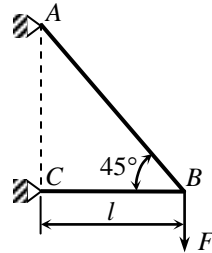


## CHAP 1. STRESS–STRAIN ANALYSIS

1. A vertical force  $F$  is applied to a two-bar truss as shown in the figure. Let cross-sectional areas of the members 1 and 2 be  $A_1$  and  $A_2$ , respectively. Determine the area ratio  $A_1/A_2$  in order to have the same magnitude of stress in both members.

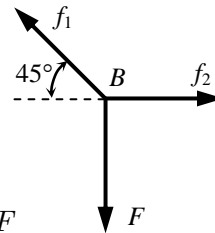


**Solution:**

From force equilibrium at  $B$ ,

$$\sum F_y = f_1 \sin 45^\circ - F = 0 \Rightarrow f_1 = \sqrt{2}F$$

$$\sum F_x = f_2 - f_1 \cos 45^\circ = 0 \Rightarrow f_2 = \sqrt{2}F \cdot \frac{1}{\sqrt{2}} = F$$



Since truss is two-force member,  $f_1 = A_1\sigma_1$  and  $f_2 = A_2\sigma_2$ . Thus,

$$\frac{f_1}{f_2} = \frac{\sqrt{2}F}{F} = \frac{A_1\sigma_1}{A_2\sigma_2} = \frac{A_1}{A_2} (\because \sigma_1 = \sigma_2),$$

$$\therefore \frac{A_1}{A_2} = \sqrt{2}$$

---

2. The stress at a point  $P$  is given below. The direction cosines of the normal  $\mathbf{n}$  to a plane that passes through  $P$  have the ratio  $n_x:n_y:n_z = 3:4:12$ . Determine (a) the traction vector  $\mathbf{T}^{(n)}$ ; (b) the magnitude  $T$  of  $\mathbf{T}^{(n)}$ ; (c) the normal stress  $\sigma_n$ ; (d) the shear stress  $\tau_n$ ; and (e) the angle between  $\mathbf{T}^{(n)}$  and  $\mathbf{n}$ .

*Hint:* Use  $n_x^2 + n_y^2 + n_z^2 = 1$ .

$$[\sigma] = \begin{bmatrix} 13 & 13 & 0 \\ 13 & 26 & -13 \\ 0 & -13 & -39 \end{bmatrix}$$

**Solution:**

- (a) First, we need unit normal vector  $\mathbf{n}$ :

$$\{\mathbf{n}\} = \frac{1}{\sqrt{3^2 + 4^2 + 12^2}} \begin{Bmatrix} 3 \\ 4 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 0.2308 \\ 0.3077 \\ 0.9231 \end{Bmatrix}$$

Then, the traction vector on this plane becomes

$$\mathbf{T}^{(n)} = [\sigma] \{\mathbf{n}\} = \begin{bmatrix} 13 & 13 & 0 \\ 13 & 26 & -13 \\ 0 & -13 & -39 \end{bmatrix} \begin{Bmatrix} 0.2308 \\ 0.3077 \\ 0.9231 \end{Bmatrix} = \begin{Bmatrix} 7 \\ -1 \\ -40 \end{Bmatrix}$$

- (b) Since  $\mathbf{T}^{(n)}$  is a vector, its magnitude can be obtained using the norm as

$$\|T^{(n)}\| = \sqrt{T_x^{(n)2} + T_y^{(n)2} + T_z^{(n)2}} = \sqrt{7^2 + (-1)^2 + (-40)^2} = 40.6202$$

$$(c) \quad \sigma_n = \mathbf{T}^{(n)} \cdot \{\mathbf{n}\} = \begin{Bmatrix} 7 & -1 & -40 \end{Bmatrix} \begin{Bmatrix} 0.2308 \\ 0.3077 \\ 0.9231 \end{Bmatrix} = -35.6154$$

$$(d) \quad \tau_n = \sqrt{\|\mathbf{T}^{(n)}\|^2 - \sigma_n^2} = \sqrt{40.6202^2 - (-35.6154)^2} = 19.5331$$

$$(e) \quad \sigma_n = \mathbf{T}^{(n)} \cdot \{\mathbf{n}\} = \|\mathbf{T}^{(n)}\| \cos \theta$$

$$\theta = \cos^{-1} \left( \frac{\sigma_n}{\|\mathbf{T}^{(n)}\|} \right) = 2.64 \approx 151.3^\circ$$


---

3. At a point  $P$  in a body, Cartesian stress components are given by  $\sigma_{xx} = 80$  MPa,  $\sigma_{yy} = -40$  MPa,  $\sigma_{zz} = -40$  MPa, and  $\tau_{xy} = \tau_{yz} = \tau_{zx} = 80$  MPa. Determine the traction vector, its normal component, and its shear component on a plane that is equally inclined to all three coordinate axes.

*Hint:* When a plane is equally inclined to all three coordinate axes, the direction cosines of the normal are equal to each other.

**Solution:**

The unit normal in this case is:

$$\{\mathbf{n}\} = \frac{1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{Bmatrix}$$

The traction vector in this direction becomes

$$\mathbf{T}^{(\mathbf{n})} = [\boldsymbol{\sigma}] \{\mathbf{n}\} = \begin{bmatrix} 80 & 80 & 80 \\ 80 & -40 & 80 \\ 80 & 80 & -40 \end{bmatrix} \begin{Bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{Bmatrix} = \begin{Bmatrix} 138.56 \\ 69.28 \\ 69.28 \end{Bmatrix} \text{ MPa}$$

The normal component of the traction vector is

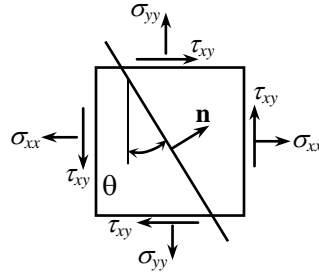
$$\sigma_n = \mathbf{T}^{(\mathbf{n})} \cdot \{\mathbf{n}\} = \{138.56 \quad 69.28 \quad 69.28\} \begin{Bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{Bmatrix} = 160 \text{ MPa}$$

The shear component of the traction vector is:

$$\tau_n = \sqrt{\|\mathbf{T}^{(\mathbf{n})}\|^2 - \sigma_n^2} = \sqrt{169.71^2 - 160^2} = 56.58 \text{ MPa}$$


---

4. If  $\sigma_{xx} = 90$  MPa,  $\sigma_{yy} = -45$  MPa,  $\tau_{xy} = 30$  MPa, and  $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$ , compute the surface traction  $\mathbf{T}^{(n)}$  on the plane shown in the figure, which makes an angle of  $\theta = 40^\circ$  with the vertical axis. What are the normal and shear components of stress on this plane?



**Solution:**

Unit normal vector:  $\mathbf{n}^T = \{ \cos(40) \quad \sin(40) \quad 0 \}$

Traction vector:  $\mathbf{T}^{(n)} = [\boldsymbol{\sigma}] \cdot \mathbf{n} = \begin{bmatrix} 90 & 30 & 0 \\ 30 & -45 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} .776 \\ .643 \\ 0 \end{bmatrix} = \begin{bmatrix} 88.23 \\ -5.94 \\ 0 \end{bmatrix} \text{MPa}$

Normal stress:  $\sigma_n = \mathbf{T}^{(n)} \cdot \mathbf{n} = 63.77 \text{MPa}$

Shear stress:  $\tau_n = \sqrt{|\mathbf{T}^{(n)}|^2 - \sigma_n^2} = 61.27 \text{MPa}$

---

5. Find the principal stresses and the corresponding principal stress directions for the following cases of plane stress.

(a)  $\sigma_{xx} = 40 \text{ MPa}$ ,  $\sigma_{yy} = 0 \text{ MPa}$ ,  $\tau_{xy} = 80 \text{ MPa}$

(b)  $\sigma_{xx} = 140 \text{ MPa}$ ,  $\sigma_{yy} = 20 \text{ MPa}$ ,  $\tau_{xy} = -60 \text{ MPa}$

(c)  $\sigma_{xx} = -120 \text{ MPa}$ ,  $\sigma_{yy} = 50 \text{ MPa}$ ,  $\tau_{xy} = 100 \text{ MPa}$

**Solution:**

(a) The stress matrix becomes

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 40 & 80 \\ 80 & 0 \end{bmatrix} \text{MPa}$$

To find the principal stresses, the standard eigen value problem can be written as

$$[\boldsymbol{\sigma} - \sigma \mathbf{I}]\{\mathbf{n}\} = 0$$

The above problem will have non-trivial solution when the determinant of the coefficient matrix becomes zero:

$$\begin{vmatrix} \sigma_{xx} - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} - \sigma \end{vmatrix} = \begin{vmatrix} 40 - \sigma & 80 \\ 80 & 0 - \sigma \end{vmatrix} = 0$$

The equation of the determinant becomes:

$$((40 - \sigma) \cdot -\sigma) - (80 \cdot 80) = \sigma^2 - 40\sigma - 6400 = 0$$

The above quadratic equation yields two principal stresses, as

$$\sigma_1 = 102.46 \text{ MPa} \text{ and } \sigma_2 = -62.46 \text{ MPa}.$$

To determine the orientation of the first principal stresses, substitute  $\sigma_1$  in the original eigen value problem to obtain

$$\begin{bmatrix} 40 - 102.46 & 80 \\ 80 & 0 - 102.46 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Since the determinant is zero, two equations are not independent

$$62.46 \cdot n_x = 80 \cdot n_y \text{ and } 80 \cdot n_x = -102.46 \cdot n_y.$$

Thus, we can only get the relation between  $n_x$  and  $n_y$ . Then using the condition  $|\mathbf{n}| = 1$  we obtain

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.788 \\ 0.615 \end{Bmatrix}$$

To determine the orientation of the second principal stress, substitute  $\sigma_2$  in the original eigen value problem to obtain

$$\begin{bmatrix} 40 + 62.46 & 80 \\ 80 & 0 + 62.46 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$102.46 \cdot n_x = -80 \cdot n_y \text{ and } 80 \cdot n_x = -62.46 \cdot n_y.$$

Using similar procedures as above, the eigen vector of  $\sigma_2$  can be obtained as

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.615 \\ -0.788 \end{Bmatrix}$$

Note that if  $\mathbf{n}$  is a principal direction,  $-\mathbf{n}$  is also a principal direction

(b) Repeat the procedure in (a) to obtain

$$\sigma_1 = 164.85 \text{ MPa and } \sigma_2 = -4.85 \text{ MPa}.$$

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} -0.924 \\ 0.383 \end{Bmatrix} \text{ and } \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.383 \\ 0.924 \end{Bmatrix}$$

(c) Repeat the procedure in (a) to obtain

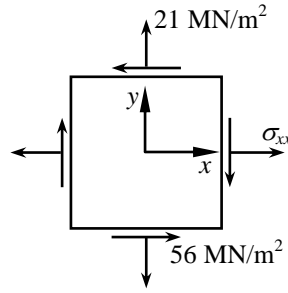
$$\sigma_1 = 96.24 \text{ MPa and } \sigma_2 = -166.24 \text{ MPa}.$$

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.420 \\ 0.908 \end{Bmatrix} \text{ and } \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} -0.908 \\ 0.420 \end{Bmatrix}$$

Note that for the case of plane stress  $\sigma_3=0$  is also a principal stress and the corresponding principal stress direction is given by  $\mathbf{n}^{(3)}=(0,0,1)$

---

6. If the minimum principal stress is  $-7$  MPa, find  $\sigma_{xx}$  and the angle that the principal stress axes make with the  $x$  and  $y$  axes for the case of plane stress illustrated



**Solution:**

With unknown  $x$ -component, the eigen value problem can be written as

$$\begin{bmatrix} \sigma_{xx} - \sigma & -56 \\ -56 & 21 - \sigma \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The principal stresses can be determined by making the determinant zero

$$\begin{vmatrix} \sigma_{xx} - \sigma & -56 \\ -56 & 21 - \sigma \end{vmatrix} = 0 \Rightarrow (\sigma_{xx} - \sigma)(21 - \sigma) - 56^2 = 0$$

Since  $-7$  MPa is one of the roots of the above equation, we can find  $\sigma_{xx}$  by substituting in the above equation as

$$(\sigma_{xx} + 7)(21 + 7) - 56^2 = 0$$

By solving the above equation, we can get  $\sigma_{xx} = 105$  MPa. Then, the other principal stress can be found from the original determinant, as

$$\sigma_1 = 133 \text{ MPa} \quad \sigma_2 = -7 \text{ MPa}$$

Principal direction for the first principal stress: From the original eigen value problem,

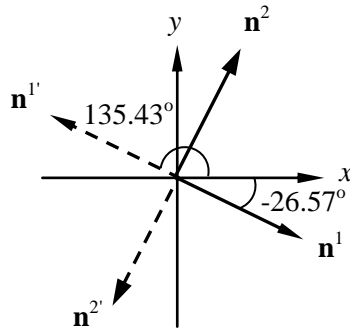
$$\begin{aligned} (105 - 133)n_x^1 - 56n_y^1 &= 0 \\ -56n_x^1 + (21 - 133)n_y^1 &= 0 \end{aligned}$$

The solution of the above equations is not unique. By putting  $|\mathbf{n}^1| = 1$ , we have  $\mathbf{n}^1 = \{\pm 0.8944, \mp 0.4472\}$ , which is principal direction corresponding to  $\sigma_1$

Principal direction for the second principal stress: From the original eigen value problem,

$$(105 + 7)n_x^2 - 56n_y^2 = 0$$
$$-56n_x^2 + (21 + 7)n_y^2 = 0$$

The solution of above equations is  $\mathbf{n}^2 = \{\pm 0.4472, \pm 0.8944\}$ , which is principal direction corresponding to  $\sigma_2$ . Two principal directions are plotted on the following graph. Note that the two principal directions are perpendicular each other.



7. Determine the principal stresses and their associated directions, when the stress matrix at a point is given by

$$[\sigma] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \text{MPa}$$

**Solution:**

Use the Eq. (0.46) of Chapter 0 with the coefficients of  $I_1=3$ ,  $I_2=-3$ , and  $I_3=-1$ ,

$$\lambda^3 - 3\lambda^2 - 3\lambda + 1 = 0$$

By solving the above cubic equation using the method described in Section 0.4,

$$\sigma_1 = 3.73 \text{ MPa}, \quad \sigma_2 = 0.268 \text{ MPa}, \quad \sigma_3 = -1.00 \text{ MPa}$$

- (a) Principal direction corresponding to  $\sigma_1$ :

$$\begin{aligned} (1 - 3.7321)n_x^1 + n_y^1 + n_z^1 &= 0 \\ n_x^1 + (1 - 3.7321)n_y^1 + 2n_z^1 &= 0 \\ n_x^1 + 2n_y^1 + (1 - 3.7321)n_z^1 &= 0 \end{aligned}$$

Solving the above equations with  $|\mathbf{n}^1| = 1$  yields

$$\mathbf{n}^1 = \{\pm 0.4597, \pm 0.6280, \pm 0.6280\}$$

- (b) Principal direction corresponding to  $\sigma_2$ :

$$\begin{aligned} (1 - 0.2679)n_x^2 + n_y^2 + n_z^2 &= 0 \\ n_x^2 + (1 - 0.2679)n_y^2 + 2n_z^2 &= 0 \\ n_x^2 + 2n_y^2 + (1 - 0.2679)n_z^2 &= 0 \end{aligned}$$

Solving the above equations with  $|\mathbf{n}^2| = 1$  yields

$$\mathbf{n}^2 = \{\pm 0.8881, \mp 0.3251, \mp 0.3251\}$$

- (c) Principal direction corresponding to  $\sigma_3$ :

$$\begin{aligned} (1 + 1)n_x^3 + n_y^3 + n_z^3 &= 0 \\ n_x^3 + (1 + 1)n_y^3 + 2n_z^3 &= 0 \\ n_x^3 + 2n_y^3 + (1 + 1)n_z^3 &= 0 \end{aligned}$$

Solving the above equations with  $|\mathbf{n}^2| = 1$  yields

$$\mathbf{n}^3 = \{0, \pm 0.7071, \mp 0.7071\}$$

---

8. Let  $x'y'z'$  coordinate system be defined using the three principal directions obtained from Problem 7. Determine the transformed stress matrix  $[\sigma]_{x'y'z'}$  in the new coordinates system.

**Solution:**

The three principal directions in Problem 6 can be used for the coordinate transformation matrix:

$$[\mathbf{N}] = \begin{bmatrix} n_x^{(1)} & n_x^{(2)} & n_x^{(3)} \\ n_y^{(1)} & n_y^{(2)} & n_y^{(3)} \\ n_z^{(1)} & n_z^{(2)} & n_z^{(3)} \end{bmatrix} = \begin{bmatrix} 0.460 & -0.888 & 0 \\ 0.628 & 0.325 & 0.707 \\ 0.628 & 0.325 & -0.707 \end{bmatrix}$$

To determine the stress components in the new coordinates we use Eq. (1.30):

$$[\sigma]_{x'y'z'} = [\mathbf{N}]^T [\sigma] [\mathbf{N}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & .268 & 0 \\ 0 & 0 & 3.732 \end{bmatrix}$$

Note that the transformed stress matrix is a diagonal matrix with the original principal stresses on the diagonal.

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9. For the stress matrix below, the two principal stresses are given as  $\sigma_3 = -3$  and  $\sigma_1 = 2$ , respectively. In addition, the two principal stress directions corresponding to the two principal stresses are also given below.

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}, \quad \mathbf{n}^1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad \mathbf{n}^3 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$$

- (a) What is the normal and shear stress on a plane whose normal vector is parallel to (2, 1, 2)?
- (b) Calculate the missing principal stress  $\sigma_2$  and the principal direction  $\mathbf{n}^2$ .
- (c) Write stress matrix in the new coordinates system that is aligned with  $\mathbf{n}^1$ ,  $\mathbf{n}^2$ , and  $\mathbf{n}^3$ .

**Solution:**

(a) Normal vector:  $\mathbf{n}^T = \frac{1}{3}\{2 \quad 1 \quad 2\}$

Traction vector  $\mathbf{T}^{(\mathbf{n})} = [\boldsymbol{\sigma}]\mathbf{n} = \begin{bmatrix} 2 \\ \frac{1}{3} \\ 0 \end{bmatrix}$

The normal component of the stress vector on the plane can be calculated

$$\sigma_n = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} = 1.4444$$

$$\tau_n = \sqrt{|\mathbf{T}^{(\mathbf{n})}|^2 - \sigma_n^2} = 1.4229$$

- (b) Using Eq. (0.46) of Chapter 0, the eigen values are governed by

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

We can find the coefficients of the above cubic equation from Eq. (0.47) by  $I_1 = 0$ ,  $I_2 = -7$ , and  $I_3 = -6$ . Thus, we have

$$\lambda^3 - 7\lambda + 6 = (\lambda - 1)(\lambda^2 + \lambda - 6) = 0$$

Thus, the missing principal stress is  $\sigma_2 = 1$ .

Since three principal directions are mutually orthogonal, the third principal direction can be calculated using the cross product. To establish a defined sign convention for the principal axes, we require them to form a right-handed triad. If  $\mathbf{n}^1$  and  $\mathbf{n}^3$  are unit vectors that define the directions of the first and third principal axes, then the unit vector  $\mathbf{n}^2$  for the second principal axis is determined by the right-hand rule of the vector multiplication. Thus we have

$$\mathbf{n}^2 = \mathbf{n}^3 \times \mathbf{n}^1 = \{0 \ 1 \ 0\}^T$$

(c) Coordinate transformation matrix can be obtained from three principal directions as

$$[\mathbf{N}] = [\mathbf{n}^1 \ \mathbf{n}^2 \ \mathbf{n}^3] = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

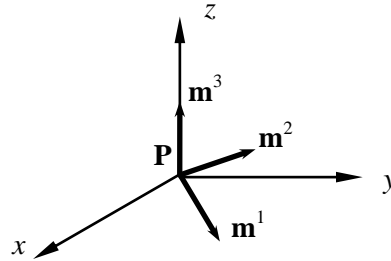
The stress matrix at the transformed coordinates becomes

$$[\mathbf{N}]^T [\boldsymbol{\sigma}] [\mathbf{N}] = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$


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10. With respect to the coordinate system  $xyz$ , the state of stress at a point  $P$  in a solid is

$$[\sigma] = \begin{bmatrix} -20 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} \text{ MPa}$$



- $\mathbf{m}^1$ ,  $\mathbf{m}^2$  and  $\mathbf{m}^3$  are three mutually perpendicular vectors such that  $\mathbf{m}^1$  makes  $45^\circ$  with both  $x$ - and  $y$ -axes and  $\mathbf{m}^3$  is aligned with the  $z$ -axis. Compute the normal stresses on planes normal to  $\mathbf{m}^1$ ,  $\mathbf{m}^2$ , and  $\mathbf{m}^3$ .
- Compute two components of shear stress on the plane normal to  $\mathbf{m}^1$  in the directions  $\mathbf{m}^2$  and  $\mathbf{m}^3$ .
- Is the vector  $\mathbf{n} = \{0, 1, 1\}^T$  a principal direction of stress? Explain. What is the normal stress in the direction  $\mathbf{n}$ ?
- Draw an infinitesimal cube with faces normal to  $\mathbf{m}^1$ ,  $\mathbf{m}^2$  and  $\mathbf{m}^3$  and display the stresses on the positive faces of this cube.
- Express the state of stress at the point  $P$  with respect to the  $x'y'z'$  coordinates system that is aligned with the vectors  $\mathbf{m}^1$ ,  $\mathbf{m}^2$  and  $\mathbf{m}^3$ ?
- What are the principal stress and principal directions of stress at the point  $P$  with respect to the  $x'y'z'$  coordinates system? Explain.
- Compute the maximum shear stress at the point  $P$ . Which plane(s) does this maximum shear stress act on?

**Solution:**

(a)

$$\mathbf{m}^1 = \frac{1}{\sqrt{2}}(1, 1, 0)^T \quad \mathbf{m}^2 = \frac{1}{\sqrt{2}}(-1, 1, 0)^T \quad \mathbf{m}^3 = (0, 0, 1)^T$$

$$\sigma_{\mathbf{m}^1\mathbf{m}^1} = \mathbf{m}^1 \cdot [\sigma] \cdot \mathbf{m}^1 = 15 \text{ MPa}$$

$$\sigma_{\mathbf{m}^2\mathbf{m}^2} = \mathbf{m}^2 \cdot [\sigma] \cdot \mathbf{m}^2 = 15 \text{ MPa}$$

$$\sigma_{\mathbf{m}^3\mathbf{m}^3} = \mathbf{m}^3 \cdot [\sigma] \cdot \mathbf{m}^3 = 50 \text{ MPa}$$

(b)

$$\mathbf{T}^{(\mathbf{m}^1)} = [\sigma] \cdot \mathbf{m}^1 = \frac{1}{\sqrt{2}} \{-20 \quad 50 \quad 0\}^T$$

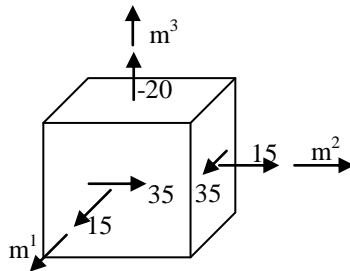
$$\begin{aligned}\tau_{\mathbf{m}^1\mathbf{m}^2} &= \mathbf{T}^{(\mathbf{m}^1)} \cdot \mathbf{m}^2 = 35 \text{ MPa} \\ \tau_{\mathbf{m}^1\mathbf{m}^3} &= \mathbf{T}^{(\mathbf{m}^1)} \cdot \mathbf{m}^3 = 0 \text{ MPa}\end{aligned}$$

(c) Yes,

$$\begin{aligned}\mathbf{n} &= \frac{1}{\sqrt{2}}(0, 1, 1) \\ \mathbf{T}^{(\mathbf{n})} &= [\sigma] \cdot \mathbf{n} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 50 \\ 50 \\ 0 \end{Bmatrix} = 50 \times \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = 50\mathbf{n}\end{aligned}$$

Since  $\mathbf{T}^{(\mathbf{n})} // \mathbf{n}$ ,  $\mathbf{n}$  is a principal direction with principal stress = 50 MPa.

(d)



(e)

$$\begin{aligned}[\sigma]_{x'y'z'} &= [N]^T [\sigma] [N] = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -20 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ [\sigma]_{x'y'z'} &= \begin{bmatrix} 15 & 35 & 0 \\ 35 & 15 & 0 \\ 0 & 0 & 50 \end{bmatrix} \text{ MPa}\end{aligned}$$

(f) Principal stresses = 50, 50, and -20 MPa

$$\mathbf{n}^3 = \frac{1}{\sqrt{2}}(1, -1, 0)$$

$\mathbf{n}^1$  and  $\mathbf{n}^2$  are any two perpendicular unit vectors that is on the plane perpendicular to  $\mathbf{n}^3$ .

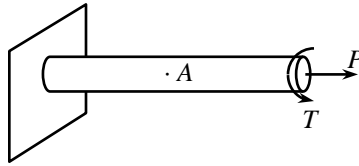
(g) The maximum shear stress occurs on a plane whose normal is at  $45^\circ$  from the principal stress direction. Since  $\sigma_1 = \sigma_2$ , all directions that are  $45^\circ$  from  $x$ -axis ( $\sigma_3$  axis) will have the maximum shear stress whose value is

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 35 \text{ MPa}$$

The maximum shear stress planes are in the shape of a cone whose axis is parallel to  $x$ -axis and has an angle of  $45^\circ$ .

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11. A solid shaft of diameter  $d = 5$  cm, as shown in the figure, is subjected to tensile force  $P = 13,000$  N and a torque  $T = 6,000$  N·cm. At point A on the surface, what is the state of stress (write in matrix form), the principal stresses, and the maximum shear stress? Show the coordinate system you are using.



**Solution:**

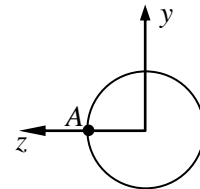
Let us establish a coordinate system as shown in the figure. The axial force will cause normal stress  $\sigma_{xx}$ , while the torque will cause shear stress  $\tau_{xy}$ . Their magnitudes are

$$\frac{P}{A} = 6.62 \text{ MPa}$$

$$\frac{T \cdot r}{J} = 2.44 \text{ MPa}$$

Then, the stress matrix becomes

$$[\sigma]_A = \begin{bmatrix} 6.62 & -2.44 & 0 \\ -2.44 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa}$$



By solving the eigen value problem, the principal stress can be obtained as

$$\sigma_1 = 7.43, \quad \sigma_2 = 0, \quad \sigma_3 = -0.81 \text{ MPa}$$

Maximum shear stress is

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = 4.11 \text{ MPa}$$


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12. If the displacement field is given by

$$\begin{cases} u_x = x^2 + 2y^2 \\ u_y = -y^2 - 2x(y - z) \\ u_z = -z^2 - 2xy \end{cases}$$

(a) Write down  $3 \times 3$  strain matrix.

(b) What is the normal strain component in the direction of  $(1,1,1)$  at point  $(1,-3,1)$ ?

**Solution:**

(a)  $3 \times 3$  symmetric strain matrix can be calculated from its definition as

$$[\epsilon] = \begin{bmatrix} 2x & y + z & -y \\ y + z & -2(x + y) & 0 \\ -y & 0 & -2z \end{bmatrix}$$

In addition, the unit normal vector in the direction of  $(1, 1, 1)$  is

$$\mathbf{n}^T = \frac{1}{\sqrt{3}} \{1 \quad 1 \quad 1\}$$

b) Thus, the normal component of strain is

$$\mathbf{n} \cdot [\epsilon] \cdot \mathbf{n} = \frac{1}{3} (2x + y + z - y + y + z - 2x - 2y - y - 2z) = -\frac{2}{3} y$$

Thus, the normal component of strain reduces as the  $y$ -coordinate of a point increases. At point  $(1, -3, 1)$ ,  $y = -3$

$$\therefore \mathbf{n} \cdot [\epsilon] \cdot \mathbf{n} \Big|_{y=-3} = 2.$$


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13. Consider the following displacement field in a plane solid:

$$\begin{aligned}u(x, y) &= 0.04 - 0.01x + 0.006y \\v(x, y) &= 0.06 + 0.009x + 0.012y\end{aligned}$$

- Compute the strain components  $\varepsilon_{xx}$ ,  $\varepsilon_{yy}$ , and  $\gamma_{xy}$ . Is this a state of uniform strain?
- Determine the principal strains and their corresponding directions. Express the principal strain directions in terms of angles the directions make with the  $x$ -axis.
- What is the normal strain at Point  $O$  in a direction  $45^\circ$  to the  $x$ -axis?

**Solution:**

(a) Strain components:

$$\begin{aligned}\varepsilon_{xx} &= \frac{\partial u}{\partial x} = -0.01 \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} = 0.012 \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.009 + 0.006 = 0.015\end{aligned}$$

Yes, this is a state of uniform strain, because the strains are independent of position  $x, y, z$ .

(b) Principal strains and principal directions.

$$\begin{aligned}\varepsilon_{xy} &= \frac{1}{2}\gamma_{xy} = 0.0075 \\ \boldsymbol{\varepsilon} &= \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} -0.01 & 0.0075 \\ 0.0075 & 0.012 \end{bmatrix}\end{aligned}$$

Find the eigen values (principal strains) and eigen vectors (principal direction) by solving the eigen value problem:

$$\begin{bmatrix} -0.01 - \lambda & 0.0075 \\ 0.0075 & 0.012 - \lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The above equation yields two principal strains:  $\varepsilon_1 = \lambda_1 = -0.01231$  and  $\varepsilon_2 = \lambda_2 = 0.01431$ . The principal direction corresponding to the first principal strain is

$$\mathbf{n}^{(1)} = \{-0.9556 \quad 0.2948\},$$

The angle the direction makes with the  $x$ -axis can be found from the relation  $\cos \theta = -0.9556, \sin \theta = +0.2948$ . Solving  $\theta \approx 163^\circ$

The principal direction corresponding to the second principal strain is

$$\mathbf{n}^{(2)} = \{0.2948 \quad 0.9556\},$$

and the angle is found to be  $\theta \approx 73^\circ$

(c)

$$\text{Strain at point O} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} -0.01 & 0.0075 \\ 0.0075 & 0.012 \end{bmatrix},$$

$$\text{direction vector} \quad \mathbf{n} = \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Bmatrix}$$

Thus the normal strain in the direction of  $\mathbf{n}$  becomes

$$\varepsilon_{45^\circ} = \mathbf{n} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{n} = \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Bmatrix}^T \begin{bmatrix} -0.01 & 0.0075 \\ 0.0075 & 0.012 \end{bmatrix} \cdot \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Bmatrix} = 0.0085$$

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