1.6. Exercises

P1.1 Using Cartesian bases, show that $(\mathbf{u}\otimes\mathbf{v})\cdot(\mathbf{w}\otimes\mathbf{x}) = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}\otimes\mathbf{x}$ where **u**, **v**, **w**, and **x** are rank 1 tensor.

Solution: Using the Cartesian basis, $(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x}) = (u_i\mathbf{e}_i\otimes v_j\mathbf{e}_j)\cdot(w_k\mathbf{e}_k\otimes x_l\mathbf{e}_l)$. Since the dot product occurs between adjacent bases, we have

$$\begin{aligned}
&(u_i\mathbf{e}_i\otimes v_j\mathbf{e}_j)\cdot(w_k\mathbf{e}_k\otimes x_l\mathbf{e}_l)\\
&= u_iv_jw_kx_l(\mathbf{e}_j\cdot\mathbf{e}_k)(\mathbf{e}_i\otimes\mathbf{e}_l)\\
&= u_iv_jw_kx_l\delta_{jk}(\mathbf{e}_i\otimes\mathbf{e}_l)\\
&= u_iv_jw_jx_l(\mathbf{e}_i\otimes\mathbf{e}_l)\\
&= v_jw_j(u_i\mathbf{e}_i\otimes x_l\mathbf{e}_l)\\
&= (\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\otimes\mathbf{x})
\end{aligned}$$

In the above equation, we used the following properties: $\mathbf{e}_j\cdot\mathbf{e}_k = \delta_{jk}$, $w_k\delta_{jk} = w_j$, and $v_jw_j = \mathbf{v}\cdot\mathbf{w}$.

P1.2 Any rank 2 tensor **T** can be decomposed by **T** = **S** + **W**, where **S** is the symmetric part of **T** and **W** is the skew part of **T**. Let **A** be a symmetric rank 2 tensor. Show $\mathbf{A}:\mathbf{W} = 0$ and $\mathbf{A}:\mathbf{T} = \mathbf{A}:\mathbf{S}$.

Solution: Since **A** is symmetric and **W** is skew, we have

$$\mathbf{A}:\mathbf{W} = A_{ij}W_{ij} = -A_{ij}W_{ji} = -A_{ji}W_{ji}$$

Since in the above equation, the repeated indices *i* and *j* are dummy, the above equation can be rewritten as

$$A_{ij}W_{ij} = -A_{ij}W_{ij} = 0$$

In addition, from the relation **T** = **S** + **W**,

$$\mathbf{A}:\mathbf{T} = \mathbf{A}:(\mathbf{S}+\mathbf{W}) = \mathbf{A}:\mathbf{S} + \mathbf{A}:\mathbf{W} = \mathbf{A}:\mathbf{S}$$

P1.3 For a symmetric rank-two tensor $\mathbf{E}$, using the index notation, show that $\mathbf{I}:\mathbf{E} = \mathbf{E}$, where $\mathbf{I} = \frac{1}{2}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$ is a symmetric unit tensor of rank-4.

Solution: Using index notation, the contraction operator can be written as

$$(\mathbf{I}:\mathbf{E})_{ij} = \tfrac{1}{2}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]E_{kl}$$

Since the Kronecker-delta symbol replaces indices, the above equation can be written as

$$(\mathbf{I}:\mathbf{E})_{ij} = \tfrac{1}{2}[E_{ij} + E_{ji}] = E_{ij} = (\mathbf{E})_{ij}$$

The symmetric property of $\mathbf{E}$ is used.

P1.4 The deviator of a symmetric rank-2 tensor is defined as $\mathbf{A}_{dev} = \mathbf{A} - A^m\mathbf{1}$ where $A^m = \frac{1}{3}(A_{11} + A_{22} + A_{33})$. Find the rank-4 deviatoric identity tensor $\mathbf{I}_{dev}$ that satisfies $\mathbf{A}_{dev} = \mathbf{I}_{dev} : \mathbf{A}$.

Solution: From Problem P1.3, it can be shown that $\mathbf{I} : \mathbf{A} = \mathbf{A}$. In addition, A^m can be written in the tensor notation as $A^m = \frac{1}{3}\mathbf{1} : \mathbf{A}$. Therefore, $\mathbf{A}_{dev} = \mathbf{A} - A^m\mathbf{1}$ and it can be written as

$$\mathbf{A}_{dev} = \left[\mathbf{I} - \tfrac{1}{3}\mathbf{1} \otimes \mathbf{1}\right] : \mathbf{A} = \mathbf{I}_{dev} : \mathbf{A}$$

The last equality defined the rank-4 deviatoric identity tensor $\mathbf{I}_{dev}$.

P1.5 The norm of a rank-2 tensor is defined as $\|\mathbf{A}\| = \sqrt{\mathbf{A} : \mathbf{A}}$. Calculate the following derivative $\partial\|\mathbf{A}\| / \partial\mathbf{A}$. What is the rank of the derivative?

Solution: From the definition

$$\frac{\partial\|\mathbf{A}\|}{\partial\mathbf{A}} = \frac{\partial}{\partial\mathbf{A}}\left[(\mathbf{A} : \mathbf{A})^{1/2}\right] = \frac{1}{2}(\mathbf{A} : \mathbf{A})^{-1/2}(2\mathbf{A} : \mathbf{I}) = \frac{\mathbf{A}}{\|\mathbf{A}\|}$$

The result is a rank-2 tensor. Note that the property that $\partial\mathbf{A} / \partial\mathbf{A} = \mathbf{I}$ is used.

P1.6 A unit rank-2 tensor in the direction of rank-2 tensor $\mathbf{A}$ can be defined as $\mathbf{N} = \mathbf{A} / \|\mathbf{A}\|$. Show that $\partial\mathbf{N} / \partial\mathbf{A} = [\mathbf{I} - \mathbf{N} \otimes \mathbf{N}] / \|\mathbf{A}\|$.

Solution: Using chain-rule of differentiation, the unit normal tensor can be differentiated as

$$\frac{\partial\mathbf{N}}{\partial\mathbf{A}} = \frac{\partial}{\partial\mathbf{A}}\left(\frac{\mathbf{A}}{\|\mathbf{A}\|}\right) = \frac{1}{\|\mathbf{A}\|^2}\left(\frac{\partial\mathbf{A}}{\partial\mathbf{A}}\|\mathbf{A}\| - \mathbf{A} \otimes \frac{\partial\|\mathbf{A}\|}{\partial\mathbf{A}}\right)$$

It is straightforward to show that $\partial\mathbf{A} / \partial\mathbf{A} = \mathbf{I}$. From Problem 1.5, we have

$$\frac{\partial\|\mathbf{A}\|}{\partial\mathbf{A}} = \frac{\partial}{\partial\mathbf{A}}\left[(\mathbf{A} : \mathbf{A})^{1/2}\right] = \frac{1}{2}(\mathbf{A} : \mathbf{A})^{-1/2}(2\mathbf{A}) = \frac{\mathbf{A}}{\|\mathbf{A}\|}$$

Therefore, we have

$$\frac{\partial \mathbf{N}}{\partial \mathbf{A}} = \frac{1}{\|\mathbf{A}\|}\left(\mathbf{I} - \mathbf{N} \otimes \mathbf{N}\right)$$

P1.7 Through direct calculation of a rank-2 tensor, show that the following identity $e_{rst}\det[\mathbf{A}] = e_{ijk}A_{ir}A_{js}A_{kt}$ is true

Solution: In the index notation, (r, s, t) are real indices, while (i, j, k) are dummy indices. Since (r, s, t) only appears in the permutation symbol, it is enough to show the cases of even and odd permutation. Consider the following case of even permutation: (r, s, t) = (1, 2, 3). In such a case, non-zero components of the right-hand side can be written as

$$\begin{aligned} e_{ijk}A_{i1}A_{j2}A_{k3} &= e_{123}A_{11}A_{22}A_{33} + e_{132}A_{11}A_{32}A_{23} \\ &+ e_{231}A_{21}A_{32}A_{13} + e_{213}A_{21}A_{12}A_{33} \\ &+ e_{312}A_{31}A_{12}A_{23} + e_{321}A_{31}A_{22}A_{13} \end{aligned}$$

In the above equation, we have $e_{123} = e_{231} = e_{312} = 1$ and $e_{132} = e_{213} = e_{321} = -1$. Therefore, the above equation becomes

$$e_{ijk}A_{i1}A_{j2}A_{k3} = A_{11}(A_{22}A_{33} - A_{32}A_{23}) + A_{21}(A_{32}A_{13} - A_{12}A_{33}) + A_{31}(A_{12}A_{23} - A_{22}A_{13})$$

which is the definition of $\det[\mathbf{A}]$. By following a similar approach, it can be shown that the odd permutation of (r, s, t) will yield $-\det[\mathbf{A}]$.

P1.8 For a vector $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ and its norm $r = |\mathbf{r}|$, prove $\nabla \cdot (r\mathbf{r}) = 4r$.

Solution: From the product rule,

$$\nabla \cdot (r\mathbf{r}) = \nabla r \cdot \mathbf{r} + r\nabla \cdot \mathbf{r}$$

Now consider

$$(\nabla r)_i = \frac{\partial}{\partial x_i}(x_j x_j)^{1/2} = \frac{1}{2(x_k x_k)^{1/2}}\frac{\partial}{\partial x_i}(x_j x_j) = \frac{1}{2r}\left(\frac{\partial x_j}{\partial x_i}x_j + x_j\frac{\partial x_j}{\partial x_i}\right) = \frac{1}{r}\delta_{ij}x_j = \frac{x_i}{r}$$

Therefore,

$$\nabla \cdot (r\mathbf{r}) = \nabla r \cdot \mathbf{r} + r\nabla \cdot \mathbf{r} = \frac{x_i}{r}x_i + r\frac{\partial x_i}{\partial x_i} = \frac{r^2}{r} + 3r = 4r$$

This completes the proof.

P1.9 A velocity gradient is decomposed into symmetric and skew parts, $\nabla \mathbf{v} = \mathbf{d} + \boldsymbol{\omega}$, where

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

Show that

(a) For a symmetric stress tensor, $\boldsymbol{\sigma} : \nabla \mathbf{v} = \boldsymbol{\sigma} : \mathbf{d}$.

(b) $w_{ij} = \frac{1}{2} e_{ijk} e_{mnk} \frac{\partial v_m}{\partial x_n}$

Solution:

(a) From Prob. 1.2, Since stress tensor is symmetric, $\boldsymbol{\sigma} : \boldsymbol{\omega} = 0$. Therefore, it is obvious that $\boldsymbol{\sigma} : \nabla \mathbf{v} = \boldsymbol{\sigma} : \boldsymbol{\omega} + \boldsymbol{\sigma} : \mathbf{d} = \boldsymbol{\sigma} : \mathbf{d}$.

(b) The direct substitution method can be used to show the identity. We will show the case when $i = 1, j = 2$. The other cases can also be shown in the same way. Knowing that the permutation symbol becomes zero when indices are repeated, in this case the only nonzero situation happens when $k = 3$. For the second permutation symbol, the only non-zero situations are $m = 1, n = 2$ and $m = 2, n = 1$, where the former is even permutation and the latter is odd permutation. Therefore,

$$w_{12} = \frac{1}{2} e_{123} e_{mn3} \frac{\partial v_m}{\partial x_n} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right)$$

Other cases can also be shown in the same way.

P1.10 A symmetric rank four tensor is defined by $\mathbf{D} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$ where $\mathbf{1} = [\delta_{ij}]$ is a unit tensor of rank-two and $\mathbf{I} = \frac{1}{2}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$ is a symmetric unit tensor of rank-four. When $\mathbf{E}$ is an arbitrary symmetric rank-two tensor, calculate $\mathbf{S} = \mathbf{D} : \mathbf{E}$ in terms of $\mathbf{E}$.

Solution: Using index notation, the contraction can be written as

$$S_{ij} = D_{ijkl} E_{kl} = \left[\lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right] E_{kl}$$

Since the Kronecker-delta symbol replaces indices, the above equation can be simplified as

$$S_{ij} = D_{ijkl} E_{kl} = \lambda E_{kk} \delta_{ij} + \mu(E_{ij} + E_{ji}) = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij}$$

In the tensor notation, the above relation can be written as

$$\mathbf{S} = \mathbf{D} : \mathbf{E} = \lambda tr(\mathbf{E}) \mathbf{1} + 2\mu \mathbf{E}$$

P1.11 Using integration by parts, calculate $I = \int x\cos(x)\,\mathrm{d}x$.

Solution: Let $u = x$ and $v' = \cos(x)$. Then

$$\begin{aligned}\int x\cos(x)\,\mathrm{d}x &= \int uv'\,\mathrm{d}x \\ &= uv - \int u'v\,\mathrm{d}x \\ &= x\sin(x) - \int \sin(x)\,\mathrm{d}x \\ &= x\sin(x) + \cos(x) + C\end{aligned}$$

P1.12 Using integration by parts, calculate $I = \int e^x\cos(x)\,\mathrm{d}x$.

Solution: Let $u = \cos(x)$ and $v' = e^x$. Then

$$\int e^x\cos(x)\,\mathrm{d}x = e^x\cos(x) + \int e^x\sin(x)\,\mathrm{d}x$$

Now, to evaluate the second terms on the right-hand side using additional integration by parts with $u = \sin(x)$ and $v' = e^x$, as

$$\int e^x\sin(x)\,\mathrm{d}x = e^x\sin(x) - \int e^x\cos(x)\,\mathrm{d}x$$

Therefore, putting these together, we have

$$\int e^x\cos(x)\,\mathrm{d}x = e^x\cos(x) + e^x\sin(x) - \int e^x\cos(x)\,\mathrm{d}x$$

After rearranging, the original integral can be obtained as

$$\int e^x\cos(x)\,\mathrm{d}x = \frac{1}{2}(e^x\cos(x) + e^x\sin(x)) + C$$

P1.13 Calculate the surface integral of the vector function $\mathbf{F} = x\mathbf{e}_1 + y\mathbf{e}_2$ over the portion of the surface of the unit sphere, $S : x^2 + y^2 + z^2 = 1$, above the xy plane; i.e., $z \geq 0$.

$$\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$$

Solution: If we close the surface of integration by adding the portion of the xy plane which spans the hemisphere, we notice that the surface integral of $\mathbf{F}$ over the added surface is zero, since

$$\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot(-\mathbf{e}_3) = 0$$

over this area. Thus, the divergence theorem states that we may calculate the required surface integral of **F** by evaluating

$$\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V$$

where V is the volume interior of the hemisphere. Since $\nabla\cdot\mathbf{F} = 2$, the result is merely twice the volume of the unit hemisphere, or $4\pi/3$.

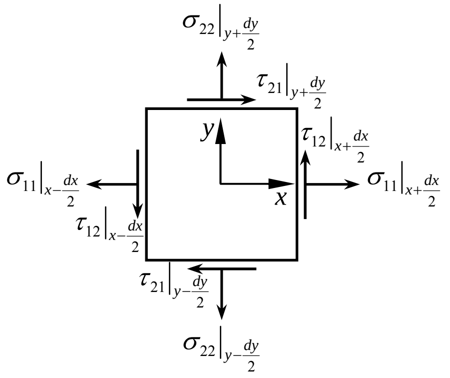
P1.14 Evaluate the surface integral of a vector, $\mathbf{F} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$, over the closed surface of the cube bounded by the planes, $x = \pm 1, y = \pm 1, z = \pm 1$, using the divergence theorem.

$$\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$$

Solution: Using the divergence theorem and $\nabla\cdot\mathbf{F} = 3$,

$$\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V = \iint_V 3\,\mathrm{d}V = 24$$

P1.15 Consider a unit-depth (in *z*-axis) infinitesimal element as shown in the figure. Using force equilibrium, derive the governing differential equation in two-dimension (equilibrium in *x*- and *y*-directions). Assume that a uniform body force, $\mathbf{f}^B = [f_1^B, f_2^B]$, is applied to the infinitesimal element.



Solution: Equilibrium in the *x*–direction yields the following equation:

$$\left(\sigma_{11}\Big|_{x+\frac{dx}{2}}\right)dy - \left(\sigma_{11}\Big|_{x-\frac{dx}{2}}\right)dy + \left(\tau_{21}\Big|_{y+\frac{dy}{2}}\right)dx - \left(\tau_{21}\Big|_{y-\frac{dy}{2}}\right)dx + f_1^B dxdy = 0$$

If the first-order Taylor series expansion is used to represent stresses on the surfaces of the rectangle in terms of stresses at the center, the first two terms in the above equation can be approximated by

$$\left(\sigma_{11}\Big|_{x+\frac{dx}{2}}\right)dy-\left(\sigma_{11}\Big|_{x-\frac{dx}{2}}\right)dy$$
$$=\left(\sigma_{11}\Big|_{x}+\frac{\partial\sigma_{11}}{\partial x}\frac{dx}{2}\right)dy-\left(\sigma_{11}\Big|_{x}-\frac{\partial\sigma_{11}}{\partial x}\frac{dx}{2}\right)dy=\frac{\partial\sigma_{11}}{\partial x}dxdy$$

Similarly, the next two terms can be approximated by

$$\left(\tau_{21}\Big|_{y+\frac{dy}{2}}\right)dx-\left(\tau_{21}\Big|_{y-\frac{dy}{2}}\right)dx$$
$$=\left(\tau_{21}\Big|_{y}+\frac{\partial\tau_{21}}{\partial y}\frac{dy}{2}\right)dx-\left(\tau_{21}\Big|_{y}-\frac{\partial\tau_{21}}{\partial y}\frac{dy}{2}\right)dx=\frac{\partial\tau_{21}}{\partial y}dxdy$$

By substituting these two equations into the original equation, we obtain an equilibrium equation in the *x*–direction as

$$\frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\tau_{21}}{\partial y}+f_1^B=0$$

Similarly, equilibrium in the *y*-direction yields the following equation:

$$\frac{\partial\tau_{12}}{\partial x}+\frac{\partial\sigma_{22}}{\partial y}+f_2^B=0$$

P1.16 In the above unit-depth (in z-axis) infinitesimal element, show that the stress tensor is symmetric using moment equilibrium.

Solution: Moment equilibrium with respect to the center of the element becomes

$$\left(\tau_{12}\Big|_{x+\frac{dx}{2}}\right)\frac{dxdy}{2}+\left(\tau_{12}\Big|_{x-\frac{dx}{2}}\right)\frac{dxdy}{2}-\left(\tau_{21}\Big|_{y+\frac{dy}{2}}\right)\frac{dxdy}{2}-\left(\tau_{21}\Big|_{y-\frac{dy}{2}}\right)\frac{dxdy}{2}=0$$

If the first-order Taylor series expansion is used to represent stresses on the surfaces of the rectangle in terms of stresses at the center,

$$\tau_{12}dxdy-\tau_{21}dxdy=0$$

Thus, the stress tensor is symmetric. The same relation can be shown for 3-D stress tensor.

P1.17 The principal stresses at a point in a body are given by $\sigma_1=4,\sigma_2=2,\sigma_3=1$, and the principal directions of the first two principal stresses are given by $\mathbf{n}^{(1)}=\frac{1}{\sqrt{2}}(0,1,-1)$ and $\mathbf{n}^{(2)}=\frac{1}{\sqrt{2}}(0,1,1)$. Determine the state of stress at the point; i.e., 6 components of stress tensor.

Solution:
Since the three principal directions are mutually orthogonal, the third principal direction can be calculated by using the cross-product of the two principal directions, as

$$\mathbf{n}^{(3)} = \mathbf{n}^{(1)} \times \mathbf{n}^{(2)} = (1,0,0)$$

Since these three principal directions are mutually orthogonal, they can be considered as a basis of coordinate system. In this new coordinate system, the stress tensor will only have diagonal components, which is the same as the three principal stresses. Then, the transformation between the two coordinate systems for a rank-2 tensor can be written as

$$[\boldsymbol{\sigma}]_{123} = [\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\sigma}]_{xyz}[\mathbf{Q}]$$

where $[\mathbf{Q}] = [\mathbf{n}^{(1)} \quad \mathbf{n}^{(2)} \quad \mathbf{n}^{(3)}]$ is the orthogonal transformation matrix between the two coordinate systems. Using the property that the inverse of an orthogonal matrix is the same as the transpose, the reverse relationship can be obtained as

$$[\boldsymbol{\sigma}]_{xyz} = [\mathbf{Q}][\boldsymbol{\sigma}]_{123}[\mathbf{Q}]^{\mathrm{T}}$$

Or,

$$[\boldsymbol{\sigma}]_{xyz} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The last matrix defines all 6 components of stress tensor.

P1.18 Find the principal stresses and the corresponding principal stress directions for the following cases of plane stress:

(a) $\sigma_{11} = 40$ MPa, $\sigma_{22} = 0$ MPa, $\sigma_{12} = 80$ MPa
(b) $\sigma_{11} = 140$ MPa, $\sigma_{22} = 20$ MPa, $\sigma_{12} = -60$ MPa
(c) $\sigma_{11} = -120$ MPa, $\sigma_{22} = 50$ MPa, $\sigma_{12} = 100$ MPa

Solution:
(a) The stress matrix becomes

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 40 & 80 \\ 80 & 0 \end{bmatrix} \text{MPa}$$

To find the principal stresses, the standard eigen value problem can be written as

$$\left[\boldsymbol{\sigma} - \sigma\mathbf{I}\right]\left\{\mathbf{n}\right\} = 0$$

The above problem will have non-trivial solution when the determinant of the coefficient matrix becomes zero:

$$\begin{vmatrix} \sigma_{xx} - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} - \sigma \end{vmatrix} = \begin{vmatrix} 40 - \sigma & 80 \\ 80 & 0 - \sigma \end{vmatrix} = 0$$

The equation of the determinant becomes:

$$\big((40 - \sigma)\cdot -\sigma\big) - \big(80 \cdot 80\big) = \sigma^2 - 40\sigma - 6400 = 0$$

The above quadratic equation yields two principal stresses, as

$$\sigma_1 = 102.46\,\mathrm{MPa} \text{ and } \sigma_2 = -62.46\,\mathrm{MPa}\,.$$

To determine the orientation of the first principal stresses, substitute σ_1 in the original eigen value problem to obtain

$$\begin{bmatrix} 40 - 102.46 & 80 \\ 80 & 0 - 102.46 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Since the determinant is zero, two equations are not independent

$$62.46 \cdot n_x = 80 \cdot n_y \text{ and } 80 \cdot n_x = -102.46 \cdot n_y\,.$$

Thus, we can only get the relation between n_x and n_y. Then using the condition $|\mathbf{n}| = 1$ we obtain

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.788 \\ 0.615 \end{Bmatrix}$$

To determine the orientation of the second principal stress, substitute σ_2 in the original eigen value problem to obtain

$$\begin{bmatrix} 40 + 62.46 & 80 \\ 80 & 0 + 62.46 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$102.46 \cdot n_x = -80 \cdot n_y \text{ and } 80 \cdot n_x = -62.46 \cdot n_y\,.$$

Using similar procedures as above, the eigen vector of σ_2 can be obtained as

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.615 \\ -0.788 \end{Bmatrix}$$

Note that if $\mathbf{n}$ is a principal direction, $-\mathbf{n}$ is also a principal direction

(b) Repeat the procedure in (a) to obtain

$$\sigma_1 = 164.85\,\mathrm{MPa} \text{ and } \sigma_2 = -4.85\,\mathrm{MPa}\,.$$

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} -0.924 \\ 0.383 \end{Bmatrix} \text{ and } \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.383 \\ 0.924 \end{Bmatrix}$$

(c) Repeat the procedure in (a) to obtain

$$\sigma_1 = 96.24\,\mathrm{MPa} \quad \text{and} \quad \sigma_2 = -166.24\,\mathrm{MPa}\,.$$

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.420 \\ 0.908 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} -0.908 \\ 0.420 \end{Bmatrix}$$

Note that for the case of plane stress $\sigma_3=0$ is also a principal stress and the corresponding principal stress direction is given by $n^{(3)}=(0,0,1)$

P1.19 Determine the principal stresses and their associated directions, when the stress matrix at a point is given by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \mathrm{MPa}$$

Solution:
Use Eq. (1.50) with the coefficients of $I_1=3$, $I_2=-3$, and $I_3=-1$,

$$\lambda^3 - 3\lambda^2 - 3\lambda + 1 = 0$$

By solving the above cubic equation,

$$\sigma_1 = 3.73\,\mathrm{MPa}, \quad \sigma_2 = 0.268\,\mathrm{MPa}, \quad \sigma_3 = -1.00\,\mathrm{MPa}$$

(a) Principal direction corresponding to σ_1:

$$\begin{aligned} (1-3.7321)n_x^1 + n_y^1 + n_z^1 &= 0 \\ n_x^1 + (1-3.7321)n_y^1 + 2n_z^1 &= 0 \\ n_x^1 + 2n_y^1 + (1-3.7321)n_z^1 &= 0 \end{aligned}$$

Solving the above equations with $|\mathbf{n}^1| = 1$ yields

$$\mathbf{n}^1 = \{\pm 0.4597, \pm 0.6280, \pm 0.6280\}$$

(b) Principal direction corresponding to σ_2:

$$\begin{aligned} (1-0.2679)n_x^1 + n_y^1 + n_z^1 &= 0 \\ n_x^2 + (1-0.2679)n_y^2 + 2n_z^2 &= 0 \\ n_x^2 + 2n_y^2 + (1-0.2679)n_z^2 &= 0 \end{aligned}$$

Solving the above equations with $|\mathbf{n}^2| = 1$ yields

$$\mathbf{n}^2 = \{\pm 0.8881, \mp 0.3251, \mp 0.3251\}$$

(c) Principal direction corresponding to σ_3: