vector space. The area under the sum of two such functions is 2, not 1, violating closure under addition.

2-2. (a) This is a vector space. Consider two such vectors, (x_1, x_2, x_3) and (y_1, y_2, y_3). The product with a scalar c is (cx_1, cx_2, cx_3), and $cx_1 + cx_2 + cx_3 = c(x_1 + x_2 + x_3) = 0$.

The sum of two vectors is $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$, and $(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 0$.

One of the vector components is determined by the other two, *e.g.*, $x_3 = -x_1 - x_2$, which reduces the dimensionality of the space to two. Two basis vectors are $(1, -1, 0)$ and $(0, 1, -1)$. They are linearly independent, but not orthogonal.

(b) This is not a vector space. The sum of two such vectors, (x_1, x_2, x_3) and (y_1, y_2, y_3), is $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$, and $(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 2$, not 1.

(c) {Real $m \times n$ matrices A} is a vector space. The key point is closure under addition and scalar multiplication, which holds.

The space is mn-dimensional. A basis consists of mn matrices, with 1 in the ij location and 0 in all the other locations, for $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$.

(d) {Real $n \times n$ matrices A with $\det A = 0$} is not a vector space, because the determinant is not a linear operation. For a counterexample, consider $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Their determinants are each 0, but their sum is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which has determinant 1.

(e) Square symmetric matrices are a vector space. If you add two symmetric matrices, the result is symmetric: $(A + B)' = A' + B' = A + B$. If you multiply a symmetric matrix by a scalar, the result is symmetric: $(cA)' = c'A' = cA' = cA$. The additive inverse, $-A$, is also symmetric. These are the key points.

For a basis, we seek a set of square symmetric matrices that are linearly independent and span the space of square symmetric matrices. One such set, for $n = 3$, is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

Here there are six basis matrices and the dimension of the space is, correspondingly, six. In general, for $n \times n$ matrices, there are $n + (n-1) + (n-2) + \ldots + 1 = n(n+1)/2$ basis matrices (dimensions).

(f) n^{th}-degree polynomials are a vector space. The sum of two such polynomials is an n^{th}-degree polynomial. Multiplying by a scalar results in an n^{th}-degree polynomial. The additive inverse, obtained by multiplying all coefficients by -1, is an n^{th}-degree polynomial. These are the key points.

For a basis, we need linearly independent n^{th}-degree polynomials. Here is a set that works: $\{x^n, x^n + 1, x^n + x, x^n + x^2, \ldots, x^n + x^{n-1}\}$ (n dimensions).

(g) These polynomials do not comprise a vector space. The sum of two such polynomials, $f + g$, at $x = 0$, is $(f + g)(0) = f(0) + g(0) = 2 \neq 1$.

(h) These polynomials do comprise a vector space. The sum of two such polynomials, $f + g$, at $x = 1$, is $(f + g)(1) = f(1) + g(1) = 0$. Multiplying by a scalar, $cf(1) = c \cdot 0 = 0$. The additive inverse, $-f$, has $-f(1) = 0$. These are the key points.

For a basis, we seek linearly independent polynomials of degree $\leq n$, with $p(1) = 0$. Here is a set that works: $\left\{x^n - 1, x^{n-1} - 1, \ldots, x - 1\right\}$. The dimension of the space is n. There is no nontrivial $n = 0$ (constant) member of the space and no constant basis element. If there were, because $p(1) = 0$, the constant would be zero everywhere, which is trivial.

2-3. For continuous functions $f, g, h \in C[0, 1]$: (a) The sum $(f + g)(x)$ is continuous and the product $(cf)(x) = cf(x)$ is continuous. Because these functions are real-valued, at every x they satisfy commutative, associative and distributive properties (b-d) and then, by (a), these combinations are continuous. (e) The additive identity element is 0. (f) Each function f has an additive inverse $(-f)(x) = -1 \cdot f(x)$ which is continuous, by (a). (g) The multiplicative identity element is 1.

2-4. For $\|v\| = \sqrt{|v_1|^2 + |v_2|^2} = \sqrt{v_1 v_1^* + v_2 v_2^*}$, applying Definition 2.2,

(a) The squared magnitudes $|v_1|^2$ and $|v_2|^2$ are nonnegative, so the square root of their sum must be nonnegative.

(b) Clearly, if $v = 0$ then $\|v\| = 0$. On the other hand, if $\|v\| = 0$, then $|v_1|^2 + |v_2|^2 = 0$, and because $|v_1|^2$ and $|v_2|^2$ are nonnegative, they can sum to zero only if they are both zero, hence $v = 0$.

(c) $\|cv\| = \sqrt{|cv_1|^2 + |cv_2|^2} = \sqrt{c^2\left(|v_1|^2 + |v_2|^2\right)} = |c|\sqrt{|v_1|^2 + |v_2|^2} = |c|\|v\|$.

(d) To verify the triangle inequality, write

$$\|v + w\|^2 = \langle v + w, v + w\rangle = \|v\|^2 + \|w\|^2 + \langle v, w\rangle + \langle v, w\rangle^*$$

Now, by the triangle inequality (since each term is a real or complex number),

$$\|v + w\|^2 = \left|\|v + w\|^2\right| \leq \|v\|^2 + \|w\|^2 + 2\,|\langle v, w\rangle| \leq \|v\|^2 + \|w\|^2 \ .$$

2-5.

$$\|x\|_1 = |1| + |2i| = 1 + 2 = 3$$

$$\|x\|_2 = \sqrt{|1|^2 + |2i|^2} = \sqrt{1 + 4} = \sqrt{5}$$

$$\|x\|_\infty = \max\{|1|, |2i|\} = 2$$

2-6. Write $x = (x - y) + y$ and apply the triangle inequality,

$$\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\|$$

then rearrange terms,

$$\|x - y\| \geq \|x\| - \|y\| \ .$$

By similar reasoning, exchanging x and y,

$$\|y - x\| \geq \|y\| - \|x\| = -\left(\|x\| - \|y\|\right)$$

Together, these imply that

$$\|x - y\| \geq \Big| \|x\| - \|y\| \Big| \ .$$

2-7. In $\mathbb{R}^n$, $d_1(x,y) = \displaystyle\sum_{k=1}^{n} |x_k - y_k|$. Squaring both sides,

$$\begin{aligned} d_1^2(x,y) &= \left(\sum_{k=1}^{n} |x_k - y_k|\right)^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \left|x_j - y_j\right| \, |x_k - y_k| \\ &= \underbrace{\sum_{k=1}^{n} |x_k - y_k|^2}_{d_2^2(x,y)} + \underbrace{\sum_{j \neq k} \left|x_j - y_j\right| \, |x_k - y_k|}_{\geq 0} \\ \implies \ & d_1(x,y) \geq d_2(x,y) \end{aligned}$$

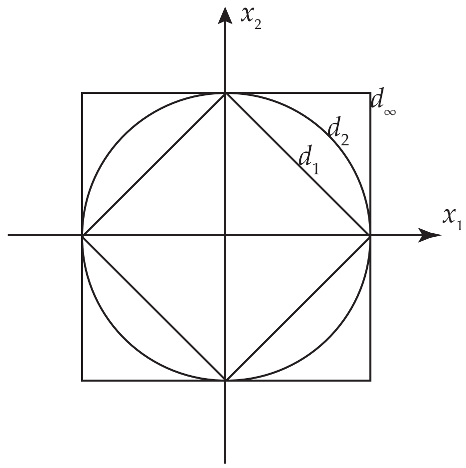
Next,

$$\begin{aligned} d_2^2(x,y) &= \sum_{k=1}^{n} |x_k - y_k|^2 = \underbrace{\left(\max_k |x_k - y_k|\right)^2}_{d_\infty^2(x,y)} + \underbrace{n-1 \text{ other terms}}_{\geq 0} \\ \implies \ & d_2(x,y) \geq d_\infty(x,y) \end{aligned}$$

2-8.
- For $d_1(0,x) = |x_1| + |x_2| = 1$, the "unit circle" is a diamond:

$$x_2 = \pm\left(1 - |x_1|\right) = \begin{cases} \pm\left(1 - x_1\right), & 1 \geq x \geq 0 \\ \pm\left(1 + x_1\right), & 0 \geq x \geq -1 \end{cases}$$

- For $d_2(0,x) = \sqrt{|x_1|^2 + |x_2|^2} = 1$, the "unit circle" is the unit circle.
- For $d_\infty(0,x) = \max\{|x_1|, |x_2|\} = 1$, the "unit circle" is a square: $|x_2| = 1$ for $|x_1| < 1$, and $|x_1| = 1$ for $|x_2| < 1$.



Comparing with Problem 2-7, $d_1(0,x) \geq d_2(0,x) \geq d_\infty(0,x)$, the "unit circles" are ordered with d_1 inside d_2 inside d_∞. d_∞ is the smallest of the metrics, so the points for which $d_\infty(0,x) = 1$ will be the farthest from the origin.

2-9. (a) $\langle v, cw\rangle = \langle cw, v\rangle^* = c^* \langle w, v\rangle^* = c^* \langle v, w\rangle$

(b) $\langle u, v+w\rangle = \langle v+w, u\rangle^* = \langle v, u\rangle^* + \langle w, u\rangle^* = \langle u, v\rangle + \langle u, w\rangle$

(c) For any vector u, $0u = 0$, so $\langle v, 0\rangle = \langle v, 0u\rangle = 0\,\langle v, u\rangle = 0$, and similarly for $\langle 0, v\rangle$.

(d) If $\langle u, v\rangle = \langle u, w\rangle$, then $\langle u, v\rangle - \langle u, w\rangle = \langle u, v-w\rangle = 0$. That is, $v - w$ is orthogonal to every $u \in V$, which implies $v - w = 0$, hence $v = w$.

2-10. With reference to Figure 2.2, using the Law of Cosines,

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\| \cos(\text{angle opposite } u-v)$$
$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\| \cos(\text{angle opposite } u+v)$$

Adding the two,

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 - 2\|u\|\|v\| \left[\cos(\text{angle opposite } u+v) + \cos(\text{angle opposite } u-v)\right] .$$

In a parallelogram, the angles sum to 360°, so the angle opposite $u+v$ is 180°− the angle opposite $u-v$. Now, $\cos(\pi - A) = -\cos A$, so the sum of the two cosines is 0, leaving the parallelogram law.

2-11. (a) In an inner product space, $\langle u, u\rangle = \|u\|^2$. Then,

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle$$
$$\|u-v\|^2 = \langle u-v, u-v\rangle = \langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle$$

Now add these two equations.

(b) In the parallelogram law, $\|u+v\|^2+\|u-v\|^2 = 2\|u\|^2+2\|v\|^2$. Since $\|u-v\|^2 \geq 0$, subtracting it from the left hand side makes the left hand side $\leq$ the right hand side.

2-12. (a) For real u, v, use the equations for $\|u+v\|^2$ and $\|u-v\|^2$ from the previous proof, and subtract them instead of adding, giving

$$\|u+v\|^2 - \|u-v\|^2 = 2\langle u, v\rangle + 2\langle v, u\rangle = 4\langle u, v\rangle \ .$$

For complex u, v, we cannot equate the two inner products,

$$\|u+v\|^2 - \|u-v\|^2 = 2\langle u, v\rangle + 2\langle v, u\rangle$$

Derive a similar expression for $\|u+iv\|^2 - \|u-iv\|^2$,

$$\|u+iv\|^2 - \|u-iv\|^2 = -2i\langle u, v\rangle + 2i\langle v, u\rangle$$

Then eliminate the $\langle v, u\rangle$ term by combining the two expressions,

$$\begin{aligned}\|u+v\|^2 - \|u-v\|^2 + i\left(\|u+iv\|^2 - \|u-iv\|^2\right) &= 2\langle u, v\rangle + 2\langle v, u\rangle + i\left(-2i\langle u, v\rangle + 2i\langle v, u\rangle\right)\\ &= 4\langle u, v\rangle\end{aligned}$$

(b) Apply the triangle inequality, then the parallelogram law,

$$\begin{aligned}|4\langle u, v\rangle| &\leq \underbrace{\|u+v\|^2 + \|u-v\|^2}_{2\|u\|^2+2\|v\|^2} + \underbrace{\|u+iv\|^2 + \|u-iv\|^2}_{2\|u\|^2+2\|v\|^2}\\ \implies |\langle u, v\rangle| &\leq \|u\|^2 + \|v\|^2\end{aligned}$$

For real u, v, begin with the real result,

$$\begin{aligned}|4\langle u, v\rangle| &\leq \underbrace{\|u+v\|^2 + \|u-v\|^2}_{2\|u\|^2+2\|v\|^2}\\ \implies |\langle u, v\rangle| &\leq \frac{\|u\|^2 + \|v\|^2}{2}\end{aligned}$$

2-13. Assume that the parallelogram law holds. Then it holds in particular for the vectors $u = (a, 0, 0, \ldots, 0)$ and $v = (0, b, 0, \ldots, 0)$. Calculating the various norms,

$$\begin{aligned}\|u\|_1 &= \|(a, 0, 0, \ldots, 0)\|_1 = |a|\\ \|v\|_1 &= \|(0, b, 0, \ldots, 0)\|_1 = |b|\\ \|u+v\|_1 &= \|(a, b, 0, \ldots, 0)\|_1 = |a| + |b|\\ \|u-v\|_1 &= \|(a, -b, 0, \ldots, 0)\|_1 = |a| + |b|\end{aligned}$$

and substituting into the parallelogram law,

$$\|u+v\|_1^2 + \|u-v\|_1^2 = 2\|u\|_1^2 + 2\|v\|_1^2 \ ,$$

we obtain

$$2(|a| + |b|)^2 = 2|a|^2 + 2|b|^2 \ ,$$

which is only true in the trivial case $a = 0$ or $b = 0$. A similar calculation shows that $\|u\|_\infty$ also cannot be calculated as an inner product.

2-14. (a) $\langle au, bv\rangle = ab^* \langle u, v\rangle = 0$

(b) $\langle u+v, u-v\rangle = \underbrace{\langle u,u\rangle}_{\|u\|^2} + \underbrace{\langle v,u\rangle}_{=0} - \underbrace{\langle u,v\rangle}_{=0} - \underbrace{\langle v,v\rangle}_{\|v\|^2} = 0.$

2-15. (a) Norms of u:

$$\|u\|_1 = |2| + |1+i| + |i| + |1-i| = 2 + \sqrt{2} + 1 + \sqrt{2} = 3 + 2\sqrt{2}$$
$$\|u\|_2 = \sqrt{|2|^2 + |1+i|^2 + |i|^2 + |1-i|^2} = \sqrt{4+2+1+2} = 3$$
$$\|u\|_\infty = 2$$

(b) Norms of v:

$$\|v\|_1 = |1| + |i| + |-1| + |i| = 4$$
$$\|v\|_2 = \sqrt{4} = 2$$
$$\|v\|_\infty = 1$$

(c) $\langle u, v\rangle = 2\times 1 + (1+i)\times -i + i \times -1 + (1-i)\times -i = 2 - 3i$

(d) $\langle v, u\rangle = 1 \times 2 + i \times (1-i) - 1 \times (-i) + i \times (1+i) = 2 + 3i$

Note $\|u\|_1 > \|u\|_2 > \|u\|_\infty$, $\|v\|_1 > \|v\|_2 > \|v\|_\infty$, and $\langle u, v\rangle = \langle v, u\rangle^*$.

2-16. If the vectors u and v are orthogonal, then

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \underbrace{\langle u,v\rangle}_{0} + \underbrace{\langle v,u\rangle}_{0} + \langle v,v\rangle = \|u\|^2 + \|v\|^2.$$

Conversely, if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, then

$$\langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle = \langle u,u\rangle + \langle v,v\rangle$$
$$\implies \langle u,v\rangle + \langle v,u\rangle = 2\mathcal{R}e\,\langle u,v\rangle = 0$$

If u and v are real, then their inner product has no imaginary part, and $2\mathcal{R}e\,\langle u,v\rangle = 2\langle u,v\rangle = 0$, which implies u and v are orthogonal. If u and v are complex, then their inner product can still have a nonzero imaginary part and satisfy $2\mathcal{R}e\,\langle u,v\rangle = 0$; that is, they can be nonorthogonal.

2-17.

$$\|e_1 - e_2\|_2^2 = \langle e_1 - e_2, e_1 - e_2\rangle = \underbrace{\langle e_1, e_1\rangle}_{1} - \underbrace{\langle e_1, e_2\rangle}_{0} - \underbrace{\langle e_2, e_1\rangle}_{0} + \underbrace{\langle e_2, e_2\rangle}_{1} = 2$$

$$\implies \|e_1 - e_2\|_2 = \sqrt{2}$$

2-18.

$$\|e_m - e_n\|^2 = \langle e_m - e_n, e_m - e_n\rangle = \|e_m\|^2 + \|e_n\|^2 - \langle e_m, e_n\rangle - \langle e_m, e_n\rangle^*$$

$$\|e_m\|^2 = \int_{-\pi}^{\pi} \left|\frac{1}{\sqrt{2\pi}} e^{imx}\right|^2 dx = \int_{-\pi}^{\pi} \frac{1}{2\pi}\, dx = 1$$

$$\langle e_m, e_n\rangle = \int_{-\pi}^{\pi} \frac{1}{2\pi} e^{i(m-n)x}\, dx = \frac{1}{2\pi}\frac{1}{i(m-n)} e^{i(m-n)x}\Big|_{-\pi}^{\pi}$$

$$= \frac{\sin \pi(m-n)}{\pi(m-n)} = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}$$

$$\implies \|e_m - e_n\| = \begin{cases} 0, & m = n \\ \sqrt{2}, & \text{otherwise} \end{cases}$$

This is the same as the distance between $\mathbf{e}_x$ and $\mathbf{e}_y$ in the plane.

2-19. (a) In $\mathbb{R}^4$, four vectors are needed for a basis.
(b) The four chosen vectors must be linearly independent in order to comprise a basis.

2-20. By the triangle inequality,

$$|\langle f, g\rangle + \langle g, f\rangle| \le |\langle f, g\rangle| + |\langle g, f\rangle| = 2\,|\langle f, g\rangle| = 2\|f\|_2\ \|g\|_2,$$

using the Cauchy-Schwarz inequality (Theorem 2.1).

2-21.

$$\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \langle x, y\rangle^*$$

$$\le \|x\|^2 + \|y\|^2 + |\langle x, y\rangle| + \left|\langle x, y\rangle^*\right| \qquad \text{(triangle inequality)}$$

$$\le \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2 \qquad \text{(Cauchy-Schwarz)}$$

$$\implies \|x + y\| \le \|x\| + \|y\|$$

2-22. For ℓ^1, $\|x\|_1 = \sum_n |x_n|$. The first three properties of the norm are obvious. For the triangle inequality, consider $x, y \in \ell_1$, and

$$\sum_{n=1}^{N} |x_n + y_n| \le \sum_{n=1}^{N} |x_n| + |y_n| \qquad \text{(triangle inequality in } \mathbb{C}\text{)}$$

$$= \sum_{n=1}^{N} |x_n| + \sum_{n=1}^{N} |y_n|$$