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## Chapter 1

# The Wave Function

### Problem 1.1

(a)

$$\langle j \rangle^2 = 21^2 = \boxed{441.}$$

$$\begin{aligned} \langle j^2 \rangle &= \frac{1}{N} \sum j^2 N(j) = \frac{1}{14} [(14^2) + (15^2) + 3(16^2) + 2(22^2) + 2(24^2) + 5(25^2)] \\ &= \frac{1}{14} (196 + 225 + 768 + 968 + 1152 + 3125) = \frac{6434}{14} = \boxed{459.571.} \end{aligned}$$

(b)

$j$	$\Delta j = j - \langle j \rangle$
14	$14 - 21 = -7$
15	$15 - 21 = -6$
16	$16 - 21 = -5$
22	$22 - 21 = 1$
24	$24 - 21 = 3$
25	$25 - 21 = 4$

$$\begin{aligned} \sigma^2 &= \frac{1}{N} \sum (\Delta j)^2 N(j) = \frac{1}{14} [(-7)^2 + (-6)^2 + (-5)^2 \cdot 3 + (1)^2 \cdot 2 + (3)^2 \cdot 2 + (4)^2 \cdot 5] \\ &= \frac{1}{14} (49 + 36 + 75 + 2 + 18 + 80) = \frac{260}{14} = \boxed{18.571.} \end{aligned}$$

$$\sigma = \sqrt{18.571} = \boxed{4.309.}$$

(c)

$$\langle j^2 \rangle - \langle j \rangle^2 = 459.571 - 441 = 18.571. \quad [\text{Agrees with (b).}]$$


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**Problem 1.2**

(a)

$$\langle x^2 \rangle = \int_0^h x^2 \frac{1}{2\sqrt{hx}} dx = \frac{1}{2\sqrt{h}} \left( \frac{2}{5} x^{5/2} \right) \Big|_0^h = \frac{h^2}{5}.$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{h^2}{5} - \left( \frac{h}{3} \right)^2 = \frac{4}{45} h^2 \Rightarrow \sigma = \boxed{\frac{2h}{3\sqrt{5}} = 0.2981h}.$$

(b)

$$P = 1 - \int_{x_-}^{x_+} \frac{1}{2\sqrt{hx}} dx = 1 - \frac{1}{2\sqrt{h}} (2\sqrt{x}) \Big|_{x_-}^{x_+} = 1 - \frac{1}{\sqrt{h}} (\sqrt{x_+} - \sqrt{x_-}).$$

$$x_+ \equiv \langle x \rangle + \sigma = 0.3333h + 0.2981h = 0.6315h; \quad x_- \equiv \langle x \rangle - \sigma = 0.3333h - 0.2981h = 0.0352h.$$

$$P = 1 - \sqrt{0.6315} + \sqrt{0.0352} = \boxed{0.393}.$$

**Problem 1.3**

(a)

$$1 = \int_{-\infty}^{\infty} A e^{-\lambda(x-a)^2} dx. \quad \text{Let } u \equiv x - a, \quad du = dx, \quad u : -\infty \rightarrow \infty.$$

$$1 = A \int_{-\infty}^{\infty} e^{-\lambda u^2} du = A \sqrt{\frac{\pi}{\lambda}} \Rightarrow \boxed{A = \sqrt{\frac{\lambda}{\pi}}}.$$

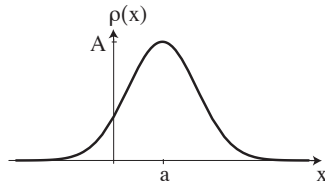
(b)

$$\begin{aligned} \langle x \rangle &= A \int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2} dx = A \int_{-\infty}^{\infty} (u+a) e^{-\lambda u^2} du \\ &= A \left[ \int_{-\infty}^{\infty} u e^{-\lambda u^2} du + a \int_{-\infty}^{\infty} e^{-\lambda u^2} du \right] = A \left( 0 + a \sqrt{\frac{\pi}{\lambda}} \right) = \boxed{a}. \end{aligned}$$

$$\begin{aligned} \langle x^2 \rangle &= A \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} dx \\ &= A \left\{ \int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du + 2a \int_{-\infty}^{\infty} u e^{-\lambda u^2} du + a^2 \int_{-\infty}^{\infty} e^{-\lambda u^2} du \right\} \\ &= A \left[ \frac{1}{2\lambda} \sqrt{\frac{\pi}{\lambda}} + 0 + a^2 \sqrt{\frac{\pi}{\lambda}} \right] = \boxed{a^2 + \frac{1}{2\lambda}}. \end{aligned}$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda}; \quad \boxed{\sigma = \frac{1}{\sqrt{2\lambda}}}.$$

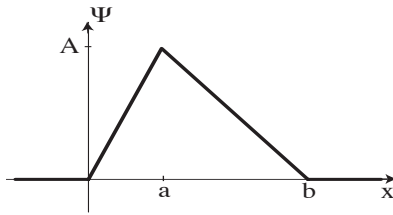
(c)

**Problem 1.4**

(a)

$$\begin{aligned}
 1 &= \frac{|A|^2}{a^2} \int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2} \int_a^b (b-x)^2 dx = |A|^2 \left\{ \frac{1}{a^2} \left( \frac{x^3}{3} \right) \Big|_0^a + \frac{1}{(b-a)^2} \left( -\frac{(b-x)^3}{3} \right) \Big|_a^b \right\} \\
 &= |A|^2 \left[ \frac{a}{3} + \frac{b-a}{3} \right] = |A|^2 \frac{b}{3} \Rightarrow \boxed{A = \sqrt{\frac{3}{b}}}.
 \end{aligned}$$

(b)

(c) At  $\boxed{x = a}$ .

(d)

$$P = \int_0^a |\Psi|^2 dx = \frac{|A|^2}{a^2} \int_0^a x^2 dx = |A|^2 \frac{a}{3} = \boxed{\frac{a}{b}} \cdot \begin{cases} P = 1 & \text{if } b = a, \checkmark \\ P = 1/2 & \text{if } b = 2a, \checkmark \end{cases}$$

(e)

$$\begin{aligned}
 \langle x \rangle &= \int x |\Psi|^2 dx = |A|^2 \left\{ \frac{1}{a^2} \int_0^a x^3 dx + \frac{1}{(b-a)^2} \int_a^b x(b-x)^2 dx \right\} \\
 &= \frac{3}{b} \left\{ \frac{1}{a^2} \left( \frac{x^4}{4} \right) \Big|_0^a + \frac{1}{(b-a)^2} \left( b^2 \frac{x^2}{2} - 2b \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_a^b \right\} \\
 &= \frac{3}{4b(b-a)^2} [a^2(b-a)^2 + 2b^4 - 8b^4/3 + b^4 - 2a^2b^2 + 8a^3b/3 - a^4] \\
 &= \frac{3}{4b(b-a)^2} \left( \frac{b^4}{3} - a^2b^2 + \frac{2}{3}a^3b \right) = \frac{1}{4(b-a)^2} (b^3 - 3a^2b + 2a^3) = \boxed{\frac{2a+b}{4}}.
 \end{aligned}$$

**Problem 1.5**

(a)

$$1 = \int |\Psi|^2 dx = 2|A|^2 \int_0^\infty e^{-2\lambda x} dx = 2|A|^2 \left( \frac{e^{-2\lambda x}}{-2\lambda} \right) \Big|_0^\infty = \frac{|A|^2}{\lambda}; \quad \boxed{A = \sqrt{\lambda}.}$$

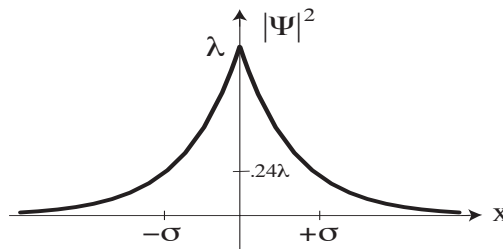
(b)

$$\langle x \rangle = \int x |\Psi|^2 dx = |A|^2 \int_{-\infty}^\infty x e^{-2\lambda|x|} dx = \boxed{0}. \quad [\text{Odd integrand.}]$$

$$\langle x^2 \rangle = 2|A|^2 \int_0^\infty x^2 e^{-2\lambda x} dx = 2\lambda \left[ \frac{2}{(2\lambda)^3} \right] = \boxed{\frac{1}{2\lambda^2}}.$$

(c)

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2}; \quad \boxed{\sigma = \frac{1}{\sqrt{2\lambda}}}. \quad |\Psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda/\sqrt{2\lambda}} = \lambda e^{-\sqrt{2}} = 0.2431\lambda.$$

*Probability outside:*

$$2 \int_\sigma^\infty |\Psi|^2 dx = 2|A|^2 \int_\sigma^\infty e^{-2\lambda x} dx = 2\lambda \left( \frac{e^{-2\lambda x}}{-2\lambda} \right) \Big|_\sigma^\infty = e^{-2\lambda\sigma} = \boxed{e^{-\sqrt{2}} = 0.2431}.$$

**Problem 1.6**

For integration by parts, the differentiation has to be with respect to the *integration* variable – in this case the differentiation is with respect to  $t$ , but the integration variable is  $x$ . It's true that

$$\frac{\partial}{\partial t}(x|\Psi|^2) = \frac{\partial x}{\partial t}|\Psi|^2 + x \frac{\partial}{\partial t}|\Psi|^2 = x \frac{\partial}{\partial t}|\Psi|^2,$$

but this does *not* allow us to perform the integration:

$$\int_a^b x \frac{\partial}{\partial t}|\Psi|^2 dx = \int_a^b \frac{\partial}{\partial t}(x|\Psi|^2) dx \neq (x|\Psi|^2) \Big|_a^b.$$

**Problem 1.7**

From Eq. 1.33,  $\frac{d\langle p \rangle}{dt} = -i\hbar \int \frac{\partial}{\partial t} (\Psi^* \frac{\partial \Psi}{\partial x}) dx$ . But, noting that  $\frac{\partial^2 \Psi}{\partial x \partial t} = \frac{\partial^2 \Psi}{\partial t \partial x}$  and using Eqs. 1.23-1.24:

$$\begin{aligned} \frac{\partial}{\partial t} \left( \Psi^* \frac{\partial \Psi}{\partial x} \right) &= \frac{\partial \Psi^*}{\partial t} \frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x} \left( \frac{\partial \Psi}{\partial t} \right) = \left[ -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V \Psi^* \right] \frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x} \left[ \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V \Psi \right] \\ &= \frac{i\hbar}{2m} \left[ \Psi^* \frac{\partial^3 \Psi}{\partial x^3} - \frac{\partial^2 \Psi^*}{\partial x^2} \frac{\partial \Psi}{\partial x} \right] + \frac{i}{\hbar} \left[ V \Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial}{\partial x} (V \Psi) \right] \end{aligned}$$

The first term integrates to zero, using integration by parts twice, and the second term can be simplified to  $V \Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* V \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial V}{\partial x} \Psi = -|\Psi|^2 \frac{\partial V}{\partial x}$ . So

$$\frac{d\langle p \rangle}{dt} = -i\hbar \left( \frac{i}{\hbar} \right) \int -|\Psi|^2 \frac{\partial V}{\partial x} dx = \left\langle -\frac{\partial V}{\partial x} \right\rangle. \quad \text{QED}$$

**Problem 1.8**

Suppose  $\Psi$  satisfies the Schrödinger equation *without*  $V_0$ :  $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi$ . We want to find the solution  $\Psi_0$  *with*  $V_0$ :  $i\hbar \frac{\partial \Psi_0}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_0}{\partial x^2} + (V + V_0) \Psi_0$ .

*Claim:*  $\Psi_0 = \Psi e^{-iV_0 t/\hbar}$ .

$$\begin{aligned} \text{Proof: } i\hbar \frac{\partial \Psi_0}{\partial t} &= i\hbar \frac{\partial \Psi}{\partial t} e^{-iV_0 t/\hbar} + i\hbar \Psi \left( -\frac{iV_0}{\hbar} \right) e^{-iV_0 t/\hbar} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi \right] e^{-iV_0 t/\hbar} + V_0 \Psi e^{-iV_0 t/\hbar} \\ &= -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_0}{\partial x^2} + (V + V_0) \Psi_0. \quad \text{QED} \end{aligned}$$

This has *no* effect on the expectation value of a dynamical variable, since the extra phase factor, being independent of  $x$ , cancels out in Eq. 1.36.

**Problem 1.9**

(a)

$$1 = 2|A|^2 \int_0^\infty e^{-2amx^2/\hbar} dx = 2|A|^2 \frac{1}{2} \sqrt{\frac{\pi}{(2am/\hbar)}} = |A|^2 \sqrt{\frac{\pi\hbar}{2am}}; \quad \boxed{A = \left( \frac{2am}{\pi\hbar} \right)^{1/4}}$$

(b)

$$\frac{\partial \Psi}{\partial t} = -ia\Psi; \quad \frac{\partial \Psi}{\partial x} = -\frac{2amx}{\hbar} \Psi; \quad \frac{\partial^2 \Psi}{\partial x^2} = -\frac{2am}{\hbar} \left( \Psi + x \frac{\partial \Psi}{\partial x} \right) = -\frac{2am}{\hbar} \left( 1 - \frac{2amx^2}{\hbar} \right) \Psi.$$

Plug these into the Schrödinger equation,  $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi$ :

$$\begin{aligned} V \Psi &= i\hbar(-ia)\Psi + \frac{\hbar^2}{2m} \left( -\frac{2am}{\hbar} \right) \left( 1 - \frac{2amx^2}{\hbar} \right) \Psi \\ &= \left[ \hbar a - \hbar a \left( 1 - \frac{2amx^2}{\hbar} \right) \right] \Psi = 2a^2 m x^2 \Psi, \quad \text{so } \boxed{V(x) = 2ma^2 x^2}. \end{aligned}$$

(c)

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi|^2 dx = \boxed{0}. \quad [\text{Odd integrand.}]$$

$$\langle x^2 \rangle = 2|A|^2 \int_0^{\infty} x^2 e^{-2amx^2/\hbar} dx = 2|A|^2 \frac{1}{2^2(2am/\hbar)} \sqrt{\frac{\pi\hbar}{2am}} = \boxed{\frac{\hbar}{4am}}.$$

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt} = \boxed{0}.$$

$$\begin{aligned} \langle p^2 \rangle &= \int \Psi^* \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 \Psi dx = -\hbar^2 \int \Psi^* \frac{\partial^2 \Psi}{\partial x^2} dx \\ &= -\hbar^2 \int \Psi^* \left[ -\frac{2am}{\hbar} \left( 1 - \frac{2amx^2}{\hbar} \right) \Psi \right] dx = 2am\hbar \left\{ \int |\Psi|^2 dx - \frac{2am}{\hbar} \int x^2 |\Psi|^2 dx \right\} \\ &= 2am\hbar \left( 1 - \frac{2am}{\hbar} \langle x^2 \rangle \right) = 2am\hbar \left( 1 - \frac{2am}{\hbar} \frac{\hbar}{4am} \right) = 2am\hbar \left( \frac{1}{2} \right) = \boxed{am\hbar}. \end{aligned}$$

(d)

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{\hbar}{4am} \implies \sigma_x = \sqrt{\frac{\hbar}{4am}}; \quad \sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 = am\hbar \implies \sigma_p = \sqrt{am\hbar}.$$

$$\sigma_x \sigma_p = \sqrt{\frac{\hbar}{4am}} \sqrt{am\hbar} = \frac{\hbar}{2}. \quad \text{This is (just barely) consistent with the uncertainty principle.}$$

**Problem 1.10**From Math Tables:  $\pi = 3.141592653589793238462643 \dots$ 

$$(a) \quad \begin{array}{cccccc} P(0) = 0 & P(1) = 2/25 & P(2) = 3/25 & P(3) = 5/25 & P(4) = 3/25 \\ P(5) = 3/25 & P(6) = 3/25 & P(7) = 1/25 & P(8) = 2/25 & P(9) = 3/25 \end{array}$$

In general,  $P(j) = \frac{N(j)}{N}$ .(b) *Most probable:*  $\boxed{3}$ . *Median:* 13 are  $\leq 4$ , 12 are  $\geq 5$ , so median is  $\boxed{4}$ .

$$\begin{aligned} \text{Average: } \langle j \rangle &= \frac{1}{25} [0 \cdot 0 + 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 5 + 4 \cdot 3 + 5 \cdot 3 + 6 \cdot 3 + 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3] \\ &= \frac{1}{25} [0 + 2 + 6 + 15 + 12 + 15 + 18 + 7 + 16 + 27] = \frac{118}{25} = \boxed{4.72}. \end{aligned}$$

$$\begin{aligned} (c) \quad \langle j^2 \rangle &= \frac{1}{25} [0 + 1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 5 + 4^2 \cdot 3 + 5^2 \cdot 3 + 6^2 \cdot 3 + 7^2 \cdot 1 + 8^2 \cdot 2 + 9^2 \cdot 3] \\ &= \frac{1}{25} [0 + 2 + 12 + 45 + 48 + 75 + 108 + 49 + 128 + 243] = \frac{710}{25} = \boxed{28.4}. \\ \sigma^2 &= \langle j^2 \rangle - \langle j \rangle^2 = 28.4 - 4.72^2 = 28.4 - 22.2784 = 6.1216; \quad \sigma = \sqrt{6.1216} = \boxed{2.474}. \end{aligned}$$

**Problem 1.11**

(a)

$$\frac{1}{2}mv^2 + V = E \quad \rightarrow \quad \boxed{v(x) = \sqrt{\frac{2}{m}(E - V(x))}}$$

(b)

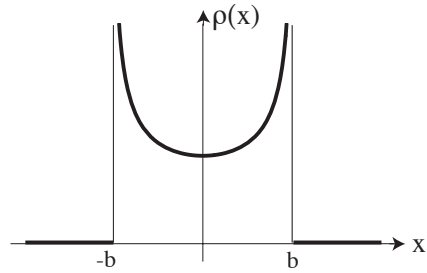
$$T = \int_a^b \frac{1}{\sqrt{\frac{2}{m}(E - \frac{1}{2}kx^2)}} dx = \sqrt{\frac{m}{k}} \int_a^b \frac{1}{\sqrt{(2E/k) - x^2}} dx.$$

Turning points:  $v = 0 \Rightarrow E = V = \frac{1}{2}kb^2 \Rightarrow b = \sqrt{2E/k}$ ;  $a = -b$ .

$$\begin{aligned} T &= 2\sqrt{\frac{m}{k}} \int_0^b \frac{1}{\sqrt{b^2 - x^2}} dx = 2\sqrt{\frac{m}{k}} \sin^{-1}\left(\frac{x}{b}\right) \Big|_0^b = 2\sqrt{\frac{m}{k}} \sin^{-1}(1) \\ &= 2\sqrt{\frac{m}{k}} \left(\frac{\pi}{2}\right) = \pi\sqrt{\frac{m}{k}}. \end{aligned}$$

$$\rho(x) = \frac{1}{\pi\sqrt{\frac{m}{k}}\sqrt{\frac{2}{m}(E - \frac{1}{2}kx^2)}} = \boxed{\frac{1}{\pi\sqrt{b^2 - x^2}}}.$$

$$\int_a^b \rho(x) dx = \frac{2}{\pi} \int_0^b \frac{1}{\sqrt{b^2 - x^2}} dx = \frac{2}{\pi} \left(\frac{\pi}{2}\right) = 1. \quad \checkmark$$

(c)  $\boxed{\langle x \rangle = 0}$ .

$$\begin{aligned} \langle x^2 \rangle &= \frac{1}{\pi} \int_{-b}^b \frac{x^2}{\sqrt{b^2 - x^2}} dx = \frac{2}{\pi} \int_0^b \frac{x^2}{\sqrt{b^2 - x^2}} dx \\ &= \frac{2}{\pi} \left[ -\frac{x}{2}\sqrt{b^2 - x^2} + \frac{b^2}{2} \sin^{-1}\left(\frac{x}{b}\right) \right] \Big|_0^b = \frac{b^2}{\pi} \sin^{-1}(1) = \frac{b^2}{\pi} \frac{\pi}{2} = \frac{b^2}{2} = \boxed{\frac{E}{k}}. \end{aligned}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle x^2 \rangle} = \frac{b}{\sqrt{2}} = \boxed{\sqrt{\frac{E}{k}}}.$$

**Problem 1.12**

(a)

$$\rho(p) dp = \frac{dt}{T} = \frac{|dt/dp| dp}{T}$$

where  $dt$  is now the time it spends with momentum in the range  $dp$  ( $dt$  is intrinsically positive, but  $dp/dt = F = -kx$  runs negative—hence the absolute value). Now

$$\frac{p^2}{2m} + \frac{1}{2}kx^2 = E \Rightarrow x = \pm \sqrt{\frac{2}{k} \left( E - \frac{p^2}{2m} \right)},$$

so

$$\rho(p) = \frac{1}{\pi \sqrt{\frac{m}{k}} k \sqrt{\frac{2}{k} \left( E - \frac{p^2}{2m} \right)}} = \boxed{\frac{1}{\pi \sqrt{2mE - p^2}}} = \frac{1}{\pi \sqrt{c^2 - p^2}},$$

where  $c \equiv \sqrt{2mE}$ . This is the same as  $\rho(x)$  (Problem 1.11(b)), with  $c$  in place of  $b$  (and, of course,  $p$  in place of  $x$ ).

(b) From Problem 1.11(c),  $\langle p \rangle = 0$ ,  $\langle p^2 \rangle = \frac{c^2}{2}$ ,  $\sigma_p = \frac{c}{\sqrt{2}} = \sqrt{mE}$ .

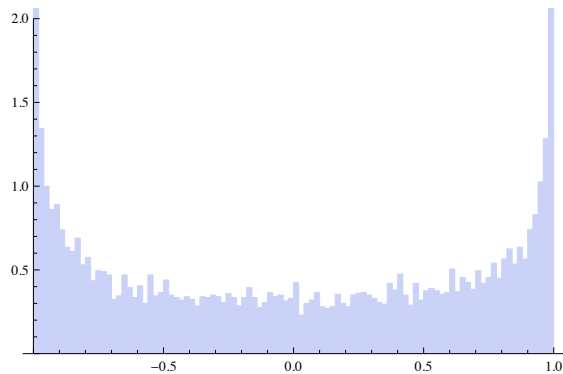
(c)  $\sigma_x \sigma_p = \sqrt{\frac{E}{k}} \sqrt{mE} = \boxed{\sqrt{\frac{m}{k}} E} = \frac{E}{\omega}$ . If  $E \geq \frac{1}{2} \hbar \omega$ , then  $\sigma_x \sigma_p \geq \frac{1}{2} \hbar$ , which is precisely the Heisenberg uncertainty principle!

### Problem 1.13

```
x[t_] := Cos[t]
```

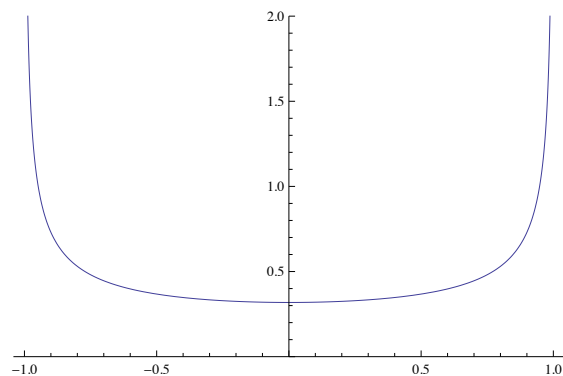
```
snapshots = Table[x[π RandomReal[j]], {j, 10000}]
```

```
Histogram[snapshots, 100, "PDF", PlotRange -> {0, 2}]
```

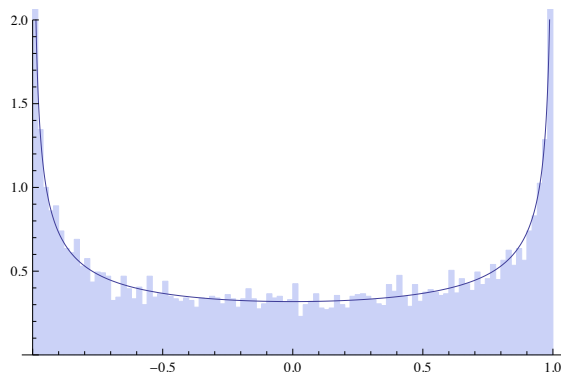


$$r[x_] := \frac{1}{\pi \sqrt{1-x^2}}$$

```
Plot[r[x], {x, -1, 1}, PlotRange -> {0, 2}]
```



```
Show[Histogram[snapshots, 100, "PDF", PlotRange -> {0, 2}],  
Plot[r[x], {x, -1, 1}, PlotRange -> {0, 2}]]
```



### Problem 1.14

(a)  $P_{ab}(t) = \int_a^b |\Psi(x, t)|^2 dx$ , so  $\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial}{\partial t} |\Psi|^2 dx$ . But (Eq. 1.25):

$$\frac{\partial |\Psi|^2}{\partial t} = \frac{\partial}{\partial x} \left[ \frac{i\hbar}{2m} \left( \Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \right] = -\frac{\partial}{\partial x} J(x, t).$$

$$\therefore \frac{dP_{ab}}{dt} = - \int_a^b \frac{\partial}{\partial x} J(x, t) dx = - [J(x, t)]_a^b = J(a, t) - J(b, t). \quad \text{QED}$$

Probability is dimensionless, so  $J$  has the dimensions 1/time, and units  $\boxed{\text{seconds}^{-1}}$ .

(b) Here  $\Psi(x, t) = f(x)e^{-iat}$ , where  $f(x) \equiv Ae^{-amx^2/\hbar}$ , so  $\Psi \frac{\partial \Psi^*}{\partial x} = f e^{-iat} \frac{df}{dx} e^{iat} = f \frac{df}{dx}$ ,

and  $\Psi^* \frac{\partial \Psi}{\partial x} = f \frac{df}{dx}$  too, so  $\boxed{J(x, t) = 0}$ .

**Problem 1.15**

Use Eqs. [1.23] and [1.24], and integration by parts:

$$\begin{aligned}
 \frac{d}{dt} \int_{-\infty}^{\infty} \Psi_1^* \Psi_2 dx &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} (\Psi_1^* \Psi_2) dx = \int_{-\infty}^{\infty} \left( \frac{\partial \Psi_1^*}{\partial t} \Psi_2 + \Psi_1^* \frac{\partial \Psi_2}{\partial t} \right) dx \\
 &= \int_{-\infty}^{\infty} \left[ \left( \frac{-i\hbar}{2m} \frac{\partial^2 \Psi_1^*}{\partial x^2} + \frac{i}{\hbar} V \Psi_1^* \right) \Psi_2 + \Psi_1^* \left( \frac{i\hbar}{2m} \frac{\partial^2 \Psi_2}{\partial x^2} - \frac{i}{\hbar} V \Psi_2 \right) \right] dx \\
 &= -\frac{i\hbar}{2m} \int_{-\infty}^{\infty} \left( \frac{\partial^2 \Psi_1^*}{\partial x^2} \Psi_2 - \Psi_1^* \frac{\partial^2 \Psi_2}{\partial x^2} \right) dx \\
 &= -\frac{i\hbar}{2m} \left[ \frac{\partial \Psi_1^*}{\partial x} \Psi_2 \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial \Psi_1^*}{\partial x} \frac{\partial \Psi_2}{\partial x} dx - \Psi_1^* \frac{\partial \Psi_2}{\partial x} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\partial \Psi_1^*}{\partial x} \frac{\partial \Psi_2}{\partial x} dx \right] = 0. \text{ QED}
 \end{aligned}$$


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**Problem 1.16**

(a)

$$\begin{aligned}
 1 &= |A|^2 \int_{-a}^a (a^2 - x^2)^2 dx = 2|A|^2 \int_0^a (a^4 - 2a^2x^2 + x^4) dx = 2|A|^2 \left[ a^4x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right] \Big|_0^a \\
 &= 2|A|^2 a^5 \left( 1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{16}{15} a^5 |A|^2, \text{ so } A = \sqrt{\frac{15}{16a^5}}.
 \end{aligned}$$

(b)

$$\langle x \rangle = \int_{-a}^a x |\Psi|^2 dx = \boxed{0}. \quad (\text{Odd integrand.})$$

(c)

$$\langle p \rangle = \frac{\hbar}{i} A^2 \int_{-a}^a (a^2 - x^2) \underbrace{\frac{d}{dx} (a^2 - x^2)}_{-2x} dx = \boxed{0}. \quad (\text{Odd integrand.})$$

Since we only know  $\langle x \rangle$  at  $t = 0$  we cannot calculate  $d\langle x \rangle/dt$  directly.

(d)

$$\begin{aligned}
 \langle x^2 \rangle &= A^2 \int_{-a}^a x^2 (a^2 - x^2)^2 dx = 2A^2 \int_0^a (a^4x^2 - 2a^2x^4 + x^6) dx \\
 &= 2 \frac{15}{16a^5} \left[ a^4 \frac{x^3}{3} - 2a^2 \frac{x^5}{5} + \frac{x^7}{7} \right] \Big|_0^a = \frac{15}{8a^5} (a^7) \left( \frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right) \\
 &= \frac{15a^2}{8} \left( \frac{35 - 42 + 15}{\cancel{3} \cdot \cancel{5} \cdot 7} \right) = \frac{a^2}{8} \cdot \frac{8}{7} = \boxed{\frac{a^2}{7}}.
 \end{aligned}$$

(e)

$$\begin{aligned}\langle p^2 \rangle &= -A^2 \hbar^2 \int_{-a}^a (a^2 - x^2) \underbrace{\frac{d^2}{dx^2} (a^2 - x^2)}_{-2} dx = 2A^2 \hbar^2 2 \int_0^a (a^2 - x^2) dx \\ &= 4 \cdot \frac{15}{16a^5} \hbar^2 \left( a^2 x - \frac{x^3}{3} \right) \Big|_0^a = \frac{15\hbar^2}{4a^5} \left( a^3 - \frac{a^3}{3} \right) = \frac{15\hbar^2}{4a^2} \cdot \frac{2}{3} = \boxed{\frac{5}{2} \frac{\hbar^2}{a^2}}.\end{aligned}$$

(f)

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{7} a^2} = \boxed{\frac{a}{\sqrt{7}}}.$$

(g)

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{5}{2} \frac{\hbar^2}{a^2}} = \boxed{\sqrt{\frac{5}{2}} \frac{\hbar}{a}}.$$

(h)

$$\sigma_x \sigma_p = \frac{a}{\sqrt{7}} \cdot \sqrt{\frac{5}{2}} \frac{\hbar}{a} = \sqrt{\frac{5}{14}} \hbar = \sqrt{\frac{10}{7}} \frac{\hbar}{2} > \frac{\hbar}{2}. \checkmark$$

**Problem 1.17**(a) Eq. 1.24 now reads  $\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V^* \Psi^*$ , and Eq. 1.25 picks up an extra term:

$$\frac{\partial}{\partial t} |\Psi|^2 = \dots + \frac{i}{\hbar} |\Psi|^2 (V^* - V) = \dots + \frac{i}{\hbar} |\Psi|^2 (V_0 + i\Gamma - V_0 + i\Gamma) = \dots - \frac{2\Gamma}{\hbar} |\Psi|^2,$$

and Eq. 1.27 becomes  $\frac{dP}{dt} = -\frac{2\Gamma}{\hbar} \int_{-\infty}^{\infty} |\Psi|^2 dx = -\frac{2\Gamma}{\hbar} P$ . QED

(b)

$$\frac{dP}{P} = -\frac{2\Gamma}{\hbar} dt \implies \ln P = -\frac{2\Gamma}{\hbar} t + \text{constant} \implies \boxed{P(t) = P(0)e^{-2\Gamma t/\hbar}}, \text{ so } \boxed{\tau = \frac{\hbar}{2\Gamma}}.$$

**Problem 1.18**

$$\frac{\hbar}{\sqrt{3mk_B T}} > d \implies T < \frac{\hbar^2}{3mk_B d^2}.$$

(a) Electrons ( $m = 9.1 \times 10^{-31}$  kg):

$$T < \frac{(6.6 \times 10^{-34})^2}{3(9.1 \times 10^{-31})(1.4 \times 10^{-23})(3 \times 10^{-10})^2} = \boxed{1.3 \times 10^5 \text{ K.}}$$

Silicon nuclei ( $m = 28m_p = 28(1.7 \times 10^{-27}) = 4.8 \times 10^{-26}$  kg):

$$T < \frac{(6.6 \times 10^{-34})^2}{3(4.8 \times 10^{-26})(1.4 \times 10^{-23})(3 \times 10^{-10})^2} = \boxed{2.4 \text{ K.}}$$

(b)  $PV = Nk_B T$ ; volume occupied by one molecule ( $N = 1$ ,  $V = d^3$ )  $\Rightarrow d = (k_B T/P)^{1/3}$ .

$$T < \frac{h^2}{3mk_B} \left( \frac{P}{k_B T} \right)^{2/3} \Rightarrow T^{5/3} < \frac{h^2}{3m} \frac{P^{2/3}}{k_B^{5/3}} \Rightarrow T < \frac{1}{k_B} \left( \frac{h^2}{3m} \right)^{3/5} P^{2/5}.$$

For helium ( $m = 4m_p = 6.8 \times 10^{-27}$  kg) at 1 atm =  $1.0 \times 10^5$  N/m<sup>2</sup>:

$$T < \frac{1}{(1.4 \times 10^{-23})} \left( \frac{(6.6 \times 10^{-34})^2}{3(6.8 \times 10^{-27})} \right)^{3/5} (1.0 \times 10^5)^{2/5} = \boxed{2.8 \text{ K.}}$$

For atomic hydrogen ( $m = m_p = 1.7 \times 10^{-27}$  kg) with  $d = 0.01$  m:

$$T < \frac{(6.6 \times 10^{-34})^2}{3(1.7 \times 10^{-27})(1.4 \times 10^{-23})(10^{-2})^2} = \boxed{6.2 \times 10^{-14} \text{ K.}}$$

At 3 K it is definitely in the classical regime.

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