

THIRD EDITION

HEAT

CONDUCTION

David W. Hahn M. Necati Özışık

Solution Manual

(R)

1-1 Derive the heat conduction equation (1-43) in cylindrical coordinates using the differential control approach beginning with the general statement of conservation of energy. Show all steps and list all assumptions. Consider Fig. 1-7.

Assume quiescent medium with no mass flow in or out of the control volume.

Assume no work by control volume.

$$\dot{Q} + \dot{E}_{gen} = \frac{dE_{cv}}{dt} \quad \left. \vphantom{\dot{Q} + \dot{E}_{gen} = \frac{dE_{cv}}{dt}} \right\} \begin{array}{l} \text{cons.} \\ \text{of} \\ \text{energy} \end{array}$$

per figure 1-7:

$$dV = r d\phi \cdot dr \cdot dz$$

$$dm = \rho dV = \rho r d\phi dr dz$$

$$q_r = -k A_r \frac{\partial T}{\partial r} \quad \text{with } A_r = r d\phi dz$$

$$q_z = -k A_z \frac{\partial T}{\partial z} \quad \text{with } A_z = r d\phi dr$$

$$q_\phi = -k A_\phi \left(\frac{1}{r}\right) \frac{\partial T}{\partial \phi} \quad \text{with } A_\phi = dr dz$$

↳ Scale factor.

$$q_{r+dr} = q_r + \frac{\partial}{\partial r} (q_r) dr$$

$$q_{z+dz} = q_z + \frac{\partial}{\partial z} (q_z) dz$$

$$q_{\phi+d\phi} = q_\phi + \frac{\partial}{\partial \phi} (q_\phi) d\phi$$

$$\delta \dot{Q} = \dot{q}_r - \dot{q}_{r+dr} + \dot{q}_z - \dot{q}_{z+dz} + \dot{q}_\phi - \dot{q}_{\phi+d\phi}$$

using the above:

$$\begin{aligned} \rightarrow \delta \dot{Q} &= \frac{\partial}{\partial r} \left(k \cdot r d\phi dz \frac{\partial T}{\partial r} \right) dr + \frac{\partial}{\partial z} \left(k \cdot r d\phi dr \frac{\partial T}{\partial z} \right) dz \\ &+ \frac{\partial}{\partial \phi} \left(k \cdot \frac{dr dz}{r} \frac{\partial T}{\partial \phi} \right) d\phi \end{aligned}$$

$\delta \dot{E}_{sen} = g \cdot dV$, with g equal to rate of internal energy per unit volume (W/m^3)

$$\rightarrow \delta \dot{E}_{sen} = g \cdot r d\phi dr dz$$

$$\frac{dE}{dt}_{cv} = \frac{d}{dt} (dm \cdot u) = \rho dV \cdot \frac{du}{dt}$$

with u equal to internal energy per unit mass (J/kg)

$$\rightarrow u = cT + \text{constant}$$

$$\rightarrow \frac{dE}{dt} = \rho r d\phi dr dz \cdot c \frac{\partial T}{\partial t}$$

with c equal to the specific heat ($J/kg K$)

1-1

Substituting the above three terms into the energy equation:

$$\begin{aligned} & \frac{\partial}{\partial r} \left(k \cdot r d\phi dz \frac{\partial T}{\partial r} \right) dr + \frac{\partial}{\partial z} \left(k \cdot r d\phi dr \frac{\partial T}{\partial z} \right) dz \\ & + \frac{\partial}{\partial \phi} \left(k \cdot \frac{dr dz}{r} \frac{\partial T}{\partial \phi} \right) d\phi + g \cdot r dr \cdot d\phi \cdot dz \\ & = \rho c (r dr d\phi dz) \frac{\partial T}{\partial t} \end{aligned}$$

→ Assume $k = \text{constant}$, and divide all terms by $k \cdot r dr d\phi dz$.

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

- 1-2 Derive the heat conduction equation (1-46) in spherical coordinates using the differential control approach beginning with the general statement of conservation of energy. Show all steps and list all assumptions. Consider Fig. 1-8.

Assume quiescent medium with no mass flow in or out of the control volume.

Assume no work by control volume.

$$\left. \begin{array}{l} \text{Cons.} \\ \text{of} \\ \text{energy} \end{array} \right\} \delta \dot{Q} + \delta \dot{E}_{\text{gen}} = \frac{d\dot{E}_{\text{cv}}}{dt}$$

per Figure 1-8: $dV = (r d\theta)(r \sin\theta d\phi)(dr)$

$$dV = r^2 \sin\theta dr d\phi d\theta$$

$$dm = \rho dV$$

$$q_r = -k A_r \frac{\partial T}{\partial r} \quad \text{with } A_r = r^2 \sin\theta d\theta d\phi$$

$$q_\theta = -\frac{k}{r} A_\theta \frac{\partial T}{\partial \theta} \quad \text{with } A_\theta = r \sin\theta d\phi dr$$

$$q_\phi = \frac{-k}{r \sin\theta} A_\phi \frac{\partial T}{\partial \phi} \quad \text{with } A_\phi = r d\theta dr$$

→ noting scale factors of $\left(\frac{1}{r}\right)$ & $\left(\frac{1}{r \sin\theta}\right)$

$$q_{r+dr} = q_r + \frac{\partial}{\partial r}(q_r) dr$$

$$q_{\theta+d\theta} = q_\theta + \frac{\partial}{\partial \theta}(q_\theta) d\theta$$

$$q_{\phi+d\phi} = q_\phi + \frac{\partial}{\partial \phi}(q_\phi) d\phi$$

1-2

2/3

$$\delta \dot{Q} = \dot{q}_r - \dot{q}_{r+dr} + \dot{q}_\theta - \dot{q}_{\theta+d\theta} + \dot{q}_\phi - \dot{q}_{\phi+d\phi}$$

using the above expressions:

$$\begin{aligned} \rightarrow \delta \dot{Q} &= \frac{\partial}{\partial r} \left(k \cdot r^2 \sin \theta \, d\theta \, d\phi \frac{\partial T}{\partial r} \right) dr \\ &+ \frac{\partial}{\partial \theta} \left(\frac{k}{r} \cdot r \sin \theta \, d\phi \, dr \frac{\partial T}{\partial \theta} \right) d\theta \\ &+ \frac{\partial}{\partial \phi} \left(\frac{k}{r \sin \theta} \cdot r \, d\theta \, dr \frac{\partial T}{\partial \phi} \right) d\phi \end{aligned}$$

$\delta \dot{E}_{gen} = g \cdot dV$, with g equal to the rate of internal energy per unit volume (W/m^3).

$$\rightarrow \delta \dot{E}_{gen} = g \cdot r^2 \sin \theta \, dr \, d\phi \, d\theta$$

$$\frac{dE_{c.v.}}{dt} = \frac{d}{dt} (dm \cdot u) = \rho \, dV \cdot \frac{du}{dt}$$

with u equal to the internal energy per unit mass (J/kg)

$$\rightarrow u = cT + \text{constant}$$

$$\rightarrow \frac{dE}{dt} = \rho \, r^2 \sin \theta \, dr \, d\phi \, d\theta \cdot c \frac{\partial T}{\partial t}$$

with c equal to the specific heat ($J/kg \cdot K$)

1-2

Substituting the above three terms
into the energy equation:

$$\begin{aligned} & \frac{\partial}{\partial r} \left(k r^2 \sin \theta \, d\theta \, d\phi \, \frac{\partial T}{\partial r} \right) dr \\ & + \frac{\partial}{\partial \theta} \left(\frac{k}{r} \cdot r \sin \theta \, d\phi \, dr \, \frac{\partial T}{\partial \theta} \right) d\theta \\ & + \frac{\partial}{\partial \phi} \left(\frac{k}{r \sin \theta} \cdot r \, d\theta \, dr \, \frac{\partial T}{\partial \phi} \right) d\phi \\ & + g \cdot r^2 \sin \theta \, dr \, d\phi \, d\theta = \rho c r^2 \sin \theta \, dr \, d\phi \, d\theta \, \frac{dT}{dt} \end{aligned}$$

Assume $k = \text{constant}$, and divide
by $k \cdot r^2 \sin \theta \, dr \, d\theta \, d\phi$:

$$\begin{aligned} & \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) \\ & + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left(\frac{\partial T}{\partial \phi} \right) + \frac{g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \end{aligned}$$

$\underbrace{\hspace{10em}}_{= \frac{\partial^2 T}{\partial \phi^2}}$

1-3 Show that the following two forms of the differential operator in the cylindrical coordinate system are equivalent:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) = \frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr}$$

$$\text{LHS) } \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) =$$

$$\frac{1}{r} \left\{ \frac{\partial r}{\partial r} \frac{\partial T}{\partial r} + r \cdot \frac{\partial^2 T}{\partial r^2} \right\}$$

$$= \frac{1}{r} \left\{ 1 \cdot \frac{\partial T}{\partial r} + r \cdot \frac{\partial^2 T}{\partial r^2} \right\}$$

$$= \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} = \text{RHS}$$

1-4 Show that the following three different forms of the differential operator in the spherical coordinate system are equivalent:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dT}{dr} \right) = \frac{1}{r} \frac{d^2}{dr^2} (rT) = \frac{d^2 T}{dr^2} + \frac{2}{r} \frac{dT}{dr}$$

LHS) $\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right)$

$$= \frac{1}{r^2} \left\{ \frac{\partial}{\partial r} r^2 \frac{\partial T}{\partial r} + r^2 \frac{\partial^2 T}{\partial r^2} \right\}$$

$$= \frac{1}{r^2} \left\{ 2r \frac{\partial T}{\partial r} + r^2 \frac{\partial^2 T}{\partial r^2} \right\} = \frac{2}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}$$

= RHS

Middle term) $\frac{1}{r} \frac{\partial^2}{\partial r^2} (rT)$

$$= \frac{1}{r} \frac{\partial}{\partial r} \left\{ \frac{\partial}{\partial r} rT + r \frac{\partial T}{\partial r} \right\}$$

$$= \frac{1}{r} \frac{\partial}{\partial r} \left\{ 1 \cdot T + r \frac{\partial T}{\partial r} \right\}$$

$$= \frac{1}{r} \left\{ \frac{\partial T}{\partial r} + 1 \cdot \frac{\partial T}{\partial r} + r \cdot \frac{\partial^2 T}{\partial r^2} \right\}$$

$$= \frac{1}{r} \left\{ 2 \cdot \frac{\partial T}{\partial r} + r \frac{\partial^2 T}{\partial r^2} \right\}$$

$$= \frac{2}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} = \text{RHS}$$

1-5 Set up the mathematical formulation of the following heat conduction problems. Formulation includes the simplified differential heat equation along with boundary and initial conditions. Do not solve the problems.

1. A slab in $0 \leq x \leq L$ is initially at a temperature $F(x)$. For times $t > 0$, the boundary at $x = 0$ is kept insulated, and the boundary at $x = L$ dissipates heat by convection into a medium at zero temperature.
2. A semi-infinite region $0 \leq x \leq \infty$ is initially at a temperature $F(x)$. For times $t > 0$, heat is generated in the medium at a constant, uniform rate of g_0 (W/m^3), while the boundary at $x = 0$ is kept at zero temperature.
3. A hollow cylinder $a \leq r \leq b$ is initially at a temperature $F(r)$. For times $t > 0$, heat is generated within the medium at a rate of $g(r)$, (W/m^3), while both the inner boundary at $r = a$ and outer boundary $r = b$ dissipate heat by convection into mediums at fluid temperature T_∞ .
4. A solid sphere $0 \leq r \leq b$ is initially at temperature $F(r)$. For times $t > 0$, heat is generated in the medium at a rate of $g(r)$, (W/m^3), while the boundary at $r = b$ is kept at a uniform temperature T_0 .

$$1) \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad 0 < x < L, \quad t > 0$$

$$\text{BC1) } \left. \frac{\partial T}{\partial x} \right|_{x=0} = 0$$

$$\text{BC2) } -k \left. \frac{\partial T}{\partial x} \right|_{x=L} = h T \Big|_{x=L}$$

$$\text{IC) } T(x=0) = F(x)$$

$$2) \frac{\partial^2 T}{\partial x^2} + \frac{g_0}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad 0 < x < \infty, t > 0$$

$$BC1) T(x=0) = 0$$

$$IC) T(t=0) = F(x)$$

note: The I.C. is not recovered as $x \rightarrow \infty$ due to generation.

$$3) \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{g(r)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad a < r < b, t > 0$$

$$BC1) -k \left. \frac{\partial T}{\partial r} \right|_{r=a} = -h [T|_{r=a} - T_{\infty}]$$

$$BC2) -k \left. \frac{\partial T}{\partial r} \right|_{r=b} = +h [T|_{r=b} - T_{\infty}]$$

$$IC) T(t=0) = F(r)$$

$$4) \frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{g(r)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad 0 \leq r < b, t > 0$$

$$BC1) T(r \rightarrow 0) \Rightarrow \text{finite}$$

$$\text{or } \left. \frac{\partial T}{\partial r} \right|_{r=0} = 0 \quad \text{per symmetry}$$

$$BC2) T(r=b) = T_0$$

$$IC) T(t=0) = F(r)$$

1-6 A solid cube of dimension L is originally at a uniform temperature T_0 . The cube is then dropped into a large bath where the cube rapidly settles flat on the bottom. The fluid in the bath provides convection heat transfer with coefficient h ($\text{W}/\text{m}^2 \text{K}$) from the fluid at constant temperature T_∞ . Formulate the heat conduction problem. Formulation includes the simplified differential heat equation along with appropriate boundary and initial conditions. Include a sketch with your coordinate axis position. Do not solve the problem.

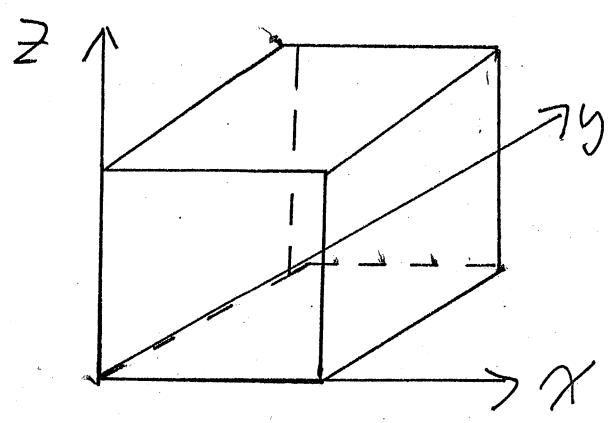
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$0 < x < L$$

$$0 < y < L$$

$$0 < z < L$$

$$t > 0$$



- BC1) $-k \frac{\partial T}{\partial x} \Big|_{x=0} = -h [T|_{x=0} - T_\infty]$
- BC2) $-k \frac{\partial T}{\partial x} \Big|_{x=L} = +h [T|_{x=L} - T_\infty]$
- BC3) $-k \frac{\partial T}{\partial y} \Big|_{y=0} = -h [T|_{y=0} - T_\infty]$
- BC4) $-k \frac{\partial T}{\partial y} \Big|_{y=L} = +h [T|_{y=L} - T_\infty]$
- BC5) $-k \frac{\partial T}{\partial z} \Big|_{z=L} = +h [T|_{z=L} - T_\infty]$
- BC6) $T(z=0) = T_\infty$ or $\frac{\partial T}{\partial z} \Big|_{z=0} = 0$

→ convection BC. not reasonable on the bottom.

IC) $T(x=0) = T_0$

1-7 For an anisotropic solid, the three components of the heat conduction vector q_x, q_y and q_z are given by equations (1-80). Write the similar expressions in the cylindrical coordinates for q_r, q_ϕ, q_z and in the spherical coordinates for q_r, q_ϕ, q_θ .

Cylinder

$$\mathcal{L}_r'' = - \left(k_{11} \frac{\partial T}{\partial r} + k_{12} \frac{1}{r} \frac{\partial T}{\partial \phi} + k_{13} \frac{\partial T}{\partial z} \right)$$

$$\mathcal{L}_\phi'' = - \left(k_{21} \frac{\partial T}{\partial r} + k_{22} \frac{1}{r} \frac{\partial T}{\partial \phi} + k_{23} \frac{\partial T}{\partial z} \right)$$

$$\mathcal{L}_z'' = - \left(k_{31} \frac{\partial T}{\partial r} + k_{32} \frac{1}{r} \frac{\partial T}{\partial \phi} + k_{33} \frac{\partial T}{\partial z} \right)$$

Sphere

$$\mathcal{L}_r'' = - \left(k_{11} \frac{\partial T}{\partial r} + k_{12} \frac{1}{r \cdot \sin \theta} \frac{\partial T}{\partial \phi} + k_{13} \frac{1}{r} \frac{\partial T}{\partial \theta} \right)$$

$$\mathcal{L}_\phi'' = - \left(k_{21} \frac{\partial T}{\partial r} + k_{22} \frac{1}{r \cdot \sin \theta} \frac{\partial T}{\partial \phi} + k_{23} \frac{1}{r} \frac{\partial T}{\partial \theta} \right)$$

$$\mathcal{L}_\theta'' = - \left(k_{31} \frac{\partial T}{\partial r} + k_{32} \frac{1}{r \cdot \sin \theta} \frac{\partial T}{\partial \phi} + k_{33} \frac{1}{r} \frac{\partial T}{\partial \theta} \right)$$

1-8 An infinitely long, solid cylinder ($D = \text{diameter}$) has the ability for uniform internal energy generation given by the rate g_o (W/m^3) by passing a current through the cylinder. Initially ($t=0$), the cylinder is at a uniform temperature T_o . The internal energy generation is then turned on (i.e. current passed) and maintained at a constant rate g_o , and at the same moment the cylinder is exposed to convection heat transfer with coefficient h ($\text{W/m}^2 \text{K}$) from a fluid at constant temperature T_∞ , noting that $T_\infty > T_o$. The cylinder has uniform and constant thermal conductivity k (W/m K). The Biot number $hD/k \ll 1$. Solve for time t at which point the surface heat flux is exactly zero. Present your answer in variable form.

For $\frac{hD}{k} \ll 1$, use lumped analysis: $T = T(x)$.

$$\cancel{Q_{in}} + E_{gen} - Q_{out} = \rho V c \frac{\partial T}{\partial t}$$

$$\left(\frac{\pi D^2 L}{4}\right) g_o - (\pi D L) h (T - T_\infty) = \rho \left(\frac{\pi D^2 L}{4}\right) c \frac{\partial T}{\partial t}$$

$$\text{I.C.) } T(t=0) = T_o$$

now cancel (πL) & let $\theta(x) = T - T_\infty$

$$\left. \begin{aligned} \frac{\partial \theta}{\partial t} + \frac{4h}{\rho D c} \theta &= \frac{g_o}{\rho c} \\ \theta(t=0) &= T_o - T_\infty \end{aligned} \right\} \begin{array}{l} \text{O.D.E.} \\ + \\ \text{I.C.} \end{array}$$

now $\theta(x) = \theta_H + \theta_p$

$$\Rightarrow \theta(x) = C_1 \cdot e^{-\frac{4h}{\rho D c} t} + \frac{g_o D}{4h}$$

per I.C) $C_1 = (T_o - T_\infty) - \frac{g_o D}{4h}$

1-8

2/2

$$\theta(x) = \left[T_0 - T_{\infty} - \frac{g_0 D}{4h} \right] e^{-\frac{4h}{pDc} x} + \frac{g_0 D}{4h}$$

then $T(x) = \left[T_0 - T_{\infty} - \frac{g_0 D}{4h} \right] e^{-\frac{4h}{pDc} x} + \frac{g_0 D}{4h} + T_{\infty}$

Surface flux is zero at $x = x_0$ if $T(x_0) = T_{\infty}$:

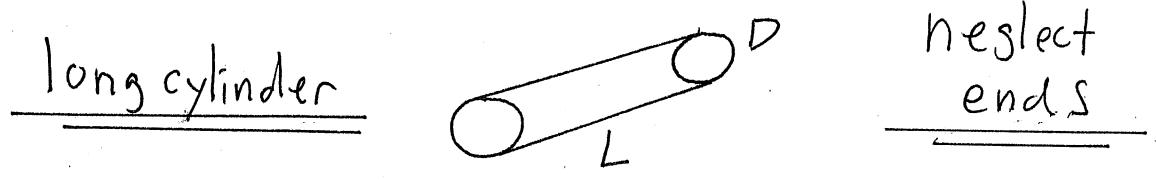
$$T_{\infty} = \left(T_0 - T_{\infty} - \frac{g_0 D}{4h} \right) e^{-\frac{4h}{pDc} x_0} + \frac{g_0 D}{4h} + T_{\infty}$$

$$e^{-\frac{4h}{pDc} x_0} = \left(\frac{-g_0 D}{4h} \right) \frac{1}{\left(T_0 - T_{\infty} - \frac{g_0 D}{4h} \right)}$$

Yields: $x_0 = \left(\frac{-pDc}{4h} \right) \ln \left[\frac{\left(\frac{-g_0 D}{4h} \right)}{\left(T_0 - T_{\infty} - \frac{g_0 D}{4h} \right)} \right]$

when the term in the brackets
is positive for $T_0 > T_{\infty}$.

1-9 A long cylindrical iron bar of diameter $D = 5$ cm, initially at temperature $T_0 = 650^\circ\text{C}$, is exposed to an air stream at $T_\infty = 50^\circ\text{C}$. The heat transfer coefficient between the air stream and the surface of the bar is $h = 80$ $\text{W}/(\text{m}^2 \cdot \text{K})$. Thermophysical properties are constant: $\rho = 7800$ kg/m^3 , $c_v = 460$ $\text{J}/(\text{kg} \cdot \text{K})$, and $k = 60$ $\text{W}/(\text{m} \cdot \text{K})$. Determine the time required for the temperature of the bar to reach 250°C by using the lumped system analysis.



$$Bi = \frac{hD}{k} = \frac{(80)(.05)}{60} = 0.06 \ll 1$$

→ lumped analysis.

$$\cancel{Q_{in}} + \cancel{E_{gen}} - Q_{out} = \rho V c \frac{\partial T}{\partial t}$$

$$-hA(T - T_\infty) = \rho V c \frac{\partial T}{\partial t}$$

let $\theta(t) = T(t) - T_\infty$

$$\left. \begin{aligned} \frac{d\theta}{dt} + \frac{hA}{\rho V c} \theta &= 0 \\ \theta(t=0) &= T_0 - T_\infty \end{aligned} \right\} \begin{array}{l} \text{O.D.E.} \\ + \\ \text{I.C.} \end{array}$$

$$\theta(t) = C_1 e^{-\frac{hA}{\rho V c} t}$$

$$C_1 = T_0 - T_\infty$$

$$\Rightarrow \theta(t) = (T_0 - T_\infty) e^{-\frac{hA}{\rho V c} t}$$

1-9

2/2

$$\frac{T(t) - T_{\infty}}{T_0 - T_{\infty}} = e^{-\frac{hA}{\rho V c} t}$$

where: $A = \pi D L$

$$V = \pi/4 D^2 L$$

$$\Rightarrow \frac{T(t) - T_{\infty}}{T_0 - T_{\infty}} = e^{-\frac{4h}{\rho D c} t}$$

$$\frac{250 - 50}{650 - 50} = e^{-\frac{4(80)}{(7800)(.05)(460)} t}$$

$$\underline{t = 616 \text{ s}}$$

1-10 A thermocouple is to be used to measure the temperature in a gas stream. The junction may be approximated as a sphere having thermal conductivity $k = 25 \text{ W/(m}\cdot\text{K)}$, $\rho = 8400 \text{ kg/m}^3$, and $c_p = 0.4 \text{ kJ/(kg}\cdot\text{K)}$. The heat transfer coefficient between the junction and the gas stream is $h = 560 \text{ W/(m}^2\cdot\text{K)}$. Calculate the diameter of the junction if the thermocouple should measure 95% of the applied temperature difference in 3 s.

→ Assume lumped analysis ($\frac{hD}{k} \ll 1$)

$$\cancel{Q_{in}} + \cancel{E_{sen}} - Q_{out} = \rho V c \frac{dT}{dt}$$

$$-hA(T - T_{\infty}) = \rho V c \frac{dT}{dt}$$

let $\theta(t) = T(t) - T_{\infty}$

$$\left. \begin{aligned} \frac{d\theta}{dt} + \frac{hA}{\rho V c} \theta &= 0 \\ \theta(t=0) &= T_0 - T_{\infty} \end{aligned} \right\} \begin{array}{l} \text{O.D.E.} \\ + \\ \text{I.C.} \end{array}$$

$$\theta(t) = C_1 \cdot e^{-\frac{hA}{\rho V c} t}$$

$$C_1 = T_0 - T_{\infty}$$

$$\Rightarrow \theta(t) = (T_0 - T_{\infty}) e^{-\frac{hA}{\rho V c} t}$$

$$A = \pi D^2$$

$$V = \frac{\pi}{6} D^3 \Rightarrow \frac{A}{V} = \frac{6}{D}$$

$$\Rightarrow \frac{T(t) - T_{\infty}}{T_0 - T_{\infty}} = e^{-\frac{6h}{\rho D c} t}$$

To measure 95% of applied temp. difference,

$$\frac{T(t) - T_{\infty}}{T_0 - T_{\infty}} = \frac{5}{100}$$

$$\Rightarrow \frac{5}{100} = e^{-\frac{6(560)}{(8400)(D)(400)}(3)} \quad (3)$$

$$\rightarrow D = 0.0010 \text{ m}$$

$$= \underline{1.0 \text{ mm}}$$

Check Bi #)

$$\frac{hD}{k} = \frac{(560)(.001)}{(25)}$$

$$= \underline{0.022 \ll 1}$$

\rightarrow lumped analysis valid.

1-11 Determine the constants C_1 and C_2 for the constant-area fin solution of equations (1-100) for the case of the prescribed base temperature of equation (1-102), and the following tip conditions:

1. Convective tip per equation (1-103).
2. Insulated or symmetric tip per equation (1-104).
3. Prescribed temperature tip per equation (1-105).
4. Infinitely long fin per equation (1-106).

$$1) \theta(x) = C_1 \cdot \cosh(mx) + C_2 \cdot \sinh(mx)$$

$$BC1) \theta(x=0) = \theta_b = C_1 \cdot 1 + C_2 \cdot 0$$

$$\rightarrow \underline{C_1 = \theta_b}$$

$$BC2) -k \frac{\partial \theta}{\partial x} \Big|_L = h \theta \Big|_L$$

$$-k \left[\theta_b \cdot \sinh(mL) + C_2 \cdot \cosh(mL) \right] (m) = h \left[\theta_b \cdot \cosh(mL) + C_2 \cdot \sinh(mL) \right]$$

$$\rightarrow C_2 = \frac{-\theta_b \left[mL \cdot \sinh(mL) + h \cdot \cosh(mL) \right]}{\left[h \cdot \sinh(mL) + mL \cdot \cosh(mL) \right]}$$

$$2) \theta(x) = C_1 \cdot \cosh(mx) + C_2 \cdot \sinh(mx)$$

$$\rightarrow \underline{C_1 = \theta_b \text{ per above}}$$

$$BC2) \frac{\partial \theta}{\partial x} \Big|_{x=L} = 0$$

1-11

$$0 = (\theta_b \cdot \sinh(mL) + C_2 \cdot \cosh(mL))(m)$$

$$\rightarrow C_2 = \frac{-\theta_b \cdot \sinh(mL)}{\cosh(mL)} = -\theta_b \cdot \tanh(mL)$$

$$3) \theta(x) = C_1 \cdot \cosh(mx) + C_2 \cdot \sinh(mx)$$

$$\rightarrow \underline{C_1 = \theta_b \text{ per above}}$$

$$\text{BC2) } \theta(x=L) = \theta_{\text{tip}}$$

$$\theta_{\text{tip}} = \theta_b \cdot \cosh(mL) + C_2 \cdot \sinh(mL)$$

$$\rightarrow C_2 = \frac{\theta_{\text{tip}} - \theta_b \cdot \cosh(mL)}{\sinh(mL)}$$

$$4) \theta(x) = C_1 e^{-mx} + C_2 e^{mx}$$

$$\text{BC1) } \theta(x=0) = \theta_b = C_1 + C_2$$

$$\text{BC2) } \theta(x \rightarrow \infty) = 0 = C_1(0) + C_2(\infty)$$

$$\rightarrow \underline{\underline{C_2 = 0}}$$

$$\text{then } \underline{\underline{C_1 = \theta_b}}$$