

## CHAPTER 2, PART A

2.1 Given

$$[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix} \text{ and } [a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Evaluate (a)  $S_{ii}$ , (b)  $S_{ij}S_{ij}$ , (c)  $S_{ji}S_{ji}$ , (d)  $S_{jk}S_{kj}$ , (e)  $a_m a_m$ , (f)  $S_{mn}a_m a_n$ , (g)  $S_{nm}a_m a_n$

Ans. (a)  $S_{ii} = S_{11} + S_{22} + S_{33} = 1 + 1 + 3 = 5$ .

(b)  $S_{ij}S_{ij} = S_{11}^2 + S_{12}^2 + S_{13}^2 + S_{21}^2 + S_{22}^2 + S_{23}^2 + S_{31}^2 + S_{32}^2 + S_{33}^2 =$   
 $1 + 0 + 4 + 0 + 1 + 4 + 9 + 0 + 9 = 28$ .

(c)  $S_{ji}S_{ji} = S_{ij}S_{ij} = 28$ .

(d)  $S_{jk}S_{kj} = S_{1k}S_{k1} + S_{2k}S_{k2} + S_{3k}S_{k3}$   
 $= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{21}S_{12} + S_{22}S_{22} + S_{23}S_{32} + S_{31}S_{13} + S_{32}S_{23} + S_{33}S_{33}$   
 $= (1)(1) + (0)(0) + (2)(3) + (0)(0) + (1)(1) + (2)(0) + (3)(2) + (0)(2) + (3)(3) = 23$ .

(e)  $a_m a_m = a_1^2 + a_2^2 + a_3^2 = 1 + 4 + 9 = 14$ .

(f)  $S_{mn}a_m a_n = S_{1n}a_1 a_n + S_{2n}a_2 a_n + S_{3n}a_3 a_n =$   
 $S_{11}a_1 a_1 + S_{12}a_1 a_2 + S_{13}a_1 a_3 + S_{21}a_2 a_1 + S_{22}a_2 a_2 + S_{23}a_2 a_3 + S_{31}a_3 a_1 + S_{32}a_3 a_2 + S_{33}a_3 a_3$   
 $= (1)(1)(1) + (0)(1)(2) + (2)(1)(3) + (0)(2)(1) + (1)(2)(2) + (2)(2)(3) + (3)(3)(1)$   
 $+ (0)(3)(2) + (3)(3)(3) = 1 + 0 + 6 + 0 + 4 + 12 + 9 + 0 + 27 = 59$ .

(g)  $S_{nm}a_m a_n = S_{mn}a_m a_n = 59$ .

2.2 Determine which of these equations have an identical meaning with  $a_i = Q_{ij}a'_j$ .

(a)  $a_p = Q_{pm}a'_m$ , (b)  $a_p = Q_{qp}a'_q$ , (c)  $a_m = a'_n Q_{mn}$ .

Ans. (a) and (c)

2.3 Given the following matrices

$$[a_i] = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, [B_{ij}] = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Demonstrate the equivalence of the subscripted equations and corresponding matrix equations in the following two problems.

(a)  $b_i = B_{ij}a_j$  and  $[b] = [B][a]$ , (b)  $s = B_{ij}a_i a_j$  and  $s = [a]^T [B][a]$

Ans. (a)

$$b_i = B_{ij}a_j \rightarrow b_1 = B_{1j}a_j = B_{11}a_1 + B_{12}a_2 + B_{13}a_3 = (2)(1) + (3)(0) + (0)(2) = 2$$

$$b_2 = B_{2j}a_j = B_{21}a_1 + B_{22}a_2 + B_{23}a_3 = 2, \quad b_3 = B_{3j}a_j = B_{31}a_1 + B_{32}a_2 + B_{33}a_3 = 2.$$

$$[b] = [B][a] = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}. \text{ Thus, } b_i = B_{ij}a_j \text{ gives the same results as } [b] = [B][a]$$

(b)

$$s = B_{ij}a_i a_j = B_{11}a_1 a_1 + B_{12}a_1 a_2 + B_{13}a_1 a_3 + B_{21}a_2 a_1 + B_{22}a_2 a_2 + B_{23}a_2 a_3 \\ + B_{31}a_3 a_1 + B_{32}a_3 a_2 + B_{33}a_3 a_3 = (2)(1)(1) + (3)(1)(0) + (0)(1)(2) + (0)(0)(1) \\ + (5)(0)(0) + (1)(0)(2) + (0)(2)(1) + (2)(2)(0) + (1)(2)(2) = 2 + 4 = 6.$$

$$\text{and } s = [a]^T [B][a] = [1 \ 0 \ 2] \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = [1 \ 0 \ 2] \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2 + 4 = 6.$$

2.4 Write in indicial notation the matrix equation (a)  $[A] = [B][C]$ , (b)  $[D] = [B]^T [C]$  and (c)  $[E] = [B]^T [C][F]$ .

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Ans. (a)  $[A] = [B][C] \rightarrow A_{ij} = B_{im} C_{mj}$ , (b)  $[D] = [B]^T [C] \rightarrow A_{ij} = B_{mi} C_{mj}$ .  
 (c)  $[E] = [B]^T [C][F] \rightarrow E_{ij} = B_{mi} C_{mk} F_{kj}$ .

2.5 Write in indicial notation the equation (a)  $s = A_1^2 + A_2^2 + A_3^2$  and (b)  $\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = 0$ .

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Ans. (a)  $s = A_1^2 + A_2^2 + A_3^2 = A_i A_i$ . (b)  $\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = 0 \rightarrow \frac{\partial^2 \phi}{\partial x_i \partial x_i} = 0$ .

2.6 Given that  $S_{ij} = a_i a_j$  and  $S'_{ij} = a'_i a'_j$ , where  $a'_i = Q_{mi} a_m$  and  $a'_j = Q_{nj} a_n$ , and  $Q_{ik} Q_{jk} = \delta_{ij}$ . Show that  $S'_{ii} = S_{ii}$ .

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Ans.  $S'_{ij} = Q_{mi} a_m Q_{nj} a_n = Q_{mi} Q_{nj} a_m a_n \rightarrow S'_{ii} = Q_{mi} Q_{ni} a_m a_n = \delta_{mn} a_m a_n = a_m a_m = S_{mm} = S_{ii}$ .

2.7 Write  $a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$  in long form.

Ans.

$$i = 1 \rightarrow a_1 = \frac{\partial v_1}{\partial t} + v_j \frac{\partial v_1}{\partial x_j} = \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} \\ i = 2 \rightarrow a_2 = \frac{\partial v_2}{\partial t} + v_j \frac{\partial v_2}{\partial x_j} = \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3} \\ i = 3 \rightarrow a_3 = \frac{\partial v_3}{\partial t} + v_j \frac{\partial v_3}{\partial x_j} = \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3}$$

2.8 Given that  $T_{ij} = 2\mu E_{ij} + \lambda E_{kk} \delta_{ij}$ , show that

(a)  $T_{ij}E_{ij} = 2\mu E_{ij}E_{ij} + \lambda(E_{kk})^2$  and (b)  $T_{ij}T_{ij} = 4\mu^2 E_{ij}E_{ij} + (E_{kk})^2 (4\mu\lambda + 3\lambda^2)$

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 Ans. (a)

$$T_{ij}E_{ij} = (2\mu E_{ij} + \lambda E_{kk} \delta_{ij})E_{ij} = 2\mu E_{ij}E_{ij} + \lambda E_{kk} \delta_{ij}E_{ij} = 2\mu E_{ij}E_{ij} + \lambda E_{kk} E_{ii} = 2\mu E_{ij}E_{ij} + \lambda(E_{kk})^2$$

(b)

$$\begin{aligned} T_{ij}T_{ij} &= (2\mu E_{ij} + \lambda E_{kk} \delta_{ij})(2\mu E_{ij} + \lambda E_{kk} \delta_{ij}) = 4\mu^2 E_{ij}E_{ij} + 2\mu\lambda E_{ij}E_{kk} \delta_{ij} + 2\mu\lambda E_{kk} \delta_{ij}E_{ij} \\ &+ \lambda^2 (E_{kk})^2 \delta_{ij} \delta_{ij} = 4\mu^2 E_{ij}E_{ij} + 2\mu\lambda E_{ii}E_{kk} + 2\mu\lambda E_{kk} E_{ii} + \lambda^2 (E_{kk})^2 \delta_{ii} \\ &= 4\mu^2 E_{ij}E_{ij} + (E_{kk})^2 (4\mu\lambda + 3\lambda^2). \end{aligned}$$

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 2.9 Given that  $a_i = T_{ij}b_j$ , and  $a'_i = T'_{ij}b'_j$ , where  $a_i = Q_{im}a'_m$  and  $T_{ij} = Q_{im}Q_{jn}T'_{mn}$ .

(a) Show that  $Q_{im}T'_{mn}b'_n = Q_{im}Q_{jn}T'_{mn}b_j$  and (b) if  $Q_{ik}Q_{im} = \delta_{km}$ , then  $T'_{kn}(b'_n - Q_{jn}b_j) = 0$ .

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 Ans. (a) Since  $a_i = Q_{im}a'_m$  and  $T_{ij} = Q_{im}Q_{jn}T'_{mn}$ , therefore,  $a_i = T_{ij}b_j \rightarrow$

$$Q_{im}a'_m = Q_{im}Q_{jn}T'_{mn}b_j \quad (1), \quad \text{Now, } a'_i = T'_{ij}b'_j \rightarrow a'_m = T'_{mj}b'_j = T'_{mn}b'_n, \text{ therefore, Eq. (1) becomes}$$

$$Q_{im}T'_{mn}b'_n = Q_{im}Q_{jn}T'_{mn}b_j. \quad (2)$$

(b) To remove  $Q_{im}$  from Eq. (2), we make use of  $Q_{ik}Q_{im} = \delta_{km}$  by multiplying the above equation, Eq.(2) with  $Q_{ik}$ . That is,

$$\begin{aligned} Q_{ik}Q_{im}T'_{mn}b'_n &= Q_{ik}Q_{im}Q_{jn}T'_{mn}b_j \rightarrow \delta_{km}T'_{mn}b'_n = \delta_{km}Q_{jn}T'_{mn}b_j \rightarrow T'_{kn}b'_n = Q_{jn}T'_{kn}b_j \\ &\rightarrow T'_{kn}(b'_n - Q_{jn}b_j) = 0. \end{aligned}$$

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 2.10 Given  $[a_i] = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$  and  $[b_i] = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$  Evaluate  $[d_i]$ , if  $d_k = \varepsilon_{ijk}a_i b_j$  and show that this result is

the same as  $d_k = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_k$ .

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 Ans.  $d_k = \varepsilon_{ijk}a_i b_j \rightarrow$

$$d_1 = \varepsilon_{ij1}a_i b_j = \varepsilon_{231}a_2 b_3 + \varepsilon_{321}a_3 b_2 = a_2 b_3 - a_3 b_2 = (2)(3) - (0)(2) = 6$$

$$d_2 = \varepsilon_{ij2}a_i b_j = \varepsilon_{312}a_3 b_1 + \varepsilon_{132}a_1 b_3 = a_3 b_1 - a_1 b_3 = (0)(0) - (1)(3) = -3$$

$$d_3 = \varepsilon_{ij3}a_i b_j = \varepsilon_{123}a_1 b_2 + \varepsilon_{213}a_2 b_1 = a_1 b_2 - a_2 b_1 = (1)(2) - (2)(0) = 2$$

Next,  $(\mathbf{a} \times \mathbf{b}) = (\mathbf{e}_1 + 2\mathbf{e}_2) \times (2\mathbf{e}_2 + 3\mathbf{e}_3) = 6\mathbf{e}_1 - 3\mathbf{e}_2 + 2\mathbf{e}_3$ .

$$d_1 = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_1 = 6, \quad d_2 = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_2 = -3, \quad d_3 = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_3 = 2.$$

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 2.11 (a) If  $\varepsilon_{ijk}T_{ij} = 0$ , show that  $T_{ij} = T_{ji}$ , and (b) show that  $\delta_{ij}\varepsilon_{ijk} = 0$

Ans. (a) for  $k=1$ ,  $\varepsilon_{ij1}T_{ij} = 0 \rightarrow \varepsilon_{231}T_{23} + \varepsilon_{321}T_{32} = 0 \rightarrow T_{23} - T_{32} \rightarrow T_{23} = T_{32}$ .

for  $k=2$ ,  $\varepsilon_{ij2}T_{ij} = 0 \rightarrow \varepsilon_{312}T_{31} + \varepsilon_{132}T_{13} = 0 \rightarrow T_{31} - T_{13} \rightarrow T_{31} = T_{13}$ .

for  $k=3$ ,  $\varepsilon_{ij3}T_{ij} = 0 \rightarrow \varepsilon_{123}T_{12} + \varepsilon_{213}T_{21} = 0 \rightarrow T_{12} - T_{21} \rightarrow T_{12} = T_{21}$ .

(b)  $\delta_{ij}\varepsilon_{ijk} = \delta_{11}\varepsilon_{11k} + \delta_{22}\varepsilon_{22k} + \delta_{33}\varepsilon_{33k} = (1)(0) + (1)(0) + (1)(0) = 0$ .

2.12 Verify the following equation:  $\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ .

(Hint): there are 6 cases to be considered (i)  $i = j$ , (2)  $i = k$ , (3)  $i = l$ , (4)  $j = k$ , (5)  $j = l$ , and (6)  $k = l$ .

Ans. There are 4 free indices in the equation. Therefore, there are the following 6 cases to consider:

(i)  $i = j$ , (2)  $i = k$ , (3)  $i = l$ , (4)  $j = k$ , (5)  $j = l$ , and (6)  $k = l$ . We consider each case below where we use LS for left side, RS for right side and repeated indices with parenthesis are not sum:

(1) For  $i = j$ ,  $LS = \varepsilon_{(i)(i)m}\varepsilon_{klm} = 0$ ,  $RS = \delta_{(i)k}\delta_{(i)l} - \delta_{(i)l}\delta_{(i)k} = 0$ .

(2) For  $i = k$ ,  $LS = \varepsilon_{(i)j1}\varepsilon_{(i)l1} + \varepsilon_{(i)j2}\varepsilon_{(i)l2} + \varepsilon_{(i)j3}\varepsilon_{(i)l3}$ ,  $RS = \delta_{(i)(i)}\delta_{jl} - \delta_{(i)l}\delta_{j(i)}$

$$LS=RS = \begin{cases} 0 & \text{if } j \neq l \\ 0 & \text{if } j = l = i \\ 1 & \text{if } j = l \neq i \end{cases}$$

(3) For  $i = l$ ,  $LS = \varepsilon_{(i)jm}\varepsilon_{k(i)m}$ ,  $RS = \delta_{(i)k}\delta_{j(i)} - \delta_{(i)(i)}\delta_{jk}$

$$LS=RS = \begin{cases} 0 & \text{if } j \neq k \\ 0 & \text{if } j = k = i \\ -1 & \text{if } j = k \neq i \end{cases}$$

(4) For  $j = k$ ,  $LS = \varepsilon_{i(j)m}\varepsilon_{(j)lm}$ ,  $RS = \delta_{i(j)}\delta_{(j)l} - \delta_{il}\delta_{(j)(j)}$

$$LS=RS = \begin{cases} 0 & \text{if } i \neq l \\ 0 & \text{if } i = l = j \\ -1 & \text{if } i = l \neq j \end{cases}$$

(5) For  $j = l$ ,  $LS = \varepsilon_{i(j)m}\varepsilon_{k(j)m}$ ,  $RS = \delta_{ik}\delta_{(j)(j)} - \delta_{i(j)}\delta_{(j)k}$

$$LS=RS = \begin{cases} 0 & \text{if } i \neq k \\ 0 & \text{if } i = k = j \\ 1 & \text{if } i = k \neq j \end{cases}$$

(6) For  $k = l$ ,  $LS = \varepsilon_{ijm}\varepsilon_{(k)(k)m} = 0$ ,  $RS = \delta_{i(k)}\delta_{j(k)} - \delta_{i(k)}\delta_{j(k)} = 0$

2.13 Use the identity  $\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$  as a short cut to obtain the following results:

(a)  $\varepsilon_{ilm}\varepsilon_{jlm} = 2\delta_{ij}$  and (b)  $\varepsilon_{ijk}\varepsilon_{ijk} = 6$ .

Ans. (a)  $\varepsilon_{ilm}\varepsilon_{jlm} = \delta_{ij}\delta_{ll} - \delta_{il}\delta_{lj} = 3\delta_{ij} - \delta_{ij} = 2\delta_{ij}$ .

(b)  $\varepsilon_{ijk}\varepsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} = (3)(3) - \delta_{ii} = 9 - 3 = 6$ .

2.14 Use the identity  $\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$  to show that  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ .

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*Ans.*  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = a_m \mathbf{e}_m \times (\varepsilon_{ijk} b_j c_k \mathbf{e}_i) = \varepsilon_{ijk} a_m b_j c_k (\mathbf{e}_m \times \mathbf{e}_i)$   
 $= \varepsilon_{ijk} a_m b_j c_k (\varepsilon_{nmi} \mathbf{e}_n) = \varepsilon_{ijk} \varepsilon_{nmi} a_m b_j c_k \mathbf{e}_n = \varepsilon_{jki} \varepsilon_{nmi} a_m b_j c_k \mathbf{e}_n$   
 $= (\delta_{jn} \delta_{km} - \delta_{jm} \delta_{kn}) a_m b_j c_k \mathbf{e}_n = \delta_{jn} \delta_{km} a_m b_j c_k \mathbf{e}_n - \delta_{jm} \delta_{kn} a_m b_j c_k \mathbf{e}_n$   
 $= a_k b_n c_k \mathbf{e}_n - a_j b_j c_n \mathbf{e}_n = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}.$

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2.15 (a) Show that if  $T_{ij} = -T_{ji}$ ,  $T_{ij} a_i a_j = 0$  and (b) if  $T_{ij} = -T_{ji}$ , and  $S_{ij} = S_{ji}$ , then  $T_{ij} S_{ij} = 0$

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*Ans.* Since  $T_{ij} a_i a_j = T_{ji} a_j a_i$  (switching the original dummy index  $i$  to  $j$  and the original index  $j$  to  $i$ ), therefore  $T_{ij} a_i a_j = T_{ji} a_j a_i = -T_{ij} a_j a_i = -T_{ij} a_i a_j \rightarrow 2T_{ij} a_i a_j = 0 \rightarrow T_{ij} a_i a_j = 0$ .  
 (b)  $T_{ij} S_{ij} = T_{ji} S_{ji}$  (switching the original dummy index  $i$  to  $j$  and the original index  $j$  to  $i$ ), therefore,  $T_{ij} S_{ij} = T_{ji} S_{ji} = -T_{ij} S_{ji} = -T_{ij} S_{ij} \rightarrow 2T_{ij} S_{ij} = 0 \rightarrow T_{ij} S_{ij} = 0$ .

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2.16 Let  $T_{ij} = (S_{ij} + S_{ji})/2$  and  $R_{ij} = (S_{ij} - S_{ji})/2$ , show that  $T_{ij} = T_{ji}$ ,  $R_{ij} = -R_{ji}$ , and  $S_{ij} = T_{ij} + R_{ij}$ .

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*Ans.*  $T_{ij} = (S_{ij} + S_{ji})/2 \rightarrow T_{ji} = (S_{ji} + S_{ij})/2 = T_{ij}$ .  
 $R_{ij} = (S_{ij} - S_{ji})/2 \rightarrow R_{ji} = (S_{ji} - S_{ij})/2 = -(S_{ij} - S_{ji})/2 = -R_{ij}$ .  
 $T_{ij} + R_{ij} = (S_{ij} + S_{ji})/2 + (S_{ij} - S_{ji})/2 = S_{ij}$ .

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2.17 Let  $f(x_1, x_2, x_3)$  be a function of  $x_1, x_2$ , and  $x_3$  and  $v_i(x_1, x_2, x_3)$  be three functions of  $x_1, x_2$ , and  $x_3$ . Express the total differential  $df$  and  $dv_i$  in indicial notation.

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*Ans.*  $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = \frac{\partial f}{\partial x_i} dx_i$ .  
 $dv_i = \frac{\partial v_i}{\partial x_1} dx_1 + \frac{\partial v_i}{\partial x_2} dx_2 + \frac{\partial v_i}{\partial x_3} dx_3 = \frac{\partial v_i}{\partial x_m} dx_m$ .

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2.18 Let  $|A_{ij}|$  denote that determinant of the matrix  $[A_{ij}]$ . Show that  $|A_{ij}| = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$

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*Ans.*  $\varepsilon_{ijk} A_{i1} A_{j2} A_{k3} = \varepsilon_{1jk} A_{11} A_{j2} A_{k3} + \varepsilon_{2jk} A_{21} A_{j2} A_{k3} + \varepsilon_{3jk} A_{31} A_{j2} A_{k3}$   
 $= \varepsilon_{123} A_{11} A_{22} A_{33} + \varepsilon_{132} A_{11} A_{32} A_{23} + \varepsilon_{231} A_{21} A_{32} A_{13} + \varepsilon_{213} A_{21} A_{12} A_{33} + \varepsilon_{312} A_{31} A_{12} A_{23} + \varepsilon_{321} A_{31} A_{22} A_{13}$   
 $= A_{11} A_{22} A_{33} - A_{11} A_{32} A_{23} + A_{21} A_{32} A_{13} - A_{21} A_{12} A_{33} + A_{31} A_{12} A_{23} - A_{31} A_{22} A_{13}$   
 $= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$

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**CHAPTER 2, PART B**

2.19 A transformation  $\mathbf{T}$  operate on any vector  $\mathbf{a}$  to give  $\mathbf{Ta} = \mathbf{a} / |\mathbf{a}|$ , where  $|\mathbf{a}|$  is the magnitude of  $\mathbf{a}$ . Show that  $\mathbf{T}$  is not a linear transformation.

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 Ans. Since  $\mathbf{Ta} = \frac{\mathbf{a}}{|\mathbf{a}|}$  for any  $\mathbf{a}$ , therefore  $\mathbf{T}(\mathbf{a} + \mathbf{b}) = \frac{\mathbf{a} + \mathbf{b}}{|\mathbf{a} + \mathbf{b}|}$ . Now  $\mathbf{Ta} + \mathbf{Tb} = \frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}$   
 therefore  $\mathbf{T}(\mathbf{a} + \mathbf{b}) \neq \mathbf{Ta} + \mathbf{Tb}$  and  $\mathbf{T}$  is not a linear transformation.

2.20 (a) A tensor  $\mathbf{T}$  transforms every vector  $\mathbf{a}$  into a vector  $\mathbf{Ta} = \mathbf{m} \times \mathbf{a}$  where  $\mathbf{m}$  is a specified vector. Show that  $\mathbf{T}$  is a linear transformation and (b) If  $\mathbf{m} = \mathbf{e}_1 + \mathbf{e}_2$ , find the matrix of the tensor  $\mathbf{T}$ .

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 Ans. (a)  $\mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = \mathbf{m} \times (\alpha\mathbf{a} + \beta\mathbf{b}) = \mathbf{m} \times \alpha\mathbf{a} + \mathbf{m} \times \beta\mathbf{b} = \alpha\mathbf{m} \times \mathbf{a} + \beta\mathbf{m} \times \mathbf{b} = \alpha\mathbf{Ta} + \beta\mathbf{Tb}$ . Thus, the given  $\mathbf{T}$  is a linear transformation.

(b)  $\mathbf{Te}_1 = \mathbf{m} \times \mathbf{e}_1 = (\mathbf{e}_1 + \mathbf{e}_2) \times \mathbf{e}_1 = -\mathbf{e}_3$ ,  $\mathbf{Te}_2 = \mathbf{m} \times \mathbf{e}_2 = (\mathbf{e}_1 + \mathbf{e}_2) \times \mathbf{e}_2 = \mathbf{e}_3$ ,  
 $\mathbf{Te}_3 = \mathbf{m} \times \mathbf{e}_3 = (\mathbf{e}_1 + \mathbf{e}_2) \times \mathbf{e}_3 = -\mathbf{e}_2 + \mathbf{e}_1$ . Thus,

$$[\mathbf{T}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

2.21 A tensor  $\mathbf{T}$  transforms the base vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$  such that  $\mathbf{Te}_1 = \mathbf{e}_1 + \mathbf{e}_2$  and  $\mathbf{Te}_2 = \mathbf{e}_1 - \mathbf{e}_2$ . If  $\mathbf{a} = 2\mathbf{e}_1 + 3\mathbf{e}_2$  and  $\mathbf{b} = 3\mathbf{e}_1 + 2\mathbf{e}_2$ , use the linear property of  $\mathbf{T}$  to find (a)  $\mathbf{Ta}$ , (b)  $\mathbf{Tb}$ , and (c)  $\mathbf{T}(\mathbf{a} + \mathbf{b})$ .

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 Ans.

(a)  $\mathbf{Ta} = \mathbf{T}(2\mathbf{e}_1 + 3\mathbf{e}_2) = 2\mathbf{Te}_1 + 3\mathbf{Te}_2 = 2(\mathbf{e}_1 + \mathbf{e}_2) + 3(\mathbf{e}_1 - \mathbf{e}_2) = 5\mathbf{e}_1 - \mathbf{e}_2$ .  
 (b)  $\mathbf{Tb} = \mathbf{T}(3\mathbf{e}_1 + 2\mathbf{e}_2) = 3\mathbf{Te}_1 + 2\mathbf{Te}_2 = 3(\mathbf{e}_1 + \mathbf{e}_2) + 2(\mathbf{e}_1 - \mathbf{e}_2) = 5\mathbf{e}_1 + \mathbf{e}_2$ .  
 (c)  $\mathbf{T}(\mathbf{a} + \mathbf{b}) = \mathbf{Ta} + \mathbf{Tb} = (5\mathbf{e}_1 - \mathbf{e}_2) + (5\mathbf{e}_1 + \mathbf{e}_2) = 10\mathbf{e}_1$ .

2.22 Obtain the matrix for the tensor  $\mathbf{T}$  which transforms the base vectors as follows:  
 $\mathbf{Te}_1 = 2\mathbf{e}_1 + \mathbf{e}_3$ ,  $\mathbf{Te}_2 = \mathbf{e}_2 + 3\mathbf{e}_3$ ,  $\mathbf{Te}_3 = -\mathbf{e}_1 + 3\mathbf{e}_2$ .

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 Ans.  $[\mathbf{T}] = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$ .

2.23 Find the matrix of the tensor  $\mathbf{T}$  which transforms any vector  $\mathbf{a}$  into a vector  $\mathbf{b} = \mathbf{m}(\mathbf{a} \cdot \mathbf{n})$  where  $\mathbf{m} = (\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2)$  and  $\mathbf{n} = (\sqrt{2}/2)(-\mathbf{e}_1 + \mathbf{e}_3)$ .

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 Ans.  $\mathbf{Te}_1 = \mathbf{m}(\mathbf{e}_1 \cdot \mathbf{n}) = n_1\mathbf{m} = (-\sqrt{2}/2) \left[ (\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2) \right] = -(\mathbf{e}_1 + \mathbf{e}_2)/2$ .

$$\mathbf{T}\mathbf{e}_2 = \mathbf{m}(\mathbf{e}_2 \cdot \mathbf{n}) = n_2 \mathbf{m} = 0 \mathbf{m} = \mathbf{0}.$$

$$\mathbf{T}\mathbf{e}_3 = \mathbf{m}(\mathbf{e}_3 \cdot \mathbf{n}) = n_3 \mathbf{m} = (\sqrt{2}/2) \left[ (\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2) \right] = (\mathbf{e}_1 + \mathbf{e}_2)/2.$$

$$\text{Thus, } [\mathbf{T}] = \begin{bmatrix} -1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.24 (a) A tensor  $\mathbf{T}$  transforms every vector into its mirror image with respect to the plane whose normal is  $\mathbf{e}_2$ . Find the matrix of  $\mathbf{T}$ . (b) Do part (a) if the plane has a normal in the  $\mathbf{e}_3$  direction.

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$$\text{Ans. (a) } \mathbf{T}\mathbf{e}_1 = \mathbf{e}_1, \quad \mathbf{T}\mathbf{e}_2 = -\mathbf{e}_2, \quad \mathbf{T}\mathbf{e}_3 = \mathbf{e}_3, \quad \text{thus, } [\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{(b) } \mathbf{T}\mathbf{e}_1 = \mathbf{e}_1, \quad \mathbf{T}\mathbf{e}_2 = \mathbf{e}_2, \quad \mathbf{T}\mathbf{e}_3 = -\mathbf{e}_3, \quad \text{thus, } [\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

2.25 (a) Let  $\mathbf{R}$  correspond to a right-hand rotation of angle  $\theta$  about the  $x_1$ -axis. Find the matrix of  $\mathbf{R}$ . (b) do part (a) if the rotation is about the  $x_2$ -axis. The coordinates are right-handed.

---

Ans.(a)  $\mathbf{R}\mathbf{e}_1 = \mathbf{e}_1$ ,  $\mathbf{R}\mathbf{e}_2 = 0\mathbf{e}_1 + \cos\theta\mathbf{e}_2 + \sin\theta\mathbf{e}_3$ ,  $\mathbf{R}\mathbf{e}_3 = 0\mathbf{e}_1 - \sin\theta\mathbf{e}_2 + \cos\theta\mathbf{e}_3$ . Thus,

$$[\mathbf{R}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

(b)  $\mathbf{R}\mathbf{e}_1 = -\sin\theta\mathbf{e}_3 + \cos\theta\mathbf{e}_1$ ,  $\mathbf{R}\mathbf{e}_2 = \mathbf{e}_2$ ,  $\mathbf{R}\mathbf{e}_3 = \cos\theta\mathbf{e}_3 + \sin\theta\mathbf{e}_1$ . Thus,

$$[\mathbf{R}] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

2.26 Consider a plane of reflection which passes through the origin. Let  $\mathbf{n}$  be a unit normal vector to the plane and let  $\mathbf{r}$  be the position vector for a point in space. (a) Show that the reflected vector for  $\mathbf{r}$  is given by  $\mathbf{T}\mathbf{r} = \mathbf{r} - 2(\mathbf{r} \cdot \mathbf{n})\mathbf{n}$ , where  $\mathbf{T}$  is the transformation that corresponds to the reflection. (b) Let  $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$ , find the matrix of  $\mathbf{T}$ . (c) Use this linear transformation to find the mirror image of the vector  $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ .

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Ans. (a) Let the vector  $\mathbf{r}$  be decomposed into two vectors  $\mathbf{r}_n$  and  $\mathbf{r}_t$ , where  $\mathbf{r}_n$  is in the direction of  $\mathbf{n}$  and  $\mathbf{r}_t$  is in a direction perpendicular to  $\mathbf{n}$ . That is,  $\mathbf{r}_n$  is normal to the plane of reflection and  $\mathbf{r}_t$  is on the plane of reflection and  $\mathbf{r} = \mathbf{r}_t + \mathbf{r}_n$ . In the reflection given by  $\mathbf{T}$ , we have,

$$\mathbf{T}\mathbf{r}_n = -\mathbf{r}_n \quad \text{and} \quad \mathbf{T}\mathbf{r}_t = \mathbf{r}_t, \quad \text{so that} \quad \mathbf{T}\mathbf{r} = \mathbf{T}\mathbf{r}_t + \mathbf{T}\mathbf{r}_n = \mathbf{r}_t - \mathbf{r}_n = (\mathbf{r} - \mathbf{r}_n) - \mathbf{r}_n = \mathbf{r} - 2\mathbf{r}_n = \mathbf{r} - 2(\mathbf{r} \cdot \mathbf{n})\mathbf{n}.$$

(b)  $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3} \rightarrow \mathbf{e}_1 \cdot \mathbf{n} = \mathbf{e}_2 \cdot \mathbf{n} = \mathbf{e}_3 \cdot \mathbf{n} = 1/\sqrt{3}$ .

$$\mathbf{T}\mathbf{e}_1 = \mathbf{e}_1 - 2(\mathbf{e}_1 \cdot \mathbf{n})\mathbf{n} = \mathbf{e}_1 - 2\left(\frac{1}{\sqrt{3}}\right)\left[\frac{(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)}{\sqrt{3}}\right] = (\mathbf{e}_1 - 2\mathbf{e}_2 - 2\mathbf{e}_3)/3.$$

$$\mathbf{T}\mathbf{e}_2 = \mathbf{e}_2 - 2(\mathbf{e}_2 \cdot \mathbf{n})\mathbf{n} = \mathbf{e}_2 - 2\left(\frac{1}{\sqrt{3}}\right)\left[\frac{(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)}{\sqrt{3}}\right] = (-2\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3)/3.$$

$$\mathbf{T}\mathbf{e}_3 = \mathbf{e}_3 - 2(\mathbf{e}_3 \cdot \mathbf{n})\mathbf{n} = \mathbf{e}_3 - 2\left(\frac{1}{\sqrt{3}}\right)\left[\frac{(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)}{\sqrt{3}}\right] = (-2\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3)/3.$$

$$[\mathbf{T}] = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

$$(c) [\mathbf{T}][\mathbf{a}] = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} \rightarrow \mathbf{T}\mathbf{a} = -(3\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3).$$

2.27 Knowing that the reflected vector for  $\mathbf{r}$  is given by  $\mathbf{T}\mathbf{r} = \mathbf{r} - 2(\mathbf{r} \cdot \mathbf{n})\mathbf{n}$  (see the previous problem), where  $\mathbf{T}$  is the transformation that corresponds to the reflection and  $\mathbf{n}$  is the normal to the mirror, show that in dyadic notation, the reflection tensor is given by  $\mathbf{T} = \mathbf{I} - 2\mathbf{nn}$  and find the matrix of  $\mathbf{T}$  if the normal of the mirror is given by  $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$ ,

-----  
*Ans.* From the definition of dyadic product, we have ,

$$\mathbf{T}\mathbf{r} = \mathbf{r} - 2(\mathbf{r} \cdot \mathbf{n})\mathbf{n} = \mathbf{r} - 2(\mathbf{nn})\mathbf{r} = (\mathbf{I}\mathbf{r} - 2(\mathbf{nn})\mathbf{r}) = (\mathbf{I} - 2\mathbf{nn})\mathbf{r} \rightarrow \mathbf{T} = \mathbf{I} - 2\mathbf{nn}.$$

$$\text{For } \mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3} \rightarrow [2\mathbf{nn}] = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1 \ 1 \ 1] = \frac{2}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\rightarrow [\mathbf{T}] = [\mathbf{I}] - [2\mathbf{nn}] = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

2.28 A rotation tensor  $\mathbf{R}$  is defined by the relation  $\mathbf{R}\mathbf{e}_1 = \mathbf{e}_2$ ,  $\mathbf{R}\mathbf{e}_2 = \mathbf{e}_3$ ,  $\mathbf{R}\mathbf{e}_3 = \mathbf{e}_1$  (a) Find the matrix of  $\mathbf{R}$  and verify that  $\mathbf{R}^T\mathbf{R} = \mathbf{I}$  and  $\det \mathbf{R} = 1$  and (b) find a unit vector in the direction of the axis of rotation that could have been used to effect this particular rotation.

$$\text{Ans. (a) } [\mathbf{R}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow [\mathbf{R}]^T[\mathbf{R}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \det[\mathbf{R}] = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1.$$

(b) Let the axis of rotation be  $\mathbf{n} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$ , then

$$\mathbf{R}\mathbf{n} = \mathbf{n} \rightarrow [\mathbf{R} - \mathbf{I}][\mathbf{n}] = [\mathbf{0}] \rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow -\alpha_1 + \alpha_3 = 0, \quad \alpha_1 - \alpha_2 = 0, \quad \alpha_2 - \alpha_3 = 0.$$

Thus,  $\alpha_1 = \alpha_2 = \alpha_3$ , so that a unit vector in the direction of the axis of rotation is

$$\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}.$$

2.29 A rigid body undergoes a right hand rotation of angle  $\theta$  about an axis which is in the direction of the unit vector  $\mathbf{m}$ . Let the origin of the coordinates be on the axis of rotation and  $\mathbf{r}$  be the position vector for a typical point in the body. (a) show that the rotated vector of  $\mathbf{r}$  is given by:  $\mathbf{Rr} = (1 - \cos\theta)(\mathbf{m} \cdot \mathbf{r})\mathbf{m} + \cos\theta\mathbf{r} + \sin\theta(\mathbf{m} \times \mathbf{r})$ , where  $\mathbf{R}$  is the rotation tensor. (b) Let  $\mathbf{m} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) / \sqrt{3}$ , find the matrix for  $\mathbf{R}$ .

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 Ans. (a) Let the vector  $\mathbf{r}$  be decomposed into two vectors  $\mathbf{r}_m$  and  $\mathbf{r}_p$ , where  $\mathbf{r}_m$  is in the direction of  $\mathbf{m}$  and  $\mathbf{r}_p$  is in a direction perpendicular to  $\mathbf{m}$ , that is,  $\mathbf{r} = \mathbf{r}_p + \mathbf{r}_m$ . Let  $\mathbf{p} \equiv \mathbf{r}_p / |\mathbf{r}_p|$  be the unit vector in the direction of  $\mathbf{r}_p$ , and let  $\mathbf{q} \equiv \mathbf{m} \times \mathbf{p}$ . Then,  $(\mathbf{m}, \mathbf{p}, \mathbf{q})$  forms an orthonormal set of vectors which rotates an angle of  $\theta$  about the unit vector  $\mathbf{m}$ . Thus,

$\mathbf{Rr}_m = \mathbf{r}_m$  and  $\mathbf{Rr}_p = |\mathbf{r}_p|(\cos\theta\mathbf{p} + \sin\theta\mathbf{q})$ . From  $\mathbf{r} = \mathbf{r}_p + \mathbf{r}_m$ , we have,

$$\begin{aligned}\mathbf{Rr} &= \mathbf{Rr}_p + \mathbf{Rr}_m = |\mathbf{r}_p|(\cos\theta\mathbf{p} + \sin\theta\mathbf{q}) + \mathbf{r}_m = \{\cos\theta|\mathbf{r}_p|\mathbf{p} + \sin\theta|\mathbf{r}_p|(\mathbf{m} \times \mathbf{p})\} + \mathbf{r}_m \\ &= \{\cos\theta\mathbf{r}_p + \sin\theta(\mathbf{m} \times \mathbf{r}_p)\} + \mathbf{r}_m = \{\cos\theta(\mathbf{r} - \mathbf{r}_m) + \sin\theta(\mathbf{m} \times (\mathbf{r} - \mathbf{r}_m))\} + \mathbf{r}_m \\ &= \mathbf{r} \cos\theta + \mathbf{r}_m(1 - \cos\theta) + \sin\theta\mathbf{m} \times (\mathbf{r} - \mathbf{r}_m) = \mathbf{r} \cos\theta + \mathbf{r}_m(1 - \cos\theta) + \sin\theta\mathbf{m} \times \mathbf{r}\end{aligned}$$

We note that  $\mathbf{r}_m = (\mathbf{r} \cdot \mathbf{m})\mathbf{m}$ , so that  $\mathbf{Rr} = \mathbf{r} \cos\theta + (\mathbf{r} \cdot \mathbf{m})\mathbf{m}(1 - \cos\theta) + \sin\theta\mathbf{m} \times \mathbf{r}$ .

(b) Use the result of (a), that is,  $\mathbf{Rr} = \mathbf{r} \cos\theta + (\mathbf{r} \cdot \mathbf{m})\mathbf{m}(1 - \cos\theta) + \sin\theta\mathbf{m} \times \mathbf{r}$ , we have,

$$\mathbf{Re}_1 = \mathbf{e}_1 \cos\theta + (\mathbf{e}_1 \cdot \mathbf{m})\mathbf{m}(1 - \cos\theta) + \sin\theta\mathbf{m} \times \mathbf{e}_1,$$

$$\mathbf{Re}_2 = \mathbf{e}_2 \cos\theta + (\mathbf{e}_2 \cdot \mathbf{m})\mathbf{m}(1 - \cos\theta) + \sin\theta\mathbf{m} \times \mathbf{e}_2,$$

$$\mathbf{Re}_3 = \mathbf{e}_3 \cos\theta + (\mathbf{e}_3 \cdot \mathbf{m})\mathbf{m}(1 - \cos\theta) + \sin\theta\mathbf{m} \times \mathbf{e}_3.$$

Now,  $\mathbf{m} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) / \sqrt{3}$ , therefore,  $\mathbf{m} \cdot \mathbf{e}_1 = \mathbf{m} \cdot \mathbf{e}_2 = \mathbf{m} \cdot \mathbf{e}_3 = 1 / \sqrt{3}$

$\mathbf{m} \times \mathbf{e}_1 = (1 / \sqrt{3})(-\mathbf{e}_3 + \mathbf{e}_2)$ ,  $\mathbf{m} \times \mathbf{e}_2 = (1 / \sqrt{3})(\mathbf{e}_3 - \mathbf{e}_1)$ ,  $\mathbf{m} \times \mathbf{e}_3 = (1 / \sqrt{3})(-\mathbf{e}_2 + \mathbf{e}_1)$ . Thus,

$$\begin{aligned}\mathbf{Re}_1 &= \mathbf{e}_1 \cos\theta + (\mathbf{e}_1 \cdot \mathbf{m})\mathbf{m}(1 - \cos\theta) + \sin\theta\mathbf{m} \times \mathbf{e}_1 \\ &= \mathbf{e}_1 \cos\theta + (1/3)(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)(1 - \cos\theta) + \sin\theta(1/\sqrt{3})(-\mathbf{e}_3 + \mathbf{e}_2) \\ &= (1/3)\{1 + 2\cos\theta\}\mathbf{e}_1 + \mathbf{e}_2\{(1/3)(1 - \cos\theta) + \sin\theta(1/\sqrt{3})\} + \mathbf{e}_3\{(1/3)(1 - \cos\theta) - \sin\theta(1/\sqrt{3})\}\end{aligned}$$

$$\begin{aligned}\mathbf{Re}_2 &= \mathbf{e}_2 \cos\theta + (\mathbf{e}_2 \cdot \mathbf{m})\mathbf{m}(1 - \cos\theta) + \sin\theta\mathbf{m} \times \mathbf{e}_2 \\ &= \mathbf{e}_2 \cos\theta + (1/3)(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)(1 - \cos\theta) + \sin\theta(1/\sqrt{3})(\mathbf{e}_3 - \mathbf{e}_1)\end{aligned}$$

$$= \{(1/3)(1 - \cos\theta) - (1/\sqrt{3})\sin\theta\}\mathbf{e}_1 + (1/3)(1 + 2\cos\theta)\mathbf{e}_2 + \{(1/3)(1 - \cos\theta) + \sin\theta(1/\sqrt{3})\}\mathbf{e}_3$$

$$\mathbf{Re}_3 = \mathbf{e}_3 \cos\theta + (\mathbf{e}_3 \cdot \mathbf{m})\mathbf{m}(1 - \cos\theta) + \sin\theta\mathbf{m} \times \mathbf{e}_3$$

$$= \mathbf{e}_3 \cos\theta + (1/3)(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)(1 - \cos\theta) + \sin\theta(1/\sqrt{3})(-\mathbf{e}_2 + \mathbf{e}_1)$$

$$= \{(1/3)(1 - \cos\theta) + (1/\sqrt{3})\sin\theta\}\mathbf{e}_1 + \{(1/3)(1 - \cos\theta) - \sin\theta(1/\sqrt{3})\}\mathbf{e}_2 + (1/3)(1 + 2\cos\theta)\mathbf{e}_3$$

Thus,

$$[\mathbf{T}] = \frac{1}{3} \begin{bmatrix} 1 + 2\cos\theta & (1 - \cos\theta) - \sqrt{3}\sin\theta & (1 - \cos\theta) + \sqrt{3}\sin\theta \\ (1 - \cos\theta) + \sqrt{3}\sin\theta & (1 + 2\cos\theta) & (1 - \cos\theta) - \sqrt{3}\sin\theta \\ (1 - \cos\theta) - \sqrt{3}\sin\theta & (1 - \cos\theta) + \sqrt{3}\sin\theta & (1 + 2\cos\theta) \end{bmatrix}.$$

2.30 For the rotation about an arbitrary axis  $\mathbf{m}$  by an angle  $\theta$ , (a) show that the rotation tensor is given by  $\mathbf{R} = (1 - \cos\theta)(\mathbf{m}\mathbf{m}) + \cos\theta\mathbf{I} + \sin\theta\mathbf{E}$ , where  $\mathbf{m}\mathbf{m}$  denotes that dyadic product of  $\mathbf{m}$  and  $\mathbf{E}$  is the antisymmetric tensor whose dual vector (or axial vector) is  $\mathbf{m}$ , (b) find the  $\mathbf{R}^A$ , the antisymmetric part of  $\mathbf{R}$  and (c) show that the dual vector for  $\mathbf{R}^A$  is given by  $(\sin\theta)\mathbf{m}$ . Hint,  $\mathbf{R}\mathbf{r} = (1 - \cos\theta)(\mathbf{m} \cdot \mathbf{r})\mathbf{m} + \cos\theta\mathbf{r} + \sin\theta(\mathbf{m} \times \mathbf{r})$  (see previous problem).

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*Ans.* (a) We have, from the previous problem,  $\mathbf{R}\mathbf{r} = (1 - \cos\theta)(\mathbf{m} \cdot \mathbf{r})\mathbf{m} + \cos\theta\mathbf{r} + \sin\theta(\mathbf{m} \times \mathbf{r})$ . Now, by the definition of dyadic product, we have  $(\mathbf{m} \cdot \mathbf{r})\mathbf{m} = (\mathbf{m}\mathbf{m})\mathbf{r}$ , and by the definition of dual vector we have,  $\mathbf{m} \times \mathbf{r} = \mathbf{E}\mathbf{r}$ , thus  $\mathbf{R}\mathbf{r} = (1 - \cos\theta)(\mathbf{m}\mathbf{m})\mathbf{r} + \cos\theta\mathbf{r} + \sin\theta\mathbf{E}\mathbf{r} = \{(1 - \cos\theta)(\mathbf{m}\mathbf{m}) + \cos\theta\mathbf{I} + \sin\theta\mathbf{E}\}\mathbf{r}$ , from which,  $\mathbf{R} = (1 - \cos\theta)(\mathbf{m}\mathbf{m}) + \cos\theta\mathbf{I} + \sin\theta\mathbf{E}$ .  
 (b)  $\mathbf{R}^A = (\mathbf{R} - \mathbf{R}^T) / 2 \rightarrow 2\mathbf{R}^A = \{(1 - \cos\theta)(\mathbf{m}\mathbf{m}) + \cos\theta\mathbf{I} + \sin\theta\mathbf{E}\} - \{(1 - \cos\theta)(\mathbf{m}\mathbf{m})^T + \cos\theta\mathbf{I} + \sin\theta\mathbf{E}^T\}$ . Now  $[\mathbf{m}\mathbf{m}] = [m_i m_j] = [m_j m_i] = [\mathbf{m}\mathbf{m}]^T$ , and the tensor  $\mathbf{E}$ , being antisymmetric,  $\mathbf{E} = -\mathbf{E}^T$ , therefore,  $2\mathbf{R}^A = 2\sin\theta\mathbf{E}$ , that is,  $\mathbf{R}^A = \sin\theta\mathbf{E}$ .  
 (c) dual vector of  $\mathbf{R}^A = (\sin\theta)(\text{dual vector of } \mathbf{E}) = \sin\theta\mathbf{m}$ .

2.31 (a) Given a mirror whose normal is in the direction of  $\mathbf{e}_2$ . Find the matrix of the tensor  $\mathbf{S}$  which first transforms every vector into its mirror image and then transforms them by a  $45^\circ$  right-hand rotation about the  $\mathbf{e}_1$ -axis. (b) Find the matrix of the tensor  $\mathbf{T}$  which first transforms every vector by a  $45^\circ$  right-hand rotation about the  $\mathbf{e}_1$ -axis, and then transforms them by a reflection with respect to the mirror (whose normal is  $\mathbf{e}_2$ ). (c) Consider the vector  $\mathbf{a} = (\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)$ , find the transformed vector by using the transformation  $\mathbf{S}$ . (d) For the same vector  $\mathbf{a} = (\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)$ , find the transformed vector by using the transformation  $\mathbf{T}$ .

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*Ans.* Let  $\mathbf{T}_1$  and  $\mathbf{T}_2$  correspond to the reflection and the rotation respectively. We have

$$\mathbf{T}_1\mathbf{e}_1 = \mathbf{e}_1, \quad \mathbf{T}_1\mathbf{e}_2 = -\mathbf{e}_2, \quad \mathbf{T}_1\mathbf{e}_3 = \mathbf{e}_3 \rightarrow [\mathbf{T}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{T}_2\mathbf{e}_1 = \mathbf{e}_1, \quad \mathbf{T}_2\mathbf{e}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3), \quad \mathbf{T}_2\mathbf{e}_3 = \frac{1}{\sqrt{2}}(-\mathbf{e}_2 + \mathbf{e}_3) \rightarrow [\mathbf{T}_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

$$\begin{aligned}
 \text{(a) } [\mathbf{S}] &= [\mathbf{T}_2][\mathbf{T}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}. \\
 \text{(b) } [\mathbf{T}] &= [\mathbf{T}_1][\mathbf{T}_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}. \\
 \text{(c) } [\mathbf{b}] &= [\mathbf{S}][\mathbf{a}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}. \\
 \text{(d) } [\mathbf{c}] &= [\mathbf{T}][\mathbf{a}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/\sqrt{2} \\ 5/\sqrt{2} \end{bmatrix}.
 \end{aligned}$$

2.32 Let  $\mathbf{R}$  correspond to a right-hand rotation of angle  $\theta$  about the  $x_3$ -axis (a) find the matrix of  $\mathbf{R}^2$ . (b) Show that  $\mathbf{R}^2$  corresponds to a rotation of angle  $2\theta$  about the same axis (c) Find the matrix of  $\mathbf{R}^n$  for any integer  $n$ .

-----

Ans. (a)  $[\mathbf{R}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

$$\rightarrow [\mathbf{R}^2] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta & 0 \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b)

$$[\mathbf{R}^2] = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta & 0 \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,  $\mathbf{R}^2$  corresponds to a rotation of angle  $2\theta$  about the same axis

$$\text{(c) } [\mathbf{R}^n] = \begin{bmatrix} \cos n\theta & -\sin n\theta & 0 \\ \sin n\theta & \cos n\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.33 Rigid body rotations that are small can be described by an orthogonal transformation  $\mathbf{R} = \mathbf{I} + \varepsilon \mathbf{R}^*$  where  $\varepsilon \rightarrow 0$  as the rotation angle approaches zero. Consider two successive small rotations  $\mathbf{R}_1$  and  $\mathbf{R}_2$ , show that the final result does not depend on the order of rotations.

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*Ans.*  $\mathbf{R}_2\mathbf{R}_1 = (\mathbf{I} + \varepsilon\mathbf{R}_2^*)(\mathbf{I} + \varepsilon\mathbf{R}_1^*) = \mathbf{I} + \varepsilon\mathbf{R}_2^* + \varepsilon\mathbf{R}_1^* + \varepsilon^2\mathbf{R}_2^*\mathbf{R}_1^* = \mathbf{I} + \varepsilon(\mathbf{R}_2^* + \mathbf{R}_1^*) + \varepsilon^2\mathbf{R}_2^*\mathbf{R}_1^*.$

As  $\varepsilon \rightarrow 0$ ,  $\mathbf{R}_2\mathbf{R}_1 \approx \mathbf{I} + \varepsilon(\mathbf{R}_2^* + \mathbf{R}_1^*) = \mathbf{R}_1\mathbf{R}_2.$

2.34 Let  $\mathbf{T}$  and  $\mathbf{S}$  be any two tensors. Show that (a)  $\mathbf{T}^T$  is a tensor, (b)  $\mathbf{T}^T + \mathbf{S}^T = (\mathbf{T} + \mathbf{S})^T$  and (c)  $(\mathbf{TS})^T = \mathbf{S}^T\mathbf{T}^T.$

*Ans.* Let  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  be three arbitrary vectors and  $\alpha, \beta$  be any two scalars, then

(a)  $\mathbf{a} \cdot \mathbf{T}^T(\alpha\mathbf{b} + \beta\mathbf{c}) = (\alpha\mathbf{b} + \beta\mathbf{c}) \cdot \mathbf{Ta} = \alpha\mathbf{b} \cdot \mathbf{Ta} + \beta\mathbf{c} \cdot \mathbf{Ta} = \alpha\mathbf{a} \cdot \mathbf{T}^T\mathbf{b} + \beta\mathbf{a} \cdot \mathbf{T}^T\mathbf{c}$   
 $= \mathbf{a} \cdot (\alpha\mathbf{T}^T\mathbf{b} + \beta\mathbf{T}^T\mathbf{c}) \rightarrow \mathbf{T}^T(\alpha\mathbf{b} + \beta\mathbf{c}) = (\alpha\mathbf{T}^T\mathbf{b} + \beta\mathbf{T}^T\mathbf{c}).$  Thus,  $\mathbf{T}^T$  is a linear transformation, i.e., tensor.

(b)  $\mathbf{a} \cdot (\mathbf{T} + \mathbf{S})^T\mathbf{b} = \mathbf{b} \cdot (\mathbf{T} + \mathbf{S})\mathbf{a} = \mathbf{b} \cdot \mathbf{Ta} + \mathbf{b} \cdot \mathbf{Sa} = \mathbf{a} \cdot \mathbf{T}^T\mathbf{b} + \mathbf{a} \cdot \mathbf{S}^T\mathbf{b}$   
 $= \mathbf{a} \cdot (\mathbf{T}^T + \mathbf{S}^T)\mathbf{b} \rightarrow (\mathbf{T} + \mathbf{S})^T = \mathbf{T}^T + \mathbf{S}^T.$

(c)  $\mathbf{a} \cdot (\mathbf{TS})^T\mathbf{b} = \mathbf{b} \cdot (\mathbf{TS})\mathbf{a} = \mathbf{b} \cdot \mathbf{T}(\mathbf{Sa}) = (\mathbf{Sa}) \cdot \mathbf{T}^T\mathbf{b} = \mathbf{a} \cdot \mathbf{S}^T\mathbf{T}^T\mathbf{b} \rightarrow (\mathbf{TS})^T = \mathbf{S}^T\mathbf{T}^T.$

2.35 For arbitrary tensors  $\mathbf{T}$  and  $\mathbf{S}$ , without relying on the component form, prove that (a)  $(\mathbf{T}^{-1})^T = (\mathbf{T}^T)^{-1}$  and (b)  $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}$

*Ans.* (a)  $\mathbf{TT}^{-1} = \mathbf{I} \rightarrow (\mathbf{TT}^{-1})^T = \mathbf{I} \rightarrow (\mathbf{T}^{-1})^T\mathbf{T}^T = \mathbf{I} \rightarrow (\mathbf{T}^{-1})^T = (\mathbf{T}^T)^{-1}.$

(b)  $(\mathbf{TS})(\mathbf{S}^{-1}\mathbf{T}^{-1}) = \mathbf{T}(\mathbf{SS}^{-1})\mathbf{T}^{-1} = \mathbf{TT}^{-1} = \mathbf{I}$ , thus,  $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}.$

2.36 Let  $\{\mathbf{e}_i\}$  and  $\{\mathbf{e}'_i\}$  be two Rectangular Cartesian base vectors. (a) Show that if  $\mathbf{e}'_i = Q_{mi}\mathbf{e}_m$ , then  $\mathbf{e}_i = Q_{im}\mathbf{e}'_m$  and (b) verify  $Q_{mi}Q_{mj} = \delta_{ij} = Q_{im}Q_{jm}.$

*Ans.* (a)  $\mathbf{e}'_i = Q_{mi}\mathbf{e}_m \rightarrow \mathbf{e}'_i \cdot \mathbf{e}_j = Q_{mi}\mathbf{e}_m \cdot \mathbf{e}_j = Q_{mi}\delta_{mj} = Q_{ji} \rightarrow \mathbf{e}_j = Q_{jm}\mathbf{e}'_m \rightarrow \mathbf{e}_i = Q_{im}\mathbf{e}'_m.$

(b) We have,  $\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ , thus,

$\delta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = Q_{mi}\mathbf{e}_m \cdot Q_{nj}\mathbf{e}_n = Q_{mi}Q_{nj}\mathbf{e}_m \cdot \mathbf{e}_n = Q_{mi}Q_{nj}\delta_{mn} = Q_{mi}Q_{mj}.$  And

$\delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = Q_{im}\mathbf{e}'_m \cdot Q_{jn}\mathbf{e}'_n = Q_{im}Q_{jn}\mathbf{e}'_m \cdot \mathbf{e}'_n = Q_{im}Q_{jn}\delta_{mn} = Q_{im}Q_{jm}.$

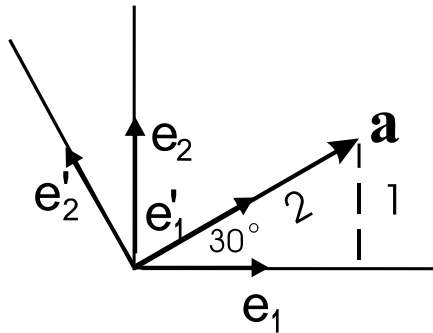
2.37 The basis  $\{\mathbf{e}'_i\}$  is obtained by a  $30^\circ$  counterclockwise rotation of the  $\{\mathbf{e}_i\}$  basis about the  $\mathbf{e}_3$  axis. (a) Find the transformation matrix  $[\mathbf{Q}]$  relating the two sets of basis, (b) by using the vector transformation law, find the components of  $\mathbf{a} = \sqrt{3}\mathbf{e}_1 + \mathbf{e}_2$  in the primed basis, i.e., find  $a'_i$  and (c) do part (b) geometrically.

*Ans.* (a)  $\mathbf{e}'_1 = \cos 30^\circ\mathbf{e}_1 + \sin 30^\circ\mathbf{e}_2$ ,  $\mathbf{e}'_2 = -\sin 30^\circ\mathbf{e}_1 + \cos 30^\circ\mathbf{e}_2$ ,  $\mathbf{e}'_3 = \mathbf{e}_3$ . Thus,

$$[\mathbf{Q}]_{\mathbf{e}_i} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$(b) [\mathbf{a}]_{\mathbf{e}'_i} = [\mathbf{Q}]^T [\mathbf{a}]_{\mathbf{e}_i} \rightarrow \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \rightarrow \mathbf{a} = 2\mathbf{e}'_1$$

(c) Clearly  $\mathbf{a} = \sqrt{3}\mathbf{e}_1 + \mathbf{e}_2$  is a vector in the same direction as  $\mathbf{e}'_1$  and has a length of 2. See figure below



2.38 Do the previous problem with the  $\{\mathbf{e}'_i\}$  basis obtained by a  $30^\circ$  clockwise rotation of the  $\{\mathbf{e}_i\}$  basis about the  $\mathbf{e}_3$  axis.

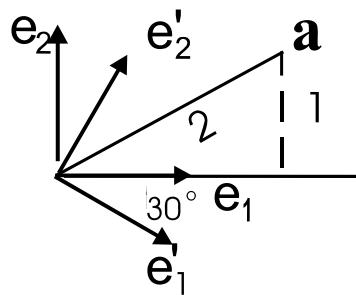
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 Ans.

(a)  $\mathbf{e}'_1 = \cos 30^\circ \mathbf{e}_1 - \sin 30^\circ \mathbf{e}_2$ ,  $\mathbf{e}'_2 = \sin 30^\circ \mathbf{e}_1 + \cos 30^\circ \mathbf{e}_2$ ,  $\mathbf{e}'_3 = \mathbf{e}_3$ . Thus,

$$[\mathbf{Q}]_{\mathbf{e}_i} = \begin{bmatrix} \cos 30^\circ & \sin 30^\circ & 0 \\ -\sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(b) [\mathbf{a}]_{\mathbf{e}'_i} = [\mathbf{Q}]^T [\mathbf{a}]_{\mathbf{e}_i} \rightarrow \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \\ 0 \end{bmatrix} \rightarrow \mathbf{a} = \mathbf{e}'_1 + \sqrt{3}\mathbf{e}'_2$$

(c) See figure below



2.39 The matrix of a tensor  $\mathbf{T}$  with respect to the basis  $\{\mathbf{e}_i\}$  is

$$[\mathbf{T}] = \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

Find  $T'_{11}, T'_{12}$  and  $T'_{31}$  with respect to a right-handed basis  $\{\mathbf{e}'_i\}$  where  $\mathbf{e}'_1$  is in the direction of  $-\mathbf{e}_2 + 2\mathbf{e}_3$  and  $\mathbf{e}'_2$  is in the direction of  $\mathbf{e}_1$ .

-----  
 Ans. The basis  $\{\mathbf{e}'_i\}$  is given by:

$$\mathbf{e}'_1 = (-\mathbf{e}_2 + 2\mathbf{e}_3)/\sqrt{5}, \quad \mathbf{e}'_2 = \mathbf{e}_1, \quad \mathbf{e}'_3 = \mathbf{e}'_1 \times \mathbf{e}'_2 = (2\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{5}.$$

$$T'_{11} = \mathbf{e}'_1 \cdot \mathbf{T}\mathbf{e}'_1 = \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = 4/5.$$

$$T'_{12} = \mathbf{e}'_1 \cdot \mathbf{T}\mathbf{e}'_2 = \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -15/\sqrt{5}.$$

$$T'_{31} = \mathbf{e}'_3 \cdot \mathbf{T}\mathbf{e}'_1 = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = 2/5.$$

2.40 (a) For the tensor of the previous problem, find  $[T'_{ij}]$ , i.e.,  $[\mathbf{T}]_{\mathbf{e}'_i}$  if  $\{\mathbf{e}'_i\}$  is obtained by a  $90^\circ$  right hand rotation about the  $\mathbf{e}_3$  axis and (b) obtain  $T'_{ii}$  and the determinant  $|T'_{ij}|$  and compare them with  $T_{ii}$  and  $|T_{ij}|$ .

-----  
 Ans. (a)  $\mathbf{e}'_1 = \mathbf{e}_2, \quad \mathbf{e}'_2 = -\mathbf{e}_1, \quad \mathbf{e}'_3 = \mathbf{e}_3 \rightarrow [\mathbf{Q}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

$$[T'_{ij}] = [\mathbf{T}]' = [\mathbf{Q}]^T [\mathbf{T}] [\mathbf{Q}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 0 \\ -5 & 1 & 5 \\ 0 & 5 & 1 \end{bmatrix}$$

(b)  $T'_{ii} = T'_{11} + T'_{22} + T'_{33} = 0 + 1 + 1 = 2, \quad |T'_{ij}| = -25.$

$$T_{ii} = T_{11} + T_{22} + T_{33} = 1 + 0 + 1 = 2, \quad |T_{ij}| = -25.$$

2.41 The dot product of two vectors  $\mathbf{a} = a_i \mathbf{e}_i$  and  $\mathbf{b} = b_i \mathbf{e}_i$  is equal to  $a_i b_i$ . Show that the dot product is a scalar invariant with respect to orthogonal transformations of coordinates.

-----  
 Ans. From  $a'_i = Q_{mi} a_m$  and  $b'_i = Q_{ni} b_n$ , we have,

$$a'_i b'_i = Q_{mi} a_m Q_{ni} b_n = Q_{mi} Q_{ni} a_m b_n = \delta_{mn} a_m b_n = a_m b_m = a_i b_i.$$

2.42 If  $T_{ij}$  are the components of a tensor (a) show that  $T_{ij}T_{ij}$  is a scalar invariant with respect to orthogonal transformations of coordinates, (b) evaluate  $T_{ij}T_{ij}$  with respect to the basis  $\{e_i\}$  for

$$[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{e_i}, \text{ (c) find } [\mathbf{T}'] \text{, if } \mathbf{e}'_i = \mathbf{Q}\mathbf{e}_i \text{, where } [\mathbf{Q}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{e_i} \text{ and}$$

(d) verify for the above  $[\mathbf{T}]$  and  $[\mathbf{T}']$  that  $T'_{ij}T'_{ij} = T_{ij}T_{ij}$ .

-----  
 Ans. (a) Since  $T_{ij}$  are the components of a tensor,  $T'_{ij} = Q_{mi}Q_{nj}T_{mn}$ . Thus,

$$T'_{ij}T'_{ij} = Q_{mi}Q_{nj}T_{mn}(Q_{pi}Q_{qj}T_{pq}) = (Q_{mi}Q_{pi})(Q_{nj}Q_{qj})T_{mn}T_{pq} = \delta_{mp}\delta_{nq}T_{mn}T_{pq} = T_{mn}T_{mn}$$

(b)  $T_{ij}T_{ij} = T_{11}^2 + T_{12}^2 + T_{13}^2 + T_{21}^2 + T_{22}^2 + T_{23}^2 + T_{31}^2 + T_{32}^2 + T_{33}^2 = 1 + 1 + 4 + 25 + 1 + 4 + 9 = 45$ .

$$(c) [\mathbf{T}'] = [\mathbf{Q}]^T [\mathbf{T}] [\mathbf{Q}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 5 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 1 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(d)  $T'_{ij}T'_{ij} = 4 + 25 + 1 + 4 + 9 + 1 + 1 = 45$ .

-----  
 2.43 Let  $[\mathbf{T}]$  and  $[\mathbf{T}']$  be two matrices of the same tensor  $\mathbf{T}$ , show that  $\det[\mathbf{T}] = \det[\mathbf{T}']$ .

-----  
 Ans.  $[\mathbf{T}'] = [\mathbf{Q}]^T [\mathbf{T}] [\mathbf{Q}] \rightarrow \det[\mathbf{T}'] = \det[\mathbf{Q}]^T \det[\mathbf{Q}] \det[\mathbf{T}] = (\pm 1)(\pm 1) \det[\mathbf{T}] = \det[\mathbf{T}]$ .

-----  
 2.44 (a) If the components of a third order tensor are  $R_{ijk}$ , show that  $R_{iik}$  are components of a vector, (b) if the components of a fourth order tensor are  $R_{ijkl}$ , show that  $R_{iikl}$  are components of a second order tensor and (c) what are components of  $R_{iik\dots}$ , if  $R_{ijk\dots}$  are components of a tensor of  $n^{th}$  order?

-----  
 Ans. (a) Since  $R_{ijk}$  are components of a third order tensor, therefore,

$$R'_{ijk} = Q_{mi}Q_{nj}Q_{pk}R_{mnp} \rightarrow R'_{iik} = Q_{mi}Q_{ni}Q_{pk}R_{mnp} = \delta_{mn}Q_{pk}R_{mnp} = Q_{pk}R_{nnp}, \text{ therefore, } R_{iik} \text{ are components of a vector.}$$

(b) Consider a  $4^{th}$  order tensor  $R_{ijkl}$ , we have,

$$R'_{ijkl} = Q_{mi}Q_{nj}Q_{pk}Q_{ql}R_{mnpq} \rightarrow R'_{iikl} = Q_{mi}Q_{ni}Q_{pk}Q_{ql}R_{mnpq} = \delta_{mn}Q_{pk}Q_{ql}R_{mnpq} = Q_{pk}Q_{ql}R_{nnpq}, \text{ therefore, } R_{iikl} \text{ are components of a second order tensor.}$$

(c)  $R_{iik\dots}$  are components of a tensor of the  $(n-2)^{th}$  order.

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 2.45 The components of an arbitrary vector  $\mathbf{a}$  and an arbitrary second tensor  $\mathbf{T}$  are related by a triply subscripted quantity  $R_{ijk}$  in the manner  $a_i = R_{ijk}T_{jk}$  for any rectangular Cartesian basis  $\{e_i\}$ . Prove that  $R_{ijk}$  are the components of a third-order tensor.

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*Ans.* Since  $a_i = R_{ijk}T_{jk}$  is true for any basis, therefore,  $a'_i = R'_{ijk}T'_{jk}$ ; Since  $\mathbf{a}$  is a vector, therefore,  $a'_i = Q_{mi}a_m$  and since  $\mathbf{T}$  is a second order tensor, therefore,  $T'_{ij} = Q_{mi}Q_{nj}T_{mn}$ . Thus,  $a'_i = Q_{mi}a_m \rightarrow R'_{ijk}T'_{jk} = Q_{mi}(R_{mjk}T_{jk})$ . Multiply the last equation with  $Q_{si}$  and noting that  $Q_{si}Q_{mi} = \delta_{sm}$ , we have,  $Q_{si}R'_{ijk}T'_{jk} = Q_{si}Q_{mi}(R_{mjk}T_{jk}) \rightarrow Q_{si}R'_{ijk}T'_{jk} = \delta_{sm}R_{mjk}T_{jk} \rightarrow Q_{si}R'_{ijk}T'_{jk} = R_{sjk}T_{jk} \rightarrow Q_{si}R'_{ijk}Q_{mj}Q_{nk}T_{mn} = R_{sjk}T_{jk} \rightarrow Q_{si}R'_{ijk}Q_{mj}Q_{nk}T_{mn} = R_{smn}T_{mn}$ . Thus,  $(R_{smn} - Q_{si}Q_{mj}Q_{nk}R'_{ijk})T_{mn} = 0$ . Since this last equation is to be true for all  $T_{mn}$ , therefore,  $R_{smn} = Q_{si}Q_{mj}Q_{nk}R'_{ijk}$ , which is the transformation law for components of a third order tensor.

2.46 For any vector  $\mathbf{a}$  and any tensor  $\mathbf{T}$ , show that (a)  $\mathbf{a} \cdot \mathbf{T}^A \mathbf{a} = 0$  and (b)  $\mathbf{a} \cdot \mathbf{T} \mathbf{a} = \mathbf{a} \cdot \mathbf{T}^S \mathbf{a}$ , where  $\mathbf{T}^A$  and  $\mathbf{T}^S$  are antisymmetric and symmetric part of  $\mathbf{T}$  respectively.

*Ans.* (a)  $\mathbf{T}^A$  is antisymmetric, therefore,  $(\mathbf{T}^A)^T = -\mathbf{T}^A$ , thus,  $\mathbf{a} \cdot \mathbf{T}^A \mathbf{a} = \mathbf{a} \cdot (\mathbf{T}^A)^T \mathbf{a} = -\mathbf{a} \cdot \mathbf{T}^A \mathbf{a} \rightarrow 2\mathbf{a} \cdot \mathbf{T}^A \mathbf{a} = 0 \rightarrow \mathbf{a} \cdot \mathbf{T}^A \mathbf{a} = 0$ .  
 (b) Since  $\mathbf{T} = \mathbf{T}^S + \mathbf{T}^A$ , therefore,  $\mathbf{a} \cdot \mathbf{T} \mathbf{a} = \mathbf{a} \cdot (\mathbf{T}^S + \mathbf{T}^A) \mathbf{a} = \mathbf{a} \cdot \mathbf{T}^S \mathbf{a} + \mathbf{a} \cdot \mathbf{T}^A \mathbf{a} = \mathbf{a} \cdot \mathbf{T}^S \mathbf{a}$ .

2.47 Any tensor can be decomposed into a symmetric part and an antisymmetric part, that is  $\mathbf{T} = \mathbf{T}^S + \mathbf{T}^A$ . Prove that the decomposition is unique. (Hint, assume that it is not true and show contradiction).

*Ans.* Suppose that the decomposition is not unique, then, we have,  $\mathbf{T} = \mathbf{T}^S + \mathbf{T}^A = \mathbf{S}^S + \mathbf{S}^A \rightarrow (\mathbf{T}^S - \mathbf{S}^S) + (\mathbf{T}^A - \mathbf{S}^A) = \mathbf{0}$ . Let  $\mathbf{a}$  be any arbitrary vector, we have,  $\mathbf{a} \cdot (\mathbf{T}^S - \mathbf{S}^S) \mathbf{a} + \mathbf{a} \cdot (\mathbf{T}^A - \mathbf{S}^A) \mathbf{a} = 0 \rightarrow \mathbf{a} \cdot \mathbf{T}^S \mathbf{a} - \mathbf{a} \cdot \mathbf{S}^S \mathbf{a} + \mathbf{a} \cdot \mathbf{T}^A \mathbf{a} - \mathbf{a} \cdot \mathbf{S}^A \mathbf{a} = 0$ . But  $\mathbf{a} \cdot \mathbf{T}^A \mathbf{a} = \mathbf{a} \cdot \mathbf{S}^A \mathbf{a} = 0$  (see the previous problem). Therefore,  $\mathbf{a} \cdot \mathbf{T}^S \mathbf{a} - \mathbf{a} \cdot \mathbf{S}^S \mathbf{a} = 0 \rightarrow \mathbf{a} \cdot (\mathbf{T}^S - \mathbf{S}^S) \mathbf{a} = 0 \rightarrow \mathbf{T}^S - \mathbf{S}^S = \mathbf{0} \rightarrow \mathbf{T}^S = \mathbf{S}^S$ . It also follows from  $(\mathbf{T}^S - \mathbf{S}^S) + (\mathbf{T}^A - \mathbf{S}^A) = \mathbf{0}$  that  $\mathbf{T}^A = \mathbf{S}^A$ . Thus, the decomposition is unique.

2.48 Given that a tensor  $\mathbf{T}$  has the matrix  $[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ , (a) find the symmetric part and the anti-symmetric part of  $\mathbf{T}$  and (b) find the dual vector (or axial vector) of the anti-symmetric part of  $\mathbf{T}$ .

*Ans.* (a)  $[\mathbf{T}^S] = \frac{1}{2}([\mathbf{T}] + [\mathbf{T}]^T) = \frac{1}{2} \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \right\} = \frac{1}{2} \begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$ .

$$[\mathbf{T}^A] = \frac{1}{2} \{ [\mathbf{T}] - [\mathbf{T}]^T \} = \frac{1}{2} \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \right\} = \frac{1}{2} \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 18 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

$$(b) \mathbf{t}^A = -(T_{23}^A \mathbf{e}_1 + T_{31}^A \mathbf{e}_2 + T_{12}^A \mathbf{e}_3) = -(-1\mathbf{e}_1 + 2\mathbf{e}_2 - 1\mathbf{e}_3) = \mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3.$$

2.49 Prove that the only possible real eigenvalues of an orthogonal tensor  $\mathbf{Q}$  are  $\lambda = \pm 1$ . Explain the direction of the eigenvectors corresponding to them for a proper orthogonal (rotation) tensor and for an improper orthogonal (reflection) tensor.

*Ans.* Since  $\mathbf{Q}$  is orthogonal, therefore, for any vector  $\mathbf{n}$ , we have,  $\mathbf{Q}\mathbf{n} \cdot \mathbf{Q}\mathbf{n} = \mathbf{n} \cdot \mathbf{n}$ . Let  $\mathbf{n}$  be an eigenvector, then  $\mathbf{Q}\mathbf{n} = \lambda\mathbf{n}$ , so that  $\mathbf{Q}\mathbf{n} \cdot \mathbf{Q}\mathbf{n} = \mathbf{n} \cdot \mathbf{n} \rightarrow$

$$\lambda^2 (\mathbf{n} \cdot \mathbf{n}) = (\mathbf{n} \cdot \mathbf{n}) \rightarrow (\lambda^2 - 1)(\mathbf{n} \cdot \mathbf{n}) = 0 \rightarrow \lambda^2 - 1 = 0 \rightarrow \lambda = \pm 1.$$

The eigenvalue  $\lambda = 1$  ( $\mathbf{Q}\mathbf{n} = \mathbf{n}$ ) corresponds to an eigenvector parallel to the axis of rotation for a proper orthogonal tensor (rotation tensor); Or, it corresponds to an eigenvector parallel to the plane of reflection for an improper orthogonal tensor (reflection tensor). The eigenvalue  $\lambda = -1$ ,

( $\mathbf{Q}\mathbf{n} = -\mathbf{n}$ ) corresponds to an eigenvector perpendicular to the axis of rotation for an  $180^\circ$  rotation; or, it corresponds to an eigenvector perpendicular to the plane of reflection.

2.50 Given the improper orthogonal tensor  $[\mathbf{Q}] = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$ . (a) Verify that  $\det[\mathbf{Q}] = -1$ .

(b) Verify that the eigenvalues are  $\lambda = 1$  and  $-1$  (c) Find the normal to the plane of reflection (i.e., eigenvectors corresponding to  $\lambda = -1$ ) and (d) find the eigenvectors corresponding  $\lambda = 1$  (vectors parallel to the plane of reflection).

*Ans.* (a)  $\det[\mathbf{Q}] = (1/3)^3 (1 - 8 - 8 - 4 - 4 - 4) = (-27)/27 = -1$ .

(b)  $I_1 = 3/3 = 1$ ,  $I_2 = (1/3)^2 \{ (1-4) + (1-4) + (1-4) \} = -1$ ,  $I_3 = -1 \rightarrow$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0 \rightarrow (\lambda - 1)(\lambda^2 - 1) = 0 \rightarrow \lambda = 1, 1, -1$$

(c) For  $\lambda = -1$ ,

$$\left(\frac{1}{3} + 1\right)\alpha_1 - \frac{2}{3}\alpha_2 - \frac{2}{3}\alpha_3 = 0, \quad -\frac{2}{3}\alpha_1 + \left(\frac{1}{3} + 1\right)\alpha_2 - \frac{2}{3}\alpha_3 = 0, \quad -\frac{2}{3}\alpha_1 - \frac{2}{3}\alpha_2 + \left(\frac{1}{3} + 1\right)\alpha_3 = 0. \text{ That}$$

is,  $2\alpha_1 - \alpha_2 - \alpha_3 = 0$ ,  $-\alpha_1 + 2\alpha_2 - \alpha_3 = 0$ ,  $-\alpha_1 - \alpha_2 + 2\alpha_3 = 0$ , thus,  $\alpha_1 = \alpha_2 = \alpha_3$ , therefore,

$\mathbf{n} = \pm(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$ , this is the normal to the plane of reflection.

(d) For  $\lambda = 1$ ,

$$\left(\frac{1}{3} - 1\right)\alpha_1 - \frac{2}{3}\alpha_2 - \frac{2}{3}\alpha_3 = 0, \quad -\frac{2}{3}\alpha_1 + \left(\frac{1}{3} - 1\right)\alpha_2 - \frac{2}{3}\alpha_3 = 0, \quad -\frac{2}{3}\alpha_1 - \frac{2}{3}\alpha_2 + \left(\frac{1}{3} - 1\right)\alpha_3 = 0$$

All three equations lead to  $\alpha_1 + \alpha_2 + \alpha_3 = 0 \rightarrow \alpha_3 = -\alpha_1 - \alpha_2$ . Thus,

$$\mathbf{n} = \frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}} [\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 - (\alpha_1 + \alpha_2) \mathbf{e}_3], \text{ e.g., } \mathbf{n} = \frac{1}{\sqrt{6}} (\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3) \text{ etc. these vectors are all}$$

perpendicular to  $\mathbf{n} = \pm(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$  and thus parallel to the plane of reflection.

2.51 Given that tensors  $\mathbf{R}$  and  $\mathbf{S}$  have the same eigenvector  $\mathbf{n}$  and corresponding eigenvalue  $r_1$  and  $s_1$  respectively. Find an eigenvalue and the corresponding eigenvector for the tensor  $\mathbf{T} = \mathbf{RS}$ .

*Ans.* We have,  $\mathbf{Rn} = r_1\mathbf{n}$  and  $\mathbf{Sn} = s_1\mathbf{n}$ , thus,  $\mathbf{Tn} = \mathbf{RSn} = \mathbf{R}s_1\mathbf{n} = s_1\mathbf{Rn} = r_1s_1\mathbf{n}$ . Thus, an eigenvalue for  $\mathbf{T} = \mathbf{RS}$  is  $r_1s_1$  with eigenvector  $\mathbf{n}$ .

2.52 Show that if  $\mathbf{n}$  is a real eigenvector of an antisymmetric tensor  $\mathbf{T}$ , then the corresponding eigenvalue vanishes.

*Ans.*  $\mathbf{Tn} = \lambda\mathbf{n} \rightarrow \mathbf{n} \cdot \mathbf{Tn} = \lambda(\mathbf{n} \cdot \mathbf{n})$ . Now, from the definition of transpose, we have  $\mathbf{n} \cdot \mathbf{Tn} = \mathbf{n} \cdot \mathbf{T}^T\mathbf{n}$ . But, since  $\mathbf{T}$  is antisymmetric, i.e.,  $\mathbf{T}^T = -\mathbf{T}$ , therefore,  $\mathbf{n} \cdot \mathbf{T}^T\mathbf{n} = -\mathbf{n} \cdot \mathbf{Tn}$ . Thus,  $\mathbf{n} \cdot \mathbf{Tn} = -\mathbf{n} \cdot \mathbf{Tn} \rightarrow 2\mathbf{n} \cdot \mathbf{Tn} = 0 \rightarrow \mathbf{n} \cdot \mathbf{Tn} = 0$ . Thus,  $\lambda(\mathbf{n} \cdot \mathbf{n}) = 0 \rightarrow \lambda = 0$ .

2.53 (a) Show that  $\mathbf{a}$  is an eigenvector for the dyadic product  $\mathbf{ab}$  of vectors  $\mathbf{a}$  and  $\mathbf{b}$  with eigenvalue  $\mathbf{a} \cdot \mathbf{b}$ , (b) find the first principal scalar invariant of the dyadic product  $\mathbf{ab}$  and (c) show that the second and the third principal scalar invariants of the dyadic product  $\mathbf{ab}$  vanish, and that zero is a double eigenvalue of  $\mathbf{ab}$ .

*Ans.* (a) From the definition of dyadic product, we have,  $(\mathbf{ab})\mathbf{a} = \mathbf{a}(\mathbf{b} \cdot \mathbf{a})$ , thus  $\mathbf{a}$  is an eigenvector for the dyadic product  $\mathbf{ab}$  with eigenvalue  $\mathbf{a} \cdot \mathbf{b}$ .

(b) Let  $\mathbf{T} \equiv \mathbf{ab}$ , then  $T_{ij} = a_i b_j$  and the first scalar invariant of  $\mathbf{ab}$  is  $T_{ii} = a_i b_i = \mathbf{a} \cdot \mathbf{b}$ .

$$(c) I_2 = \begin{vmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{vmatrix} + \begin{vmatrix} a_2 b_2 & a_2 b_3 \\ a_3 b_2 & a_3 b_3 \end{vmatrix} + \begin{vmatrix} a_1 b_1 & a_1 b_3 \\ a_3 b_1 & a_3 b_3 \end{vmatrix} = 0 + 0 + 0 = 0.$$

$$I_3 = \begin{vmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{vmatrix} = a_1 a_2 a_3 \begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

Thus, the characteristic equation is

$$\lambda^3 - I_1 \lambda^2 = 0 \rightarrow (\lambda - I_1) \lambda^2 = 0 \rightarrow \lambda_1 = I_1, \lambda_2 = \lambda_3 = 0.$$

2.54 For any rotation tensor, a set of basis  $\{\mathbf{e}'_i\}$  may be chosen with  $\mathbf{e}'_3$  along the axis of rotation so that  $\mathbf{R}\mathbf{e}'_1 = \cos\theta\mathbf{e}'_1 + \sin\theta\mathbf{e}'_2$ ,  $\mathbf{R}\mathbf{e}'_2 = -\sin\theta\mathbf{e}'_1 + \cos\theta\mathbf{e}'_2$ ,  $\mathbf{R}\mathbf{e}'_3 = \mathbf{e}'_3$ , where  $\theta$  is the angle of right hand rotation. (a) Find the antisymmetric part of  $\mathbf{R}$  with respect to the basis  $\{\mathbf{e}'_i\}$ , i.e., find  $[\mathbf{R}^A]_{\mathbf{e}'_i}$ .

(b) Show that the dual vector of  $\mathbf{R}^A$  is given by  $\mathbf{t}^A = \sin\theta\mathbf{e}'_3$  and (c) show that the first scalar invariant of  $\mathbf{R}$  is given by  $1 + 2\cos\theta$ . That is, for any given rotation tensor  $\mathbf{R}$ , its axis of rotation and the angle of rotation can be obtained from the dual vector of  $\mathbf{R}^A$  and the first scalar invariant of  $\mathbf{R}$ .

*Ans.* (a) From  $\mathbf{R}\mathbf{e}'_1 = \cos\theta\mathbf{e}'_1 + \sin\theta\mathbf{e}'_2$ ,  $\mathbf{R}\mathbf{e}'_2 = -\sin\theta\mathbf{e}'_1 + \cos\theta\mathbf{e}'_2$ ,  $\mathbf{R}\mathbf{e}'_3 = \mathbf{e}'_3$ , we have,

$$[\mathbf{R}]_{\mathbf{e}'_i} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}'_i} \rightarrow [\mathbf{R}^A]_{\mathbf{e}'_i} = \begin{bmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{e}'_i}$$

(b) the dual vector (or axial vector) of  $\mathbf{R}^A$  is given by

$$\mathbf{t}^A = -(T'_{23}\mathbf{e}'_1 + T'_{31}\mathbf{e}'_2 + T'_{12}\mathbf{e}'_3) = -(0\mathbf{e}'_1 + 0\mathbf{e}'_2 - \sin\theta\mathbf{e}'_3) = \sin\theta\mathbf{e}'_3.$$

(c) The first scalar invariant of  $\mathbf{R}$  is  $I_1 = \cos\theta + \cos\theta + 1 = 1 + 2\cos\theta$ .

2.55 The rotation of a rigid body is described by  $\mathbf{R}\mathbf{e}_1 = \mathbf{e}_2$ ,  $\mathbf{R}\mathbf{e}_2 = \mathbf{e}_3$ ,  $\mathbf{R}\mathbf{e}_3 = \mathbf{e}_1$ . Find the axis of rotation and the angle of rotation. Use the result of the previous problem.

*Ans* From the result of the previous problem, we have, the dual vector of  $\mathbf{R}^A$  is given by  $\mathbf{t}^A = \sin\theta\mathbf{e}'_3$ , where  $\mathbf{e}'_3$  is in the direction of axis of rotation and  $\theta$  is the angle of rotation. Thus, we can obtain the direction of axis of rotation and the angle of rotation  $\theta$  by obtaining the dual vector of  $\mathbf{R}^A$ . From  $\mathbf{R}\mathbf{e}_1 = \mathbf{e}_2$ ,  $\mathbf{R}\mathbf{e}_2 = \mathbf{e}_3$ ,  $\mathbf{R}\mathbf{e}_3 = \mathbf{e}_1$ , we have,

$$[\mathbf{R}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow [\mathbf{R}^A] = \frac{1}{2} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \rightarrow \mathbf{t}^A = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3). \text{ Thus,}$$

$$\mathbf{t}^A = \frac{\sqrt{3}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)}{2\sqrt{3}} = \frac{\sqrt{3}}{2}\mathbf{e}'_3, \text{ where } \mathbf{e}'_3 = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \text{ is in the direction of the axis of}$$

rotation and the angle of rotation is given by  $\sin\theta = \sqrt{3}/2$ , which gives  $\theta = 60^\circ$  or  $120^\circ$ . On the other hand, the first scalar invariant of  $\mathbf{R}$  is 0. Thus, from the result in (c) of the previous problem, we have,  $I_1 = 1 + 2\cos\theta = 0$ , so that  $\cos\theta = -1/2$  which gives  $\theta = 120^\circ$  or  $240^\circ$ . We therefore conclude that  $\theta = 120^\circ$ .

2.56 Given the tensor  $[\mathbf{Q}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . (a) Show that the given tensor is a rotation tensor. (b)

Verify that the eigenvalues are  $\lambda = 1$  and  $-1$ . (c) Find the direction for the axis of rotation (i.e., eigenvectors corresponding to  $\lambda = 1$ ). (d) Find the eigenvectors corresponding  $\lambda = -1$  and (e) obtain the angle of rotation using the formula  $I_1 = 1 + 2\cos\theta$  (see Prob. 2.54), where  $I_1$  is the first scalar invariant of the rotation tensor.

*Ans.* (a)  $\det[\mathbf{Q}] = +1$ , and  $[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{I}]$  therefore it is a rotation tensor.

(b) The principal scalar invariants are:  $I_1 = -1$ ,  $I_2 = -1$ ,  $I_3 = 1 \rightarrow$  characteristic equation is

$$\lambda^3 + \lambda^2 - \lambda - 1 = (\lambda + 1)(\lambda^2 - 1) = 0 \rightarrow \text{the eigenvalues are: } \lambda = -1, 1, 1.$$

(c) For  $\lambda = 1$ , clearly, the eigenvector are:  $\mathbf{n} = \pm\mathbf{e}_3$ , which gives the axis of rotation.

(d) For  $\lambda = -1$ , with eigenvector  $\mathbf{n} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$ , we have

$$0\alpha_1 = 0, \quad 0\alpha_2 = 0, \quad 2\alpha_3 = 0. \text{ Thus, } \alpha_1 = \text{arbitrary}, \alpha_2 = \text{arbitrary}, \alpha_3 = 0. \text{ The eigenvectors are:}$$

$\mathbf{n} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$ ,  $\alpha_1^2 + \alpha_2^2 = 1$ . That is, all vectors perpendicular to the axis of rotation are eigenvectors.

(e) The first scalar invariant of  $\mathbf{Q}$  is  $I_1 = -1$ . Thus,  $1 + 2\cos\theta = -1 \rightarrow \cos\theta = -1 \rightarrow \theta = \pi$ . (We note that for this problem, the antisymmetric part of  $\mathbf{Q} = \mathbf{0}$ , so that  $\mathbf{t}^A = \mathbf{0} = \sin\theta\mathbf{n}$ , of which  $\theta = \pi$  is a solution).

2.57 Let  $\mathbf{F}$  be an arbitrary tensor. (a) Show that  $\mathbf{F}^T\mathbf{F}$  and  $\mathbf{F}\mathbf{F}^T$  are both symmetric tensors. (b) If  $\mathbf{F} = \mathbf{Q}\mathbf{U} = \mathbf{V}\mathbf{Q}$ , where  $\mathbf{Q}$  is orthogonal and  $\mathbf{U}$  and  $\mathbf{V}$  are symmetric, show that  $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$  and  $\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T$  (c) If  $\lambda$  and  $\mathbf{n}$  are eigenvalue and the corresponding eigenvector for  $\mathbf{U}$ , find the eigenvalue and eigenvector for  $\mathbf{V}$ . [note corrections for text]

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 Ans. (a)  $(\mathbf{F}^T\mathbf{F})^T = \mathbf{F}^T(\mathbf{F}^T)^T = \mathbf{F}^T\mathbf{F}$ , thus  $\mathbf{F}^T\mathbf{F}$  is symmetric. Also  $(\mathbf{F}\mathbf{F}^T)^T = (\mathbf{F}^T)^T\mathbf{F}^T = \mathbf{F}\mathbf{F}^T$ , therefore,  $\mathbf{F}\mathbf{F}^T$  is also symmetric.

(b)  $\mathbf{F} = \mathbf{Q}\mathbf{U} \rightarrow \mathbf{F}^T = \mathbf{U}^T\mathbf{Q}^T \rightarrow \mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{Q}^T\mathbf{Q}\mathbf{U} = \mathbf{U}^T\mathbf{U} \rightarrow \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ .

$\mathbf{F} = \mathbf{V}\mathbf{Q} \rightarrow \mathbf{F}^T = \mathbf{Q}^T\mathbf{V}^T \rightarrow \mathbf{F}\mathbf{F}^T = \mathbf{V}\mathbf{Q}\mathbf{Q}^T\mathbf{V}^T = \mathbf{V}\mathbf{V}^T \rightarrow \mathbf{F}\mathbf{F}^T = \mathbf{V}^2$ .

(c) Since  $\mathbf{F} = \mathbf{Q}\mathbf{U} = \mathbf{V}\mathbf{Q}$ , and  $\mathbf{U}\mathbf{n} = \lambda\mathbf{n}$ , therefore,  $\mathbf{V}\mathbf{Q}\mathbf{n} = \mathbf{Q}\mathbf{U}\mathbf{n} = \mathbf{Q}(\lambda\mathbf{n}) \rightarrow \mathbf{V}(\mathbf{Q}\mathbf{n}) = \lambda(\mathbf{Q}\mathbf{n})$ , therefore,  $\mathbf{Q}\mathbf{n}$  is an eigenvector for  $\mathbf{V}$  with the eigenvalue  $\lambda$ .

2.58 Verify that the second principal scalar invariant of a tensor  $\mathbf{T}$  can be written:

$$I_2 = (T_{ii}T_{jj} - T_{ij}T_{ji}) / 2.$$

-----  
 Ans.  $T_{ii}T_{jj} = (T_{11} + T_{22} + T_{33})^2 = T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{11}T_{22} + 2T_{22}T_{33} + 2T_{33}T_{11}$ .

$T_{ij}T_{ji} = T_{1j}T_{j1} + T_{2j}T_{j2} + T_{3j}T_{j3} = T_{11}^2 + T_{12}T_{21} + T_{13}T_{31} + T_{21}T_{12} + T_{22}^2 + T_{23}T_{32} + T_{31}T_{13} + T_{32}T_{23} + T_{33}^2$ .

Thus,  $T_{ii}T_{jj} - T_{ij}T_{ji} = (T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{11}T_{22} + 2T_{22}T_{33} + 2T_{33}T_{11})$

$-(T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{12}T_{21} + 2T_{13}T_{31} + 2T_{23}T_{32}) = 2(T_{11}T_{22} - T_{12}T_{21} + T_{22}T_{33} - T_{23}T_{32} + T_{33}T_{11} - T_{13}T_{31})$ .

Thus,

$(T_{ii}T_{jj} - T_{ij}T_{ji}) / 2 = (T_{11}T_{22} - T_{12}T_{21} + T_{22}T_{33} - T_{23}T_{32} + T_{33}T_{11} - T_{13}T_{31})$

$$= \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} = I_2.$$

2.59 A tensor has a matrix  $[\mathbf{T}]$  given below. (a) Write the characteristic equation and find the principal values and their corresponding principal directions. (b) Find the principal scalar invariants. (c) If  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are the principal directions, write  $[\mathbf{T}]_{\mathbf{n}_i}$ . (d) Could the following matrix  $[\mathbf{S}]$  represent the same tensor  $\mathbf{T}$  with respect to some basis.

$$[\mathbf{T}] = \begin{bmatrix} 5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad [\mathbf{S}] = \begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

-----  
 Ans.

(a) The characteristic equation is:

$$\begin{vmatrix} 5-\lambda & 4 & 0 \\ 4 & -1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \rightarrow (3-\lambda)[(5-\lambda)(-1-\lambda) - 16] = (3-\lambda)(\lambda^2 - 4\lambda - 21) = (3-\lambda)(\lambda+3)(\lambda-7) = 0$$

Thus,  $\lambda_1 = 3, \lambda_2 = -3, \lambda_3 = 7$ .

For  $\lambda_1 = 3$ , clearly,  $\mathbf{n}_1 = \pm\mathbf{e}_3$ .