1.1 $f = f(x,t) = f(x(x',t'), t(t'))$

$$\frac{\partial f}{\partial t'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial t'}$$

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'}$$

$$\frac{D\rho'}{Dt'} \equiv \frac{\partial \rho'}{\partial t'} + v'\frac{\partial \rho'}{\partial x'} = \frac{\partial \rho}{\partial t'} + v'\frac{\partial \rho}{\partial x'}$$

$$= \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial \rho}{\partial t}\frac{\partial t}{\partial t'} + v'\left[\frac{\partial \rho}{\partial x}\frac{\partial x}{\partial x'}\right]$$

$$= V\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t} + v'\frac{\partial \rho}{\partial x} = (V+v')\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t}$$

$$= \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x}$$ Same form as before

$$\frac{Dv'}{Dt} \equiv \frac{\partial v'}{\partial t'} + v'\frac{\partial v'}{\partial x'} = \frac{\partial}{\partial t'}(v-V) + (v-V)\frac{\partial}{\partial x'}(v-V)$$

$$= \frac{\partial v}{\partial t'} + (v-V)\frac{\partial v}{\partial x'}$$

$$= \frac{\partial v}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial t'} + (v-V)\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial x'}\right]$$

$$= V\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - V\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}$$ Same form as before

1.2 Arbitrary Region. Vapor crosses the surface and the volume decreases.

1.3 The velocity of a material region is the velocity of the interface plus the circulation velocity of the liquid (and gas) along the interface.

1.4 Yes If $\underline{w}$ is the normal velocity of the fluid at the control surface, $\underline{w} \equiv (\underline{n} \cdot \underline{v})\,\underline{n}$, then fluid does not cross the surface. It can, however, slide along the surface

1.5
$$\underline{v}_i' \equiv \underline{v}_i - \underline{v}$$

$$m_i \underline{v}_i' = m_i \underline{v}_i - m_i \underline{v}$$

$$\sum m_i \underline{v}_i' = \sum m_i \underline{v}_i - \sum m_i \underline{v}$$

$$\sum m_i v_i' = \sum m_i \underline{v}_i - \underline{v} \sum m_i$$

$$\frac{\sum m_i v_i'}{\sum m_i} = \frac{\sum m_i \underline{v}_i}{\sum m_i} - \underline{v}$$

$$\frac{\sum n_i v_i'}{\sum m_i} = \underline{v} - \underline{v} = 0$$

1.6

$$\underline{v}_i = \underline{v}_i' + \underline{v}$$

$$\underline{v}_i \cdot \underline{v}_i = \underline{v}_i' \cdot \underline{v}_i' + 2\,\underline{v} \cdot \underline{v}_i' + \underline{v} \cdot \underline{v}$$

$$\sum m_i \tfrac{1}{2} \underline{v}_i \cdot \underline{v}_i = \sum m_i \tfrac{1}{2} \underline{v}_i' \cdot \underline{v}_i' + \underbrace{\frac{2}{2}\underline{v} \cdot \sum m_i \underline{v}_i'}_{=0 \text{ from } 1.5} + \sum m_i \tfrac{1}{2} \underline{v} \cdot \underline{v}$$

1.6

$$\frac{\text{momentum}}{\text{unit vol}} = \underline{P} = \lim \frac{\sum m_i \underline{v}_i}{V} = \lim \frac{\sum m_i}{V} \frac{\sum m_i \underline{v}_i}{\sum m_i}$$

$$= \lim \frac{\sum m_i}{V} \lim \frac{\sum m_i \underline{v}_i}{\sum m_i}$$

$$= \rho \underline{v}$$

1.7 Yes. Unsteady processes must maintain equilibrium among the particles. A time could be formed using the typical particle velocity v_p and the continuum length say $l_{\text{mean free path}}$

$$t_{\text{collision}} \sim \frac{l_{mfp}}{v_p}$$

1.8

1.8

$$\rho = \lim_{L \to 0} \frac{\sum m_i}{V}$$

$$\underline{v} = \lim_{L \to 0} \frac{\sum m_i v_i}{\sum m_i}$$

$$\rho \underline{v} = \lim_{L \to 0} \frac{\sum m_i}{V} \frac{\sum m_i v_i}{\sum m_i}$$

$$= \lim_{L \to 0} \frac{\sum m_i v_i}{V} = \lim_{L \to 0} \frac{\sum \underline{p}_i}{V}$$

Chapt 2 Problems

2.1 1 (a) no, shear forces present

2 (b) no, system not homogeneous

3 (d) Could be, neglect contraction & assume normal force only. No, if true stress state with compression, tension and shear stresses are accounted for

2.2 Yes, S/E is intensive; $S_2 = \lambda S_1$; $E_2 = \lambda E_1$

$$\frac{S_2}{E_2} = \frac{S_1}{E_1}$$

2.3 $$S = R_0 N \left(\frac{E}{E_0}\right)^{1/2} \left(\frac{V}{V_0}\right) \left(\frac{N}{N_0}\right)^{-3/2}$$

$$\frac{1}{T} = \frac{\partial S}{\partial E}\bigg)_{V,N} = R_0 N \frac{1}{2} \left(\frac{1}{E E_0}\right)^{1/2} \left(\frac{V}{V_0}\right) \left(\frac{N}{N_0}\right)^{-3/2}$$

$$\frac{P}{T} = \frac{\partial S}{\partial V}\bigg)_{E,N} = \frac{R_0 N}{V_0} \left(\frac{E}{E_0}\right)^{1/2} \left(\frac{N}{N_0}\right)^{-3/2}$$

$$\frac{\mu}{T} = \frac{\partial S}{\partial N}\bigg)_{E,V} = -\frac{1}{2} \frac{R_0 N^{-3/2}}{} \left(\frac{E}{E_0}\right)^{1/2} \underline{N_0}^{3/2}$$

Eulers Eq $$S = \frac{1}{T} E + \frac{P}{T} V + \frac{\mu}{T} N$$

$$S = \frac{1}{2} R_0 N \left(\frac{E}{E_0}\right)^{1/2} \left(\frac{V}{V_0}\right) \left(\frac{N}{N_0}\right)^{-3/2} + R_0 N \frac{V}{V_0} \left(\frac{E}{E_0}\right)^{1/2} \left(\frac{N}{N_0}\right)^{-3/2} - \frac{1}{2} R_0 N \left(\frac{E}{E_0}\right)^{\frac{1}{2}} \left(\frac{N_0}{N}\right)^{3/2}$$

$S = R_0 N \frac{V}{V_0} \left(\frac{E}{E_0}\right)^{1/2} \left(\frac{N}{N_0}\right)^{-3/2}$ which check the original equation

2.4 $$dE = TdS - pdV - \mu dN$$

so $$dS = \frac{1}{T}dE + \frac{p}{T}dV + \frac{\mu}{T}dN$$

$$\frac{1}{T} \equiv \left.\frac{\partial S}{\partial E}\right)_{V,N} \quad \text{or} \quad T = \frac{1}{\left.\frac{\partial S}{\partial E}\right|_{V,N}}$$

$$\frac{p}{T} \equiv \left.\frac{\partial S}{\partial V}\right)_{E,N} \quad \text{or} \quad p = \frac{\partial S/\partial V|_{E,N}}{\partial S/\partial E|_{V,N}}$$

$$\frac{\mu}{T} = \left.\frac{\partial S}{\partial N}\right)_{E,V} \quad \text{or} \quad \mu = \frac{\partial S/\partial N|_{E,V}}{\partial S/\partial E|_{V,N}}$$

2.5 Divide 2.7.2b by m to get the fundamental equation in a unit mass basis

$$de = T\,ds - p\,dv$$

From the definition of enthalpy 2.9.4 and $v = \frac{1}{\rho}$

$$h = e + pv$$

$$dh = de + p\,dv + v\,dp \quad \text{OR}$$

$$= T\,ds + p\,dv$$

Thus since $dh = \left.\frac{\partial h}{\partial s}\right|_v ds + \left.\frac{\partial h}{\partial v}\right|_s dv$

$$T = \left.\frac{\partial h}{\partial s}\right|_v \quad \& \quad v = \left.\frac{\partial h}{\partial v}\right|_s$$

~~At~~ $p = 30$ psia & $h = 1050\,s - 650$ Btu/lbm
$s = 1.8$ Btu/(lbm °R)

$\therefore$ $T = 1050\ °R$

$$v = \left.\frac{\partial h}{\partial v}\right|_s \approx \left.\frac{\Delta h}{\Delta v}\right|_s = \frac{-630-(-650)}{(35-30)\cdot 144}$$

$$= .0275 \frac{\text{Btu}}{\text{Lbm}} \frac{\text{ft}^2}{\text{Lb}_f}\ 778 \frac{\text{ft-Lb}_f}{\text{Btu}}$$

$$= 21.40 \frac{\text{ft}^3}{\text{Lbm}}$$

$$\rho = \frac{1}{v} = 0.0463 \frac{\text{Lbm}}{\text{ft}^3}$$

$$h = 1050\,s - 650 = 1240 \frac{\text{Btu}}{\text{Lbm}}$$

$$e = h - pv = 1240 - \frac{30 \cdot 144 \cdot 21.4}{778}$$

$$= 1240 - 118.8$$

$$= 1121\ \text{Btu/Lbm}$$

2.6

$$G = H - TS = E + PV - TS$$

but from 2.4.4

$$dE = TdS - pdV - \mu dN$$

so

$$dG = dE + PdV - VdP - TdS - SdT$$

$$= -\mu dN - VdP - SdT$$

This is the fundamental equation of thermodynamics for G and the independent variables are:

$$N, T, \& P$$

or

$$G = G(T, P, N)$$

gives all thermodynamic information about a substance.

$\boxed{2.7}$

$$p = \rho R T \qquad \rho = \frac{p}{RT}$$

$$\alpha = \frac{1}{\rho}\left.\frac{\partial \rho}{\partial p}\right)_T$$

$$= \frac{1}{\rho}\frac{1}{RT} = \frac{1}{p}$$

$$\beta = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right)_p$$

$$= -\frac{1}{\rho}\frac{p}{R}\left(-T^{-2}\right) = \frac{p}{\rho R T^2} = \frac{1}{T}$$

$\boxed{2.8}$

$$T ds = de + p\, d\rho^{-1} \qquad (2.7.2 \div M)$$

$$(2.9.3) \qquad de = C_v dT + \rho^{-2}\left[p - T\left.\frac{\partial p}{\partial T}\right)_\rho\right] d\rho$$

$$T ds = C_v dT + (-p\rho^{-2})d\rho + \rho^{-2}\left[p - T\left.\frac{\partial p}{\partial T}\right)_\rho\right] d\rho$$

$$= C_v dT - \frac{T}{\rho^2}\left.\frac{\partial p}{\partial T}\right)_\rho d\rho$$

$$ds = C_v \frac{dT}{T} - \frac{1}{\rho^2}\left.\frac{\partial p}{\partial T}\right)_\rho d\rho$$

$\boxed{2.9}$

$$p = \rho R T \qquad \left.\frac{\partial p}{\partial T}\right)_\rho = \rho R$$

$$ds = \int_{T_0, \rho_0}^{T, \rho_0} \frac{C_v(T, \rho_0 = \text{const})}{T}\, dT - R \int_{T, \rho_0}^{T, \rho} \frac{d\rho}{\rho}$$

$$s - s_0 = C_v \ln\frac{T}{T_0} - R \ln\frac{\rho}{\rho_0}$$

2.4 $\cancel{TdS = dE + pdV + \mu dN}$

or 1st ed

$\cancel{dE = +TdS - pdV - \mu dN}$

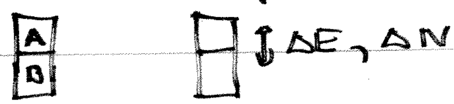
equations of state are $T = \left.\frac{\partial E}{\partial S}\right)_{V,N}$ $-p = \left.\frac{\partial E}{\partial V}\right)_{S,N}$ & $\mu = \left.\frac{\partial E}{\partial N}\right)_{S,}$

2.5 see page 2-3

2.6 ~~skipped~~ see next page

~~2.6~~ Consider fixed-adiabatic-impermeable is changed to

old 1st ed fixed-diathermal-permeable



$S = S_A + S_B$ from state ① to state ② $\Delta E, \Delta N$ change

$\Delta S = \Delta S_A + \Delta S_B = \frac{1}{T_A}\Delta E_A + \frac{1}{T_B}\Delta E_B + \frac{\mu_A \Delta N_A}{T_A} + \frac{\mu_B}{T_B}\Delta N_B$

for the manifold we find the max state by $\Delta S = 0$

Now $\Delta E_A = -\Delta E_B$ $\Delta N_A = -\Delta N_B$

$$\Delta S = 0 = \left(\frac{1}{T_A} - \frac{1}{T_B}\right)\Delta E_A + \left(\frac{\mu_A}{T_A} - \frac{\mu_B}{T_B}\right)\Delta N_A$$

for arbitrary small changes in ΔE_A & ΔN_A. Hence

$$T_A = T_B$$

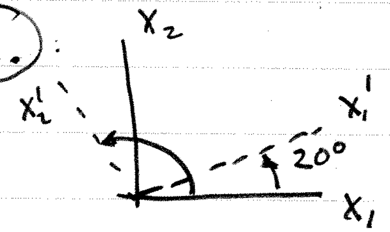
and

$$\mu_A = \mu_B$$

for the member of the manifold with maximum entropy.

Chapter 3 Solutions

1.



In new system coordinates of P are x_1', x_2', x_3'. They are

$$x_j' = c_{ij'} x_i$$

Now

$$c_{11'} = \cos 20° = .940 \qquad c_{12'} = \cos 110° = -.342 \qquad c_{13'} = \cos 90 = 0$$

$$c_{21'} = \cos 70 = .342 \qquad c_{22'} = \cos 20 = .940 \qquad c_{23'} = 0$$

$$c_{31'} = \cos 90 = 0 \qquad c_{32'} = 0 \qquad c_{33'} = \cos 0 = 1$$

$$x_{1'} = c_{i1'} x_i = c_{11'} x_1 + c_{21'} x_2 + c_{31'} x_3$$
$$= .94(5) + .342(4) + 0(0)$$
$$= \underline{6.07}$$

$$x_{2'} = c_{i2'} x_i = c_{12'} x_1 + c_{22'} x_2 + c_{32'} x_2$$
$$= -.342(5) + .940(4) + 0(0)$$
$$= \underline{2.05}$$

$$x_{3'} = c_{13'} x_1 + c_{23'} x_2 + c_{33'} x_3$$
$$= 0 + 0 + 1 \cdot 0$$
$$= \underline{0}$$

2.

$a = b_i c_{ij} d_j$ OK

$a = b_i c_i + d_j$ No free index j occurs in only one term

$a_i = \delta_{ij} b_i + c_i$ No " " " "

$a_k = b_i c_{ki}$ OK

$a_k = b_k c + d_i e_{ik}$ OK

Chapter 3

$a_i = b_i + c_{ij} d_{ji} e_i$ No i appears 3 times in last term

$a_l = \varepsilon_{ijk} b_j c_k$ No free index on left is l but on right i

$a_{ij} = b_{ji}$ OK

$a_{ij} = b_i c_j + e_{jk}$ No free index k in last term is not the same as i in other terms

$a_{kl} = b_i c_{ki} d_l + e_{ki}$ No free index i in last term is not same as l in others

3. (a) if u & v $\perp$ then $\underline{u} \cdot \underline{v} = 0$

$u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 = 3 \cdot 4 + 2 \cdot 1 + (-7) \cdot 2$

$= 12 + 2 - 14 = 0 \Rightarrow \perp$

(b)

$v = \sqrt{4^2 + 1^2 + 2^2} = 4.583$

$w = \sqrt{6^2 + 4^2 + (-5)^2} = 8.775$

(c)

$\underline{v} \cdot \underline{w} = vw \cos\theta = v_1 w_1 + v_2 w_2 + v_3 w_3$

$= 4 \cdot 6 + 1 \cdot 4 + 2(-5)$

$= 18$

$\cos\theta = \dfrac{18}{4.883 \quad 8.775} = .488$

$\theta = 63.4^\circ$

(d) $\alpha_i = \dfrac{w_i}{w}$ $\quad \alpha_1 = \dfrac{6}{8.775} = .684$

$\alpha_2 = \dfrac{4}{8.775} = .456$ $\quad \alpha_3 = \dfrac{-5}{8.775} = -.570$

(e) $\alpha \cdot u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = .684(3) + .456(2) - .570(-7)$

6.95

Chapter 3

(4.) No; the cosines are items which cannot be measured with a single coordinate system

(5.) δ_{ij} is defined as $\delta_{11} = 1$ $\delta_{12} = 0$, $\delta_{13} = 0$ etc

In a new coordinate system, it would be

$$\delta'_{ij} = c_{ki}' c_{lj}' \delta_{kl}$$

By definition as the substitution tensor we set $l = k$

$$\delta'_{ij} = c_{ki}' c_{kj}'$$

But the RHS, by the law of cosines 3.1.16, is δ_{ij} hence

$$\delta'_{ij} = \delta_{ij}$$

The components are the same in any coordinate system

(6.) $(a \times b) \cdot c = a \cdot (b \times c) = (c \times a) \cdot b$

$\varepsilon_{ijk} a_j b_k c_i = \varepsilon_{jki} a_j b_k c_i$ since $\varepsilon_{jki} = \varepsilon_{ijk}$

$= a \cdot (b \times c)$

$\varepsilon_{ijk} a_j b_k c_i = \varepsilon_{kij} c_i a_j b_k$ since $\varepsilon_{kij} = \varepsilon_{ijk}$

$= (c \cdot a) \times b$

See next page for the other parts of problem (6)

(7.) $T_{(ij)} \equiv \frac{1}{2} T_{ij} + \frac{1}{2} T_{ji}$ $\qquad$ $T_{[ij]} \equiv \frac{1}{2} T_{ij} - \frac{1}{2} T_{ji}$

$$\begin{bmatrix} 6 & 3 & 1 \\ 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & \frac{4+3}{2} & \frac{1+1}{2} \\ \frac{7}{2} & 0 & \frac{5+3}{2} \\ 1 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3-4}{2} & \frac{1-1}{2} \\ +\frac{1}{2} & 0 & \frac{5-3}{2} \\ 0 & -1 & 0 \end{bmatrix}$$

(7.) continued

$$\begin{array}{c} T_{ij} \\ \begin{bmatrix} 6 & 3 & 1 \\ 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} \end{array} = \begin{array}{c} T_{(ij)} \\ \begin{bmatrix} 6 & \frac{7}{2} & 1 \\ \frac{7}{2} & 0 & 4 \\ 1 & 4 & 2 \end{bmatrix} \end{array} + \begin{array}{c} T_{[ij]} \\ \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ +\frac{1}{2} & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \end{array}$$

Dual vector

$d_i = \varepsilon_{ijk} T_{jk}$ $\qquad d_1 = \varepsilon_{1jk} T_{jk} = \varepsilon_{123} T_{23} + \varepsilon_{132} T_{32}$

$$= -3 + 5 = \underset{\sim}{+2}$$

$$d_2 = \varepsilon_{2jk} T_{jk} = \varepsilon_{231} T_{31} + \varepsilon_{213} T_{13}$$

$$= 1 - 1 = \underset{\sim}{0}$$

$$d_3 = \varepsilon_{3jk} T_{jk} = \varepsilon_{312} T_{12} + \varepsilon_{321} T_{21}$$

$$= 3 - 4 = \underset{\sim}{-1}$$

$\underset{\sim}{d} = (-2, 0, -1)$

---

continued

(6.) $t \times (u \times v) = \underline{u}(t \cdot v) - \underline{v}(\underline{t} \cdot \underline{u})$

$$\varepsilon_{pqi} t_q \varepsilon_{ijk} u_j v_k = \varepsilon_{ipq} \varepsilon_{ijk} t_q u_j v_k$$

$$= (\delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj}) t_q u_j v_k$$

$$= v_k t_k u_p - v_p t_j u_j$$

which is the left side of the equation above

$$\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

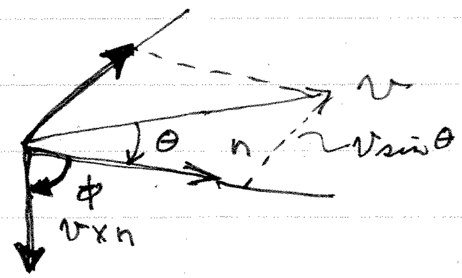
$$\varepsilon_{ijk} u_j v_k = -\varepsilon_{ikj} v_k u_j$$

(8.) $S_{ij}T_{ji}$ since $S_{ij} = S_{ji}$ & $T_{ji} = -T_{ij}$

$S_{ij}T_{ji} = S_{ji}(-T_{ij}) = -S_{ji}T_{ij} = -S_{ij}T_{ji}$

a quantity equal to its negative must be zero

(9.)



$\underset{\sim}{v} \times \underset{\sim}{n} = 1 v \sin\theta$

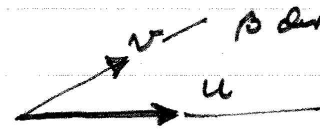
$n \times (v \times n) = 1 \cdot v \sin\theta \cdot \sin\phi$

$\sin\phi = 1$

$n \times (v \times n) = v \sin\theta$

(10.) $w_i = \varepsilon_{ijk} u_j v_k$

$w = uv \sin(\alpha, \beta)$



choose x_i system so u_i ; $u_1 = u, u_2 = 0, u_3 = 0$

choose x_i' system so v_i' ; $v_1' = v, v_2' = 0, v_3' = 0$

Now $w^2 = w_i w_i = \varepsilon_{ijk} u_j v_k \varepsilon_{ipq} u_p v_q = \varepsilon_{ijk}\varepsilon_{ipq} u_j v_k u_p v_q$

$= (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}) u_j v_k u_p v_q$

$= u_j u_j v_k v_k - u_j v_j u_k v_k$

$= u^2 v^2 - u_j v_j u_k v_k$

Next since $v_k = c_{\ell k} v_\ell'$, $u_k v_k = u_k c_{\ell k} v_\ell' = u_1 c_{11} v_1'$

$u_k v_k = uv \cos(\alpha, \beta)$

$\therefore$

$w^2 = u^2 v^2 - u^2 v^2 \cos^2(\alpha, \beta) = u^2 v^2 (1 - \cos^2(\alpha, \beta))$

$= u^2 v^2 \sin^2(\alpha, \beta)$

(11.) $\partial_i \delta_j = \partial_i x_j$ $\quad \partial_1 x_2 = 0$, $\partial_1 x_1 = 1$ etc therefore

$\partial_i x_j = \delta_{ij}$

(12)

$$\operatorname{div}(\phi \underset{\sim}{v}) = \phi \operatorname{div} \underset{\sim}{v} + \underset{\sim}{v} \cdot \operatorname{grad} \phi$$

$$\nabla\cdot(\phi \underset{\sim}{v}) = \phi\, \nabla\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\nabla\phi$$

$$\partial_i(\phi v_i) = \phi\, \partial_i v_i + v_i \partial_i \phi$$

$$\operatorname{div}(\underline{u}\times\underline{v}) = \underline{v}\cdot \operatorname{curl} \underline{u} - \underline{u}\cdot\operatorname{curl}\underline{v}$$

$$\nabla\cdot(\underline{u}\times\underline{v}) = \underline{v}\cdot(\nabla\times\underline{u}) - \underline{u}\cdot(\nabla\times\underline{v})$$

$$\partial_i(\varepsilon_{ijk} u_j v_k) = \varepsilon_{ijk}\,\partial_i u_j v_k = \varepsilon_{ijk} u_j \partial_i v_k + \varepsilon_{ijk} v_k \partial_i u_j$$

$$= u_j(-\varepsilon_{jik}\partial_i v_k) + v_k\, \varepsilon_{kij}\partial_i u_j$$

$$= -\underline{u}\cdot(\nabla\times\underline{v}) + \underline{v}\cdot(\nabla\times\underline{u})$$

$$\operatorname{curl}(\underline{u}\times\underline{v}) = \underline{v}\cdot\operatorname{grad}\underline{u} - \underline{u}\cdot\operatorname{grad}\underline{v} + \underline{u}\operatorname{div}\underline{v} - \underline{v}\operatorname{div}\underline{u}$$

$$\nabla\times(\underline{u}\times\underline{v}) = (\underline{v}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{v} + \underline{u}\,\nabla\cdot\underline{v} - \underline{v}\,\nabla\cdot\underline{u}$$

$$\varepsilon_{lmi}\partial_m(\varepsilon_{ijk} u_j v_k) = \varepsilon_{ilm}\varepsilon_{ijk}\,\partial_m(u_j v_k)$$

$$= (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj})\,\partial_m(u_j v_k)$$

$$= \partial_k(u_l v_k) - \partial_j(u_j v_l)$$

$$= u_l\partial_k v_k + v_k\partial_k u_l - u_j\partial_j v_l - v_l\partial_j u_j$$

which is $\quad \underline{u}\,\nabla\cdot\underline{v} + (\underline{v}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{v} - (\underline{v}\,\nabla)\cdot\underline{u}$

Chapter 3

(13) $\partial_i \partial_j (\ )$ is symmetric because the order of two differentiation processes may be interchanged.

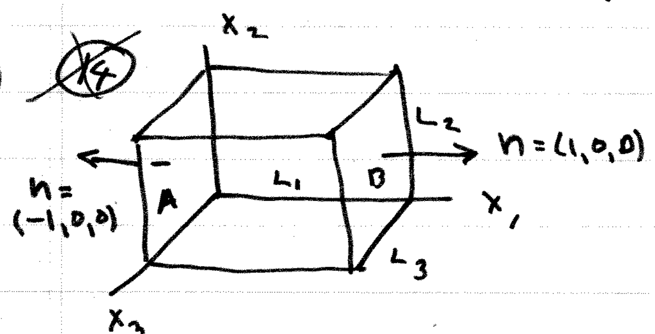
$$\text{curl grad } \phi \ , \quad \varepsilon_{kji}\, \partial_j \partial_i \phi = 0$$

This is always zero because ε_{kji} is antisymmetric and $\partial_j \partial_i$ is symmetric so the product is zero as in problem 3.8

$$\text{div curl } v \ , \quad \partial_i (\varepsilon_{ijk}\, \partial_j v_k)$$

$$= \varepsilon_{ijk}\, \partial_i \partial_j v_k = 0$$

1st ed ~~(14)~~



$$\int_R \partial_i (T_{jk})\, dV = \int_S n_i T_{jk}\, dS$$

$$T_{jk\ldots} = \phi = x_1$$

$$\int_R \partial_i (x_1)\, dV = \int_S n_i x_1\, dS$$

for $i = 1$, $\partial_1 x_1 = 1$

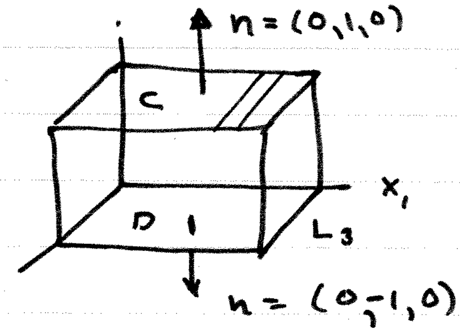
$$\int_R dV = Vol = L_1 \cdot L_2 \cdot L_3$$

$$\int_S n_1 x_1\, dS = \int_B 1 \cdot L_1\, dS + \int_A (-1)\cdot(0)\, dS \qquad n_1 = 0 \text{ on all other sides}$$

$$= L_1 \int_B dS = L_1 \cdot L_2 \cdot L_3$$

Chapter 3

(14) continued 1st ed



for $i=2$ $\quad \partial_2 x_1 = 0$

$$\int_R 0\,dV = 0$$

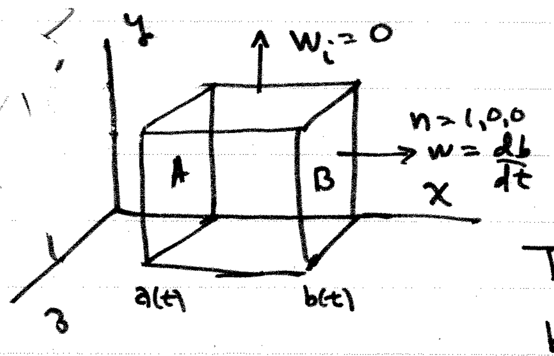
$$\int_S n_2 x_1\,dS = \int_C 1\,x_1\,dS + \int_D (-1)\,x_1\,dS$$

$$= \int x_1 L_3\,dx_1 - \int x_1 L_3\,dx_1$$

$$= 0$$

for $i=3$ $\quad \partial_3 x_1 = 0$ and both integrals are again zero

(14) 2nd 3rd



$$\frac{d}{dt}\int_R f(x,t)\,dV = \int_R \frac{\partial f}{\partial t}\,dV + \int_S n_i w_i f\,dS$$

Take $T_{ij..}(x_i,t) = f(x,t)$. Let R be $a(t), b(t)$ on x axis and one unit on y & z axes

$$\frac{d}{dt}\int_0^1\int_0^1\int_a^b f(x,t)\,dx\,dy\,dz = \frac{d}{dt}\int_a^b f\,dx$$

$$\int_0^1\int_0^1\int_a^b \frac{\partial f}{\partial t}\,dV = \int_a^b \partial f/\partial t\,dx$$

$$\int_S n_i w_i f\,dS = \int_B (1)\frac{db}{dt} f(x,t)\,dy\,dz + \int_A (-1)\frac{da}{dt} f(x,t)\,dy\,dz$$

$$= \frac{db}{dt} f(b,t) - \frac{da}{dt} f(a,t)$$

$$T_{ij} = -\frac{2}{3}\mu\,\delta_{ij}\,\partial_k v_k + 2\mu\,\partial_{(i}v_{j)}$$

1st ed

$$\partial_i T_{ij} = -\frac{2}{3}\mu\,\delta_{ij}\,\partial_i\underbrace{\partial_k v_k}_{=0} + 2\mu\,\partial_i\left[\frac{1}{2}\partial_i v_j + \frac{1}{2}\partial_j v_i\right]$$

$$= \mu\,\partial_i\partial_i v_j + \mu\,\partial_i\partial_j v_i$$

$$= \mu\,\partial_i\partial_i v_j + \mu\,\partial_j\underbrace{\partial_i v_i}_{=0}$$

2nd & 3rd edition

(15)

$$v_j\partial_j v_i = \partial_i\left(\frac{1}{2}v^2\right) - \varepsilon_{ijk}\,v_j\,\omega_k$$

$$\omega_k = \varepsilon_{klm}\,\partial_l v_m$$

$$v_j\partial_j v_i \overset{?}{=} \partial_i\left(\frac{1}{2}v^2\right) - \varepsilon_{ijk}\,\varepsilon_{klm}\,v_j\,\partial_l v_m$$

$$\overset{?}{=} \partial_i\left(\frac{1}{2}v^2\right) - \varepsilon_{kij}\,\varepsilon_{klm}\,v_j\,\partial_l v_m$$

$$\overset{?}{=} \partial_i\left(\frac{1}{2}v^2\right) - \left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right]v_j\,\partial_l v_m$$

$$\overset{?}{=} \frac{1}{2}v_k\partial_i v_k + \frac{1}{2}v_k\partial_i v_k - v_j\partial_i v_j + v_j\partial_j v_i$$

$$= v_j\partial_j v_i \qquad \text{OK}$$

Chapter 3

3.16 $\varepsilon_{ijk}\varepsilon_{ijl} \stackrel{?}{=} 2\delta_{kl}$

$$\varepsilon_{ijk}\varepsilon_{ipl} = \delta_{jp}\delta_{kl} - \delta_{jl}\delta_{kp}$$

now set $p = j$

$$\varepsilon_{ijk}\varepsilon_{ipl} = \delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj}$$

$$\delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$\delta_{jl}\delta_{kj} = \delta_{1l}\delta_{k1} + \delta_{2l}\delta_{k2} + \delta_{3l}\delta_{k3}$$

$$= \delta_{kl}$$

3.17 NEXT PAGE

$$\varepsilon_{ijk}\varepsilon_{ipl} = 3\delta_{kl} - \delta_{kl} = 2\delta_{kl}$$

3.18

$$-\nabla\times\nabla\times\underline{v} \stackrel{?}{=} \nabla^2\underline{v} = \nabla\cdot\nabla(\underline{v})$$

$$-\varepsilon_{lmi}\partial_m\varepsilon_{ijk}\partial_j v_k \stackrel{?}{=} \partial_j\partial_j v_l$$

$$-\varepsilon_{ilm}\varepsilon_{ijk}\partial_m\partial_j v_k =$$

$$-\left[\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}\right]\partial_m\partial_j v_k =$$

$$-\{\partial_k\partial_l v_k - \partial_j\partial_j v_l\} =$$

$$\partial_j\partial_j v_l - \partial_l\underbrace{\partial_k v_k}_{=0} = \partial_j\partial_j v_l$$

Chapter 3

3.17

$$\nabla v : \nabla v = S : S - \frac{1}{2}\omega^2$$

$$S_{ij} = \partial_{(i}v_{j)} \qquad \omega_i = \varepsilon_{ijk}\partial_j v_k$$

$$\nabla v \Rightarrow \partial_i v_j = \partial_{(i}v_{j)} + \frac{1}{2}\varepsilon_{ijk}d_k \qquad (3.6.13)$$

$$d_i = \varepsilon_{ijk}\partial_j v_k \text{ or } d_k = \varepsilon_{klm}\partial_l v_m \qquad (3.6.9)$$

$$\partial_i v_j = \partial_{(i}v_{j)} + \frac{1}{2}\varepsilon_{ijk}\underbrace{\varepsilon_{klm}\partial_l v_m}_{=\omega_k}$$

$$\partial_i v_j \partial_j v_i = \left[\partial_{(i}v_{j)} + \frac{1}{2}\varepsilon_{ijk}\omega_k\right]\left[\partial_{(j}v_{i)} + \frac{1}{2}\varepsilon_{jik}\omega_k\right]$$

$$= \partial_{(i}v_{j)}\partial_{(j}v_{i)} + \frac{1}{2}\varepsilon_{ijk}\omega_k\partial_{(j}v_{i)} \quad \text{ANTI} \times \text{SYM} = 0$$

$$+ \frac{1}{4}\varepsilon_{ijk}\varepsilon_{jil}\omega_k\omega_l + \frac{1}{2}\varepsilon_{jik}\omega_k\partial_{(i}v_{j)}, \quad \text{ANTI} \cdot \text{SYM} = 0$$

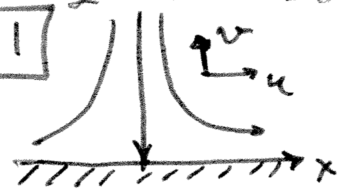
$$= \partial_{(i}v_{j)}\partial_{(j}v_{i)} - \frac{1}{4}\underbrace{\varepsilon_{ijk}\varepsilon_{ijl}}_{= 2\delta_{kl} \text{ prob 3.16}}\omega_k\omega_l$$

$$= \partial_{(i}v_{j)}\partial_{(j}v_{i)} - \frac{1}{2}\omega_k\omega_k$$

Chapter 4 Problem Solutions

2nd ed See 4.4 for 3rd ed this problem is 4.4 in 3ed

4.1



along the stagnation streamline

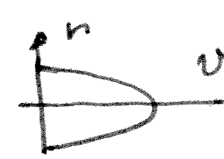
$$v = -cy$$

since $v = \frac{dy}{dt}$ where y is the particle position

$$t = \int dt = \int \frac{dy}{v} = \int_{y_1}^{0} \frac{dy}{-cy} = -\frac{1}{c} \ln \frac{0}{y_1} = \infty$$

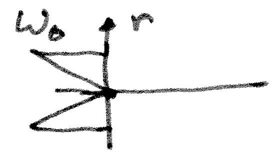
4.2 A. round tube

$$v_z = f(r) = v_0 \left[1 - \left(\frac{r}{R}\right)^2\right]$$



$\underline{\omega} = \nabla \times \underline{v}$ $\qquad \omega_r = 0, \omega_z = 0$

$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = +2 v_0 \frac{r}{R^2}$$



B. ideal vortex

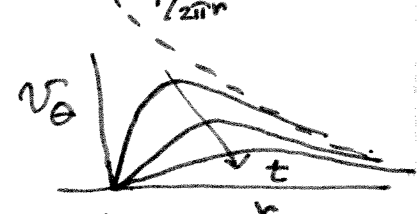
$$v_\theta = \frac{\Gamma}{2\pi r}$$

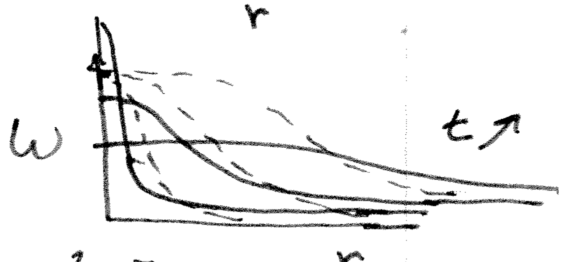
$$\omega_r = \omega_\theta = 0$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\Gamma}{2\pi r}\right) = 0$$

C. viscous vortex

$$v_\theta = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right]$$





$$\omega_r = \omega_\theta = 0$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\Gamma}{2\pi}\left(1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right)\right]$$

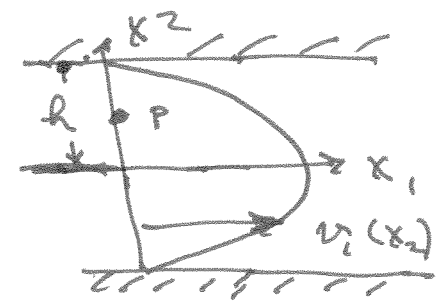
$$= \frac{\Gamma}{2\pi r}\frac{2r}{4\nu t}\exp\left(-\frac{r^2}{4\nu t}\right) = \frac{\Gamma}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right)$$

Gaussian curve that spreads with t

Amp decreases $\frac{1}{t}$

4.3 (Nondimensional form for answer)

$$v_1 = \frac{3}{2} v_{ave} \left[1 - \left(\frac{x_2}{h} \right)^2 \right]$$



let $v_1 / v_{ave} \rightarrow v_1$

and $x_2 / h \rightarrow x_2$

so that in nondimensional variables

$$v_1 = 3/2 \left[1 - x_2^2 \right]$$

VORTICITY $\quad \omega_i = \varepsilon_{ijk} \partial_j v_k$

$$\omega_1 = \omega_2 = 0 \quad ; \quad \omega_3 = -\partial_2 v_1 = 3 x_2$$

STRAIN RATE $\quad S_{ij} = \frac{1}{2} \partial_i v_j + \frac{1}{2} \partial_j v_i$

$$S_{11} = \partial_1 v_1 = 0 \quad ; \quad S_{22} = \partial_2 v_2 = 0$$

$$S_{12} = \frac{1}{2} \partial_1 v_2 + \frac{1}{2} \partial_2 v_1 = \frac{1}{2} (-3 x_2)$$

$$S_{21} = S_{12}$$

At P where $x_2 = 1/2$

$$\omega_3 = 3/2 \quad ; \quad S_{12} = -3/4$$

Solid Body Rotation $\frac{dv_j^{(r)}}{ds} = \varepsilon_{jki}\left(\frac{\omega_k}{2}\right)\alpha_i$

$$\frac{dv_j^{(r)}}{ds} = \varepsilon_{j3i}\,\frac{3}{2.2}\,\alpha_i$$

$$\frac{dv_1^{(r)}}{ds} = \varepsilon_{132}\,\frac{3}{4}\,\alpha_2 = -\frac{3}{4}\,\alpha_2 \qquad (A)$$

$$\frac{dv_2^{(r)}}{ds} = \varepsilon_{231}\,\frac{3}{4}\,\alpha_1 = \frac{3}{4}\,\alpha_1 \qquad (B)$$

Strain (Deformation)

$$\frac{dv_j^{(s)}}{ds} = d_j = \alpha_i S_{ij}$$

$$\frac{dv_1^{(s)}}{ds} = d_1 = \alpha_1 S_{11} + \alpha_2 S_{21}$$

$$= \alpha_2\left(-\frac{3}{4}\right) \qquad (C)$$

$$\frac{dv_2^{(s)}}{ds} = d_2 = \alpha_1 S_{12} + \alpha_2 S_{22}$$

$$= \alpha_1\left(-\frac{3}{4}\right) \qquad (D)$$

Strain (deformation)

$$d_j = \alpha_i S_{ij}$$
$$= \alpha_1 S_{1j} + \alpha_2 S_{2j}$$

$$d_1 = \alpha_1 S_{11} + \alpha_2 S_{21}$$
$$= \alpha_2\left(\frac{-v_0}{2h}\right) \qquad (D)$$

$$d_2 = \alpha_1 S_{12} + \alpha_2 S_{22}$$
$$= \alpha_1\left(\frac{-v_0}{2h}\right) \qquad (E)$$

$$dv_j^{(s)} = d_j\, ds \qquad (F)$$
$$= d_j\,(.1h)$$

$$dv_1^{(s)} = d_1\,(.1h) = -.05\, v_0\, \alpha_2 \qquad (G)$$
$$dv_2^{(s)} = d_2(.1h) = -.05\, v_0\, \alpha_1 \qquad (H)$$

Elongational Strain

$$dv_j^{(es)} = \alpha_j \alpha_k d_k = \alpha_j\left[\alpha_1 d_1 + \alpha_2 d_2\right] ds \qquad (I)$$
$$= \alpha_j\left[\alpha_1\alpha_2\left(\frac{-v_0}{2h}\right) + \alpha_2\alpha_1\left(\frac{-v_0}{2h}\right)\right] ds$$
$$= \alpha_j\, \alpha_1\alpha_2\left(-\frac{v_0}{h}\right) ds$$
$$= \alpha_j\, \alpha_1\alpha_2\,(-.1\, v_0)$$

$$dv_1^{(es)} = \alpha_1^2\alpha_2\,(-.1\, v_0) \qquad (J)$$

$$dv_2^{(es)} = \alpha_1\alpha_2^2\,(-.1\, v_0) \qquad (K)$$

Elongational Strain (strain $\parallel \underline{\alpha}$)

$$\frac{dv_j^{(es)}}{ds} = \alpha_j (\alpha_k d_k) = \alpha_j (\alpha_1 d_1 + \alpha_2 d_2)$$

$$= \alpha_j \left[\alpha_1 \left(-\tfrac{3}{4}\alpha_2\right) + \alpha_2 \left(-\tfrac{3}{4}\alpha_1\right)\right]$$

$$= -\tfrac{3}{2}\alpha_1 \alpha_2 \alpha_j$$

$$\frac{dv_1^{(es)}}{ds} = -\frac{3}{2}\alpha_1^2 \alpha_2 \qquad (E)$$

$$\frac{dv_2^{(es)}}{ds} = -\frac{3}{2}\alpha_1 \alpha_2^2 \qquad (F)$$

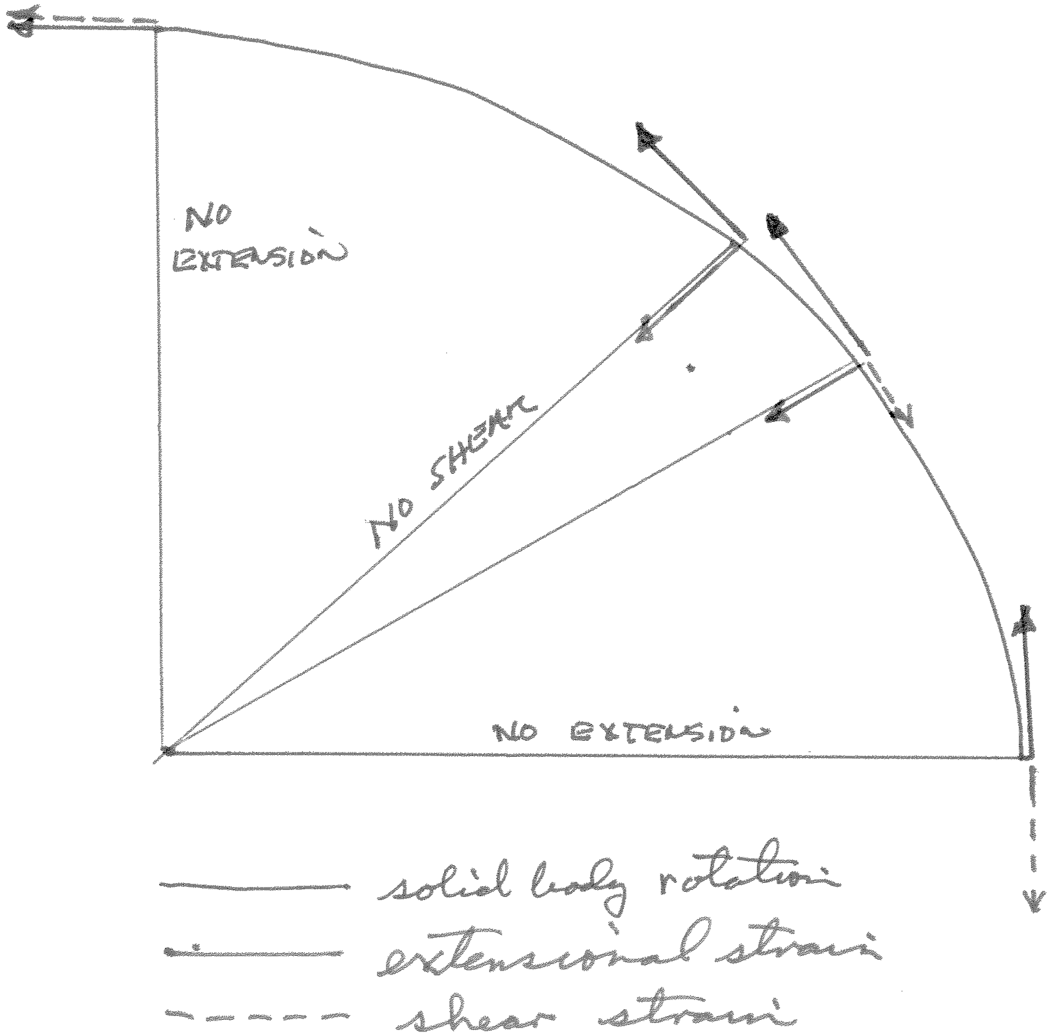
Shear Strain ($\perp \underline{\alpha}$)

$$dv_j^{(ss)} = dv_j^{(s)} - dv_j^{(es)}$$

$$\frac{dv_j^{(ss)}}{ds} = d_j + 2\alpha_1 \alpha_2 \alpha_j$$

$$\frac{dv_1^{(ss)}}{ds} = -\frac{3}{4}\alpha_2 + 2\alpha_1^2 \alpha_2 \qquad (G)$$

$$\frac{dv_2^{(ss)}}{ds} = -\frac{3}{4}\alpha_1 + 2\alpha_1 \alpha_2^2 \qquad (H)$$


NO EXTENSION
NO SHEAR
NO EXTENSION
solid body rotation
extensional strain
shear strain

Solution to problem 4.3 using nondimensional velocity profile

v1=3/2(1-x2^2)

v1 is v1/ v_ave and x2 is x2/ h where his half width and coordinate is on the centerline.

P is a point at x2=0.5. P' is at an angle Theta and distance ds'

The vorticity is nondimensionalzed Omega_3 /v_ave/h = dv1/dx2

At P: $\omega_1 := 0$ $\omega_2 := 0$ $\omega_3 := \frac{3}{2}$

The strain rate tensor is nondimensionalized Sij/v_ave/h

Sij = 1/2(dvj/dxi +dvi/dxj)

At P: $S_{1,1} := 0$ $S_{1,2} := -0.75$ $S_{2,1} := -0.75$ $S_{2,2} := 0$

The first point at 0 degrees

$\theta\_deg := 0$ $\theta := \theta\_deg \cdot \frac{\pi}{180}$ $j := 1..2$ $i := 1..2$

$\alpha_1 := \cos(\theta)$ $\alpha_2 := \sin(\theta)$ $\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$d_i := \left[\sum_{j=1}^{2} \alpha_j \cdot (S_{j,i}) \right]$ $d = \begin{bmatrix} 0 \\ -0.75 \end{bmatrix}$

dv_r is the solid rotational velocity dv(r)/ ds

$dv\_r_i := \sum_{j=1}^{2} \varepsilon(i,3,j) \cdot \frac{\omega_3}{2} \cdot \alpha_j$ $dv\_r = \begin{bmatrix} 0 \\ 0.75 \end{bmatrix}$

Magnitude

$dV\_r := (dv\_r \cdot dv\_r)^{.5}$ $dV\_r = 0.75$

dv_s is the strain velocity dv(s)/ ds

$dv\_s_i := d_i$ $dv\_s = \begin{bmatrix} 0 \\ -0.75 \end{bmatrix}$

Magnitude $dV\_s := (dv\_s \cdot dv\_s)^{.5}$ $dV\_s = 0.75$

dv_es is the elongational strain velocity dv(es)/ ds

$dv\_es_i := \left[\left[\sum_{j=1}^{2} (\alpha_j \cdot d_j) \right] \right] \cdot \alpha_i$ $dv\_es = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Magnitude $dV\_es := (dv\_es \cdot dv\_es)^{.5}$ $dV\_es = 0$

dv_ss is the shear strain velocity dv(ss)/ ds

$$dv\_ss_i := dv\_s_i - dv\_es_i \qquad dv\_ss = \begin{bmatrix} 0 \\ -0.75 \end{bmatrix}$$

Magnitude $dV\_ss := (dv\_ss \cdot dv\_ss)^{.5}$ $\quad dV\_ss = 0.75$

The second point at 30 degrees

$$\theta\_deg := 30 \qquad \theta := \theta\_deg \cdot \frac{\pi}{180} \qquad j := 1..2 \qquad i := 1..2$$

$$\alpha_1 := \cos(\theta) \qquad \alpha_2 := \sin(\theta) \qquad \alpha = \begin{bmatrix} 0.866 \\ 0.5 \end{bmatrix}$$

$$d_i := \left[\sum_{j=1}^{2} \alpha_j \cdot (S_{j,i}) \right] \qquad d = \begin{bmatrix} -0.375 \\ -0.65 \end{bmatrix}$$

dv_r is the solid rotational velocity dv(r)/ ds

$$dv\_r_i := \sum_{j=1}^{2} \varepsilon(i,3,j) \cdot \frac{\omega_3}{2} \cdot \alpha_j \qquad dv\_r = \begin{bmatrix} -0.375 \\ 0.65 \end{bmatrix}$$

Magnitude

$dV\_r := (dv\_r \cdot dv\_r)^{.5}$ $\quad dV\_r = 0.75$

dv_s is the strain velocity dv(s)/ ds

$$dv\_s_i := d_i \qquad dv\_s = \begin{bmatrix} -0.375 \\ -0.65 \end{bmatrix}$$

Magnitude $dV\_s := (dv\_s \cdot dv\_s)^{.5}$ $\quad dV\_s = 0.75$

dv_es is the elongational strain velocity dv(es)/ ds

$$dv\_es_i := \left[\left[\sum_{j=1}^{2} (\alpha_j \cdot d_j) \right] \right] \cdot \alpha_i \qquad dv\_es = \begin{bmatrix} -0.563 \\ -0.325 \end{bmatrix}$$

Magnitude $dV\_es := (dv\_es \cdot dv\_es)^{.5}$ $\quad dV\_es = 0.65$

dv_ss is the shear strain velocity dv(ss)/ ds

$$dv\_ss_i := dv\_s_i - dv\_es_i \qquad dv\_ss = \begin{bmatrix} 0.188 \\ \dots \end{bmatrix}$$

$\begin{bmatrix} -0.325 \end{bmatrix}$

Magnitude $dV\_ss := (dv\_ss \cdot dv\_ss)^{.5}$ $dV\_ss = 0.375$

The first point at 45 degrees

$\theta\_deg := 45$ $\theta := \theta\_deg \cdot \frac{\pi}{180}$ $j := 1..2$ $i := 1..2$

$\alpha_1 := \cos(\theta)$ $\alpha_2 := \sin(\theta)$ $\alpha = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$

$d_i := \left[\sum_{j=1}^{2} \alpha_j \cdot (S_{j,i}) \right]$ $d = \begin{bmatrix} -0.53 \\ -0.53 \end{bmatrix}$

dv_r is the solid rotational velocity dv(r)/ ds

$dv\_r_i := \sum_{j=1}^{2} \varepsilon(i,3,j) \cdot \frac{\omega_3}{2} \cdot \alpha_j$ $dv\_r = \begin{bmatrix} -0.53 \\ 0.53 \end{bmatrix}$

Magnitude

$dV\_r := (dv\_r \cdot dv\_r)^{.5}$ $dV\_r = 0.75$

dv_s is the strain velocity dv(s)/ ds

$dv\_s_i := d_i$ $dv\_s = \begin{bmatrix} -0.53 \\ -0.53 \end{bmatrix}$

Magnitude $dV\_s := (dv\_s \cdot dv\_s)^{.5}$ $dV\_s = 0.75$

dv_es is the elongational strain velocity dv(es)/ ds

$dv\_es_i := \left[\left[\sum_{j=1}^{2} (\alpha_j \cdot d_j) \right] \right] \cdot \alpha_i$ $dv\_es = \begin{bmatrix} -0.53 \\ -0.53 \end{bmatrix}$

$dV\_es = 0.65$

Magnitude $dV\_es := (dv\_es \cdot dv\_es)^{.5}$

dv_ss is the shear strain velocity dv(ss)/ ds

$dv\_ss_i := dv\_s_i - dv\_es_i$ $dv\_ss = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Magnitude $dV\_ss := (dv\_ss \cdot dv\_ss)^{.5}$ $dV\_ss = 0$

The last point at 90 degrees

$\theta\_deg := 90$ $\qquad \theta := \theta\_deg \cdot \frac{\pi}{180}$ $\qquad j := 1..2$ $\qquad i := 1..2$

$\alpha_1 := \cos(\theta)$ $\qquad \alpha_2 := \sin(\theta)$ $\qquad \alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$d_i := \left| \sum_{j=1}^{2} \alpha_j \cdot (S_{j,i}) \right| \qquad d = \begin{bmatrix} -0.75 \\ 0 \end{bmatrix}$$

dv_r is the solid rotational velocity dv(r)/ ds

$$dv\_r_i := \sum_{j=1}^{2} \varepsilon(i,3,j) \cdot \frac{\omega_3}{2} \cdot \alpha_j \qquad dv\_r = \begin{bmatrix} -0.75 \\ 0 \end{bmatrix}$$

Magnitude

$$dV\_r := (dv\_r \cdot dv\_r)^{.5} \qquad dV\_r = 0.75$$

dv_s is the strain velocity dv(s)/ ds

$$dv\_s_i := d_i \qquad dv\_s = \begin{bmatrix} -0.75 \\ 0 \end{bmatrix}$$

Magnitude $\quad dV\_s := (dv\_s \cdot dv\_s)^{.5} \qquad dV\_s = 0.75$

dv_es is the elongational strain velocity dv(es)/ ds

$$dv\_es_i := \left[\left[\sum_{j=1}^{2} (\alpha_j \cdot d_j) \right] \right] \cdot \alpha_i \qquad dv\_es = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Magnitude $\quad dV\_es := (dv\_es \cdot dv\_es)^{.5} \qquad dV\_es = 0$

dv_ss is the shear strain velocity dv(ss)/ ds

$$dv\_ss_i := dv\_s_i - dv\_es_i \qquad dv\_ss = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Magnitude $\quad dV\_ss := (dv\_ss \cdot dv\_ss)^{.5} \qquad dV\_ss = 0$